Real Teichmüller Spaces and Moduli of Real Algebraic Curves

J. Huisman

Abstract. This paper is an introduction to the analytic theory of moduli of real algebraic curves. We prove, in quite some detail, that the moduli space $M_{g/\mathbb{R}}$ of real algebraic curves of given genus g admits a structure of a semianalytic variety, and we study its connected components. We show that $M_{g/\mathbb{R}}$, endowed with this structure of a semianalytic variety, is a coarse moduli space.

Contents

1991 *Mathematics Subject Classification.* Primary 14H15, 14P99; Secondary 30F60.
Key words and phrases. Teichmüller space, real algebraic curve, moduli space.
Typeset using XY-pic.

1. Introduction

Moduli of real algebraic curves have known growing interest throughout the past decade [**18, 14, 17, 20, 6**]. Although the moduli problem of real algebraic curves is interesting in its own right, the principal motivation to study this moduli problem—in my opinion—is to understand the topology of real algebraic curves. By this, I do not mean the topology of single real algebraic curves, but their topology when they vary in families. To illustrate how the study of moduli of real algebraic curves can shed light on the topology of families of such curves, I'll give an example on stable real algebraic curves.

Let us recall the following remark, which is attributed to F. Klein:

REMARK 1.1 ([**11**, Proposition 2]). Let $\{C_t\}_{t\in\mathbb{R}}$ be a family of real algebraic curves, varying continuously with t. Suppose that the curves C_t are nonsingular for $t \neq 0$. Suppose, moreover, that the C_t are dividing for $t < 0$, i.e., $C_t(\mathbb{C})\backslash C_t(\mathbb{R})$ is not connected for $t < 0$. Let n_- be the number of connected components of $C_t(\mathbb{R})$, for any $t < 0$. Let, similarly, n_+ be the number of connected components of $C_t(\mathbb{R})$ for any $t > 0$. If $n_+ > n_-$, then the curve C_0 has either at least two singularities, or at least one singularity that is not an ordinary double point.

This remark can be interpreted as a statement on the structure of the moduli space of stable real algebraic curves. Let $\overline{M}_{g/\mathbb{R}}$ be the moduli space of stable real algebraic curves of genus g [**17**]. Let S be the closure of the subset of nonsingular dividing real algebraic curves $C \in \overline{M}_{g/\mathbb{R}}$ such that $C(\mathbb{R})$ has n_- connected components. Let T be the closure of the subset of nonsingular real algebraic curves $C \in \overline{M}_{g/\mathbb{R}}$ such that $C(\mathbb{R})$ has more than n_- connected components. Remark 1.1 can then be expressed by saying that the intersection $S \cap T$ is of codimension at least 2 in $\overline{M}_{g/\mathbb{R}}$. It is clear that a further study of the geometry of the moduli space of stable real algebraic curves will lead to generalizations of Remark 1.1. Other examples of how the study of moduli of real algebraic curves sheds light on the topology of real algebraic curves can easily be produced.

The study of moduli of real algebraic curves being amply justified, this paper seeks to explain in a fairly self-contained way the state of the art of the analytic theory of moduli of nonsingular complete real algebraic curves. This involves the theory of real Teichmüller spaces and leads to a natural semianalytic structure on the moduli space $M_{g/\mathbb{R}}$ of nonsingular real algebraic curves of given genus g. The paper makes precise what "natural" means, and shows that the semianalytic structure of $M_{g/\mathbb{R}}$ is natural. The latter statement should be considered as the only contribution of the paper to the theory. Although this contribution is to be qualified as a minor one, it seems to me, nevertheless, an essential one. After all, any set of cardinality not greater than the cardinality of $\mathbb{R}$ admits a semianalytic structure.

In order to be more precise, here is the statement to which the whole paper is devoted:

THEOREM 1.2. *Let g be a nonnegative integer. Let $M_{g/\mathbb{R}}$ be the set of isomorphism classes of nonsingular complete real algebraic curves of genus g. There is a unique structure of a semianalytic variety on $M_{g/\mathbb{R}}$ having the following two properties.*

1. *Let M be a real analytic manifold and let $\mathcal{C}$ be an analytic family of nonsingular complete real algebraic curves of genus g over M. Let $f\colon M \to M_{g/\mathbb{R}}$ be the map defined by letting $f(p)$ be the isomorphism class of the fiber $\mathcal{C}_p$ for any $p \in M$. Then, the map f is analytic.*
2. *If $M'_{g/\mathbb{R}}$ is another semianalytic structure on the set $M_{g/\mathbb{R}}$ satisfying condition 1, then the identity map* $\mathrm{id}\colon M_{g/R} \to M'_{g/\mathbb{R}}$ *is analytic.*

The number of connected components of the semianalytic variety $M_{g/\mathbb{R}}$ is equal to $[\frac{1}{2}(3g+4)]$. Let X and Y be nonsingular complete real algebraic curves of genus g. The curves X and Y belong to the same connected component of $M_{g/\mathbb{R}}$ if and only if they are of the same topological type, i.e., if and only if the following two conditions hold.

3. *The sets of real points $X(\mathbb{R})$ and $Y(\mathbb{R})$ have the same number of connected components.*
4. *The sets $X(\mathbb{C})\backslash X(\mathbb{R})$ and $Y(\mathbb{C})\backslash Y(\mathbb{R})$ are either both connected or both disconnected.*

A remark on what is meant by a semianalytic variety is in order. A *semianalytic* subset of $\mathbb{R}^n$ is a subset S of $\mathbb{R}^n$ such that there are a locally closed analytic subset V of $\mathbb{R}^n$ and real analytic functions $f_1, \ldots, f_k$ on V with the property that

$$S = \{\, x \in V \mid f_1(x) \geq 0, \ldots, f_k(x) \geq 0 \,\}.$$

We endow a semianalytic subset of $\mathbb{R}^n$ with the sheaf of analytic functions, so that it becomes a locally ringed space. A *semianalytic variety* is a locally ringed space that is locally isomorphic to a semianalytic subset of some $\mathbb{R}^n$. A morphism between semianalytic varieties is just a morphism between locally ringed spaces. We call such a morphism an *analytic map*. This is justified since the structure sheaf of a semianalytic variety is locally a sheaf of analytic functions, so that a morphism is essentially an analytic map.

Properties 1 and 2 of Theorem 1.2 express—to my opinion—what coarse semianalytic moduli spaces should be in real algebraic geometry (cf. [**12**, Definition 5.6] for coarse moduli spaces in algebraic geometry over algebraically closed fields). Property 1 expresses naturality of the semianalytic structure on $M_{g/\mathbb{R}}$. Property 2 is to guarantee its uniqueness. Indeed, given any semianalytic structure on $M_{g/\mathbb{R}}$ satisfying property 1, one can produce another semianalytic structure on $M_{g/\mathbb{R}}$ by introducing, for example, cuspidal singularities. That semianalytic structure on $M_{g/\mathbb{R}}$ will then also satisfy property 1, but will not satisfy property 2.

The statement of Theorem 1.2 wants to collect the most significant results in the analytic theory of moduli of real algebraic curves that are known up to the present. I would like to give an idea of the arguments that intervene in its proof.

Weichold showed that a nonsingular complete real algebraic curve of genus g can have $[\frac{1}{2}(3g+4)]$ different topological types [**21**], and Klein showed that all these topological types actually occur [**8**]. In order to rephrase the Klein-Weichold result in terms of moduli, let us choose nonsingular complete real algebraic curves X_i of genus g, for $i = 1, \ldots, [\frac{1}{2}(3g+4)]$, such that X_i and X_j have different topological types whenever $i \neq j$. For any nonsingular complete real algebraic curve X, let

$R(X)$ be the set of isomorphism classes of nonsingular complete real algebraic curves having the same topological type as X. The Klein-Weichold classification of the topological types of real algebraic curves can then be expressed by stating that the set $M_{g/\mathbb{R}}$ is the disjoint union of the sets $R(X_i)$, i.e.,

$$M_{g/\mathbb{R}} = \coprod_{i=1}^{[\frac{1}{2}(3g+4)]} R(X_i).$$

In order to endow the set $M_{g/\mathbb{R}}$ with a natural structure of a semianalytic variety, one has to study real Teichmüller spaces. Observe that a nonsingular complete real algebraic curve X is essentially a compact Riemann surface endowed with an action of the Galois group Σ of $\mathbb{C}$ over $\mathbb{R}$. Earle studied the Teichmüller space $T(X)$ of such Riemann surfaces [**4**]. The space $T(X)$ is the *real Teichmüller space* of X. Earle showed that $T(X)$ is a connected real analytic manifold, using results of Kravetz [**10**] and Rauch [**15**]. Moreover, there is a group $\mathrm{Mod}(X)$, the *real modular group* of X, that acts properly discontinuously on $T(X)$. The quotient $T(X)/\mathrm{Mod}(X)$ has then a natural structure of a connected semianalytic variety. Since the quotient $T(X)/\mathrm{Mod}(X)$ is in bijective correspondence with the set $R(X)$, the set $R(X)$ acquires the structure of a connected semianalytic variety, for any real algebraic curve X.

It is natural to define a semianalytic structure on $M_{g/\mathbb{R}}$ as the disjoint sum of the semianalytic varieties $R(X_i)$. This structure of a semianalytic variety on $M_{g/\mathbb{R}}$ turns out to satisfy properties 1 and 2 of Theorem 1.2. Therefore, real algebraic curves of given genus have coarse semianalytic moduli.

One can ask oneself whether the set $M_{g/\mathbb{R}}$ admits a richer natural structure than the one of a semianalytic variety. There has been some confusion in the literature as to whether $M_{g/\mathbb{R}}$ is a real analytic variety. The author showed that $M_{g/\mathbb{R}}$ is not a real analytic variety if $g \geq 2$ [**6**]. In fact, even more is true: every connected component $R(X_i)$ of $M_{g/\mathbb{R}}$ is a true semianalytic variety, i.e., a nonanalytic variety if $g \geq 2$. On the contrary, $M_{g/\mathbb{R}}$ has a natural structure of what is to be called a semi-Nash variety. In particular, $M_{g/\mathbb{R}}$ has a natural semialgebraic structure.

The paper is organized as follows. Sections 2, 3 and 4 recall the well known correspondence between real algebraic curves, complex algebraic curves endowed with a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action and Riemann surfaces with such an action. In Section 5 we show Weichold's result on the topological classification of real algebraic curves. In Sections 6 up to 14, we dissect further the structure of a real algebraic curve. This will lead, in Section 15, to a proof of Klein's result that any topological type of Weichold's classification actually occurs as the topological type of a real algebraic curve. It will lead as well, in Section 16, to an elementary proof of the connectedness of the moduli space $R(X)$ for a real algebraic curve X. From Section 17 up to the last section, we show that the moduli space $M_{g/\mathbb{R}}$ admits a natural structure of a semianalytic variety.

Conventions and notation. An algebraic curve is nonsingular, geometrically integral and complete [**5**]. A Riemann surface is compact. Differentiable means of class C^∞. A differentiable manifold has no boundary, unless explicit mention of a boundary is made (in which case the boundary may be empty). The Galois group of $\mathbb{C}$ over $\mathbb{R}$ is denoted by Σ. Its nontrivial element is denoted by σ. When a map is said to be equivariant, it is understood that it is equivariant with respect to actions of Σ. When Σ is said to act on a complex algebraic variety, then σ is supposed to act *antialgebraically*, i.e., its action on sections of the structure sheaf is antilinear. When Σ is said to act on a complex analytic variety $(X, \mathcal{O})$, i.e., the group Σ acts on the locally ringed space $(X, \mathcal{O})$, then σ is supposed to act *antiholomorphically*, i.e., the action of σ on sections of $\mathcal{O}$ is antilinear. Note that the present definition of an antiholomorphic action differs slightly from the usual one. There are certain advantages in considering such algebraic actions of Σ instead of the more usual geometric actions of Σ on complex analytic varieties. When Σ acts on a 2-dimensional orientable differentiable manifold, σ is supposed to act orientation-reversingly. For a real number x, the greatest integer less than or equal to x is denoted by $[x]$.

2. Complex algebraic curves and Riemann surfaces

In this section we show that the category of complex algebraic curves and the category of Riemann surfaces are isomorphic. We define actions of the Galois group Σ on both categories and show that the isomorphism is equivariant with respect to these actions.

Let X be a complex algebraic curve. Its set of complex points is denoted by $X(\mathbb{C})$. We endow $X(\mathbb{C})$ with the strong topology and its sheaf of holomorphic functions. Then, $X(\mathbb{C})$ is a Riemann surface. Obviously, a morphism of complex algebraic curves $f\colon X \to Y$ induces a holomorphic map, denoted by $f(\mathbb{C})$, from $X(\mathbb{C})$ into $Y(\mathbb{C})$. Hence, the assignment $X \mapsto X(\mathbb{C})$ defines a functor from the category of complex algebraic curves into the category of Riemann surfaces. We show that this functor is an isomorphism of categories.

If M is a Riemann surface, there is a unique complex algebraic curve X such that the induced Riemann surface $X(\mathbb{C})$ is equal to M. Indeed, let X be the union of the set M and a point η which is to be the generic point of X. Endow X with the topology for which proper subsets C of X are closed if and only if C is finite and does not contain η. The structure sheaf $\mathcal{O}$ on X is defined as follows. Let K be the field of meromorphic functions on M. For a nonempty open subset U of X define

$$\Gamma(U, \mathcal{O}) = \{\, f \in K \mid \forall p \in U \cap M\colon \operatorname{ord}_p(f) \geq 0 \,\}.$$

Then, X is a complex algebraic curve. It is clear that X is the unique complex algebraic curve such that the induced Riemann surface $X(\mathbb{C})$ is equal to M. This shows that the functor $X \mapsto X(\mathbb{C})$ is surjective.

The functor in question is clearly faithful. In order to show that it is fully faithful, let $f\colon M \to N$ be a holomorphic map of Riemann surfaces. Let X and Y be the complex algebraic curves such that $X(\mathbb{C}) = M$ and $Y(\mathbb{C}) = N$. Then, f induces a morphism $h\colon X \to Y$ of complex algebraic curves. Indeed, let K and L be the fields of meromorphic functions on M and N, respectively. Then, if f is nonconstant, f induces a morphism of fields $f^\star\colon L \to K$. Since K and L are also the function fields of X and Y, and since X and Y are nonsingular and complete, there is a morphism $h\colon X \to Y$ such that $h^\star = f^\star$. It follows that $h(\mathbb{C}) = f$. In fact, h is the unique morphism from X into Y such that the induced holomorphic map $h(\mathbb{C})$ is *equal* to f. This proves that the functor $X \mapsto X(\mathbb{C})$ is an isomorphism from the category of complex algebraic curves into the category of Riemann surfaces.

The Galois group Σ of $\mathbb{C}$ over $\mathbb{R}$ acts on the category of complex algebraic curves. Indeed, let X be a complex algebraic curve. Then, one has a structure morphism $s\colon X \to \operatorname{Spec}(\mathbb{C})$. The *conjugate algebraic curve* X^σ of X is the scheme X equipped with the structure morphism $\operatorname{Spec}(\sigma) \circ s$. It is clear that the complex algebraic curve $(X^\sigma)^\sigma$ is equal to X.

Let $f\colon X \to Y$ be a morphism of complex algebraic curves. Then, f considered as a map from the scheme X^σ into Y^σ is, in fact, a morphism of complex algebraic curves. This morphism is denoted by f^σ. It follows immediately that the assignment $X \mapsto X^\sigma$ defines an action on the category of complex algebraic curves.

One can similarly define an action of Σ on the category of Riemann surfaces. Let $M = (M, \mathcal{O})$ be a Riemann surface. In particular, $\mathcal{O}$ is a sheaf of $\mathbb{C}$-algebras. Let $\mathcal{O}^\sigma$ be the same sheaf but with the conjugate $\mathbb{C}$-algebra structure. Then, the locally ringed space $(M, \mathcal{O}^\sigma)$ is a Riemann surface, the *conjugate Riemann surface*, denoted by M^σ.

Let $f\colon M \to N$ be a holomorphic map. Then, f is automatically a holomorphic map from M^σ into N^σ. This map is denoted by f^σ. It then immediately follows that the assignment $M \mapsto M^\sigma$ defines an action of Σ on the category of Riemann surfaces.

The isomorphism $X \mapsto X(\mathbb{C})$ from the category of complex algebraic curves into the category of Riemann surfaces is equivariant with respect to the actions of Σ. Indeed, for a complex algebraic curve X one has $X(\mathbb{C})^\sigma = X^\sigma(\mathbb{C})$, and for a morphism f of complex algebraic curves one has $f(\mathbb{C})^\sigma = f^\sigma(\mathbb{C})$.

3. Antialgebraic and antiholomorphic actions

In this section we show that there is an isomorphism from the category of complex algebraic curves endowed with an action of Σ into the category of Riemann surfaces endowed with an action of Σ. If one allows oneself to use the GAGA-principle [**19**], one can prove that there is an equivalence between the category of complete algebraic varieties endowed with an action of Σ and the category of complex analytic varieties that admit an algebraic structure and which are endowed with an analytic action of Σ. But, in dimension 1, one can avoid the GAGA-principle, as we will do here.

Let X be a complex algebraic curve endowed with an action of Σ. Note that, according to our conventions, such an action is antialgebraic. It is clear that this action induces an action of Σ on the Riemann surface $X(\mathbb{C})$. Let Y be also a complex algebraic curve endowed with an action of Σ, and let $f\colon X \to Y$ be an equivariant morphism of complex algebraic curves. Then, the induced holomorphic map $f(\mathbb{C})$ is also equivariant. To put it otherwise, the assignment $X \mapsto X(\mathbb{C})$ is a functor from the category of complex algebraic curves endowed with an action of Σ into the category of Riemann surfaces endowed with an action of Σ. We show that this functor is an isomorphism of categories.

It follows from what we have seen in Section 2 that the functor under consideration is fully faithful. Hence, in order to show that the functor is an isomorphism, it suffices to show the following. Let M be any Riemann surface endowed with an action of Σ. Let X be the complex algebraic curve such that $X(\mathbb{C}) = M$. We have to show that there is an action of Σ on X such that its induced action on $M = X(\mathbb{C})$ coincides with the initial action of Σ on M.

Let the identity map from M into M^σ be denoted by σ_M. Similarly, let the identity map from X into X^σ be denoted by σ_X. Of course, σ_M is antiholomorphic and σ_X is antialgebraic. Let $\varphi\colon M \to M$ be the antiholomorphic map corresponding to the action of σ on M. Then,

$$\sigma_M \circ \varphi \colon M \longrightarrow M^\sigma$$

is a holomorphic map. According to the isomorphism between the category of complex algebraic curves and Riemann surfaces (cf. Section 2), there is a morphism of complex algebraic curves $f\colon X \to X^\sigma$ such that $f(\mathbb{C}) = \sigma_M \circ \varphi$. Then, the antialgebraic morphism

$$\psi = \sigma_{X^\sigma} \circ f \colon X \longrightarrow X$$

satisfies $\psi^2 = \mathrm{id}_X$. Indeed,

$$\begin{aligned}(\psi^2)(\mathbb{C}) &= \psi(\mathbb{C})^2\\ &= \sigma_{M^\sigma} \circ f(\mathbb{C}) \circ \sigma_{M^\sigma} \circ f(\mathbb{C})\\ &= \sigma_{M^\sigma} \circ \sigma_M \circ \varphi \circ \sigma_{M^\sigma} \circ \sigma_M \circ \varphi\\ &= \mathrm{id}_M \circ \varphi \circ \mathrm{id}_M \circ \varphi\\ &= \varphi^2\\ &= \mathrm{id}_X(\mathbb{C}).\end{aligned}$$

Therefore, there is an action of Σ on X such that σ acts as the antialgebraic map ψ. It immediately follows that this action of Σ on X induces an action on M that coincides with the given action of Σ on M. Therefore, the functor $X \mapsto X(\mathbb{C})$ is an isomorphism from the category of complex algebraic curves endowed with a Σ-action onto the category of Riemann surfaces endowed with a Σ-action.

4. Real algebraic curves and Riemann surfaces

In this section we show that there is an equivalence between the category of real algebraic curves and the category of complex algebraic curves endowed with an action of Σ. This equivalence, in such generality, does not hold in higher dimensions.

Let X be a real algebraic curve. Then, $X_\mathbb{C} = X \otimes_\mathbb{R} \mathbb{C}$ is a complex algebraic curve. This complex algebraic curve comes with a canonical antialgebraic action of Σ. Let Y be also a real algebraic curve, and let $f\colon X \to Y$ be a morphism of real algebraic curves. Then, $f_\mathbb{C} = f \otimes_\mathbb{R} \mathrm{id}$ is a morphism of complex algebraic curves from $X_\mathbb{C}$ into $Y_\mathbb{C}$. In fact, this morphism is equivariant with respect to the action of Σ. To put it otherwise, the assignment $X \mapsto X_\mathbb{C}$ is a functor from the category of real algebraic curves into the category of complex algebraic curves endowed with a Σ-action.

Observe that the functor in question is faithful. In order to see that the functor is fully faithful, note that the canonical map $X_\mathbb{C} \to X$ is a quotient map for the action of Σ on $X_\mathbb{C}$ if X is a real algebraic curve. Hence, if Y is also a real algebraic curve and $h\colon X_\mathbb{C} \to Y_C$ is an equivariant morphism, h induces a morphism $f\colon X \to Y$ such that $f_\mathbb{C} = h$. This show that the functor in question is fully faithful.

In order to show that the functor $X \mapsto X_\mathbb{C}$ is an equivalence from the category of real algebraic curves onto the category of complex algebraic curves endowed with a Σ-action, it now suffices to show the following. Let X' be a complex algebraic curve endowed with an action of Σ. Then, there is a real algebraic curve X such that $X_\mathbb{C}$ is equivariantly isomorphic to X'. To show this, one can choose, by Riemann-Roch, a very ample divisor D on X' such that $\sigma^\star D = D$. One may then assume that the associated embedding $\iota\colon X' \to \mathbb{P}^n_\mathbb{C}$ is equivariant. Of course, the quotient $\mathbb{P}^n_\mathbb{C}/\Sigma$ of $\mathbb{P}^n_\mathbb{C}$ as a locally ringed space is the scheme $\mathbb{P}^n_\mathbb{R}$. It follows that the quotient X'/Σ

of X' as a locally ringed space is isomorphic to the closed subscheme $\iota(X')/\Sigma$ of $\mathbb{P}^n_{\mathbb{R}}$. Clearly, this closed subscheme is a real algebraic curve. Therefore, $X = X'/\Sigma$ is a real algebraic curve, and, obviously, $X_{\mathbb{C}}$ is equivariantly isomorphic to X'. It follows that the assignment $X \mapsto X_{\mathbb{C}}$ is an equivalence between the category of real algebraic curves and the category of complex algebraic curves endowed with an action of Σ. In particular, by what we have seen in Section 3, the assignment $X \mapsto X(\mathbb{C})$ is an equivalence between the category of real algebraic curves and the category of Riemann surfaces endowed with an action of Σ.

5. Topology of real algebraic curves

In this section we will prove Weichold's result on the classification of topological types of real algebraic curves. We shall do this in quite some detail.

In this section, we consider, for a real algebraic curve X, its set $X(\mathbb{C})$ of complex points as a differentiable surface. The action of Σ on $X(\mathbb{C})$ is a differentiable action. The canonical map from the set of real points $X(\mathbb{R})$ into the set of complex points is injective. Its image is the set of fixed points $X(\mathbb{C})^{\Sigma}$ for the action of Σ on $X(\mathbb{C})$. We identify $X(\mathbb{R})$ with its image $X(\mathbb{C})^{\Sigma}$ in $X(\mathbb{C})$. Locally, the action of Σ can be linearized at real points of X [**2**, §4]. It follows that $X(\mathbb{R})$ is a compact 1-dimensional submanifold of $X(\mathbb{C})$ if $X(\mathbb{R})$ is not empty. More precisely, we have the following statement.

LEMMA 5.1. *Let X be a real algebraic curve and let $p \in X(\mathbb{R})$. then, there are an open neighborhood U of p in $X(\mathbb{C})$, an open neighborhood U' of 0 in $\mathbb{R}^2$ and a diffeomorphism $\varphi\colon U \to U'$ such that $\varphi(p) = 0$ and*

$$\sigma \cdot \varphi^{-1}(x, y) = \varphi^{-1}(x, -y)$$

for all $(x, y) \in U'$. In particular, $X(\mathbb{R})$ is a compact 1-dimensional submanifold of $X(\mathbb{C})$ if $X(\mathbb{R})$ is nonempty.

PROOF. Let U be an open neighborhood of p such that there is a diffeomorphism φ from U onto an open subset U' of $\mathbb{R}^2$. We may suppose that $\varphi(p) = 0$. Replacing U by $U \cap (\sigma \cdot U)$, we may assume that U is stable for the action of Σ, i.e., $\sigma \cdot U \subseteq U$. Then, Σ acts on U.

Since p is a fixed point for the action of Σ, one has the tangent action of Σ on the tangent space $T_pX(\mathbb{C})$. Then, there is a unique action of Σ on T_0U' such that the tangent map $T_p\varphi$ is equivariant. Identify the tangent space T_0U' with $\mathbb{R}^2$ in the canonical way. Since σ acts orientation-reversingly on $X(\mathbb{C})$, we may assume that σ acts as $(x, y) \mapsto (x, -y)$ on $\mathbb{R}^2$. Consider the map $\widehat{\varphi}\colon U \to \mathbb{R}^2$ defined by

$$\widehat{\varphi}(x) = \varphi(x) + \sigma \cdot \varphi(\sigma \cdot x)$$

for $x \in U$. It is clear that $\widehat{\varphi}$ is an equivariant differentiable map. Since $T_p\varphi$ is equivariant, $T_p\widehat{\varphi} = 2T_p\varphi$. In particular, $\widehat{\varphi}$ is a local diffeomorphism. Shrinking U, one may assume that $\widehat{\varphi}$ is a diffeomorphism. Replacing φ by $\widehat{\varphi}$, the map $\varphi\colon U \to U'$ is an equivariant diffeomorphism from a Σ-stable open neighborhood U of p in $X(\mathbb{C})$ onto a Σ-stable open neighborhood U' of 0 in $\mathbb{R}^2$, where the action of Σ on $\mathbb{R}^2$ is defined by $\sigma \cdot (x, y) = (x, -y)$ for $(x, y) \in \mathbb{R}^2$. This proves the lemma. □

Next, it will be convenient to show in quite some detail that the quotient $X(\mathbb{C})/\Sigma$ of the differentiable manifold $X(\mathbb{C})$ in the category of locally ringed spaces is a differentiable manifold with boundary. Denote the sheaf of differentiable functions on $X(\mathbb{C})$ by $\mathcal{E}$. Let $\pi\colon X(\mathbb{C}) \to X(\mathbb{C})/\Sigma$ be the quotient map of topological spaces. One has an action of Σ on the sheaf $\pi_\star\mathcal{E}$ on $X(\mathbb{C})/\Sigma$. Then, the structure sheaf of the quotient $X(\mathbb{C})/\Sigma$ in the category of locally ringed spaces is the sheaf $(\pi_\star\mathcal{E})^\Sigma$ of invariant sections of $\pi_\star\mathcal{E}$. It is clear that $X(\mathbb{C})/\Sigma$ is a differentiable manifold at the π-images of nonreal points of $X(\mathbb{C})$. That the quotient $X(\mathbb{C})/\Sigma$ is a differentiable manifold with boundary will then follow from a local presentation of the quotient map at a real point.

Let $\mathbb{H}$ denote the closed real upper-half plane $\{\,(x,y)\in\mathbb{R}^2 \mid y\geq 0\,\}$ considered as a differentiable manifold with boundary.

PROPOSITION 5.2. *Let $p\in X(\mathbb{R})$ and let $q=\pi(p)$ be its image. There are an open neighborhood V of q, open subsets U' of $\mathbb{R}^2$ and V' of $\mathbb{H}$, a diffeomorphism $\varphi\colon U=\pi^{-1}(V)\to U'$ and an isomorphism of locally ringed spaces $\psi\colon V\to V'$ such that*

$$\psi\circ\pi\circ\varphi^{-1}(x,y)=(x,y^2)$$

for $(x,y)\in U'$. In particular, the locally ringed space $(X(\mathbb{C})/\Sigma,(\pi_\star\mathcal{E})^\Sigma)$ is a differentiable surface with boundary, its boundary is equal to the π-image of the set of real points $X(\mathbb{R})$ of X, and the quotient map $\pi\colon X(\mathbb{C})\to X(\mathbb{C})/\Sigma$ is a differentiable map.

PROOF. By Lemma 5.1, there are an open neighborhood U of p in $X(\mathbb{C})$, an open neighborhood U' of 0 in $\mathbb{R}^2$ and a diffeomorphism $\varphi\colon U\to U'$ such that $\varphi(p)=0$ and $\sigma\cdot\varphi^{-1}(x,y)=\varphi^{-1}(x,-y)$. To put it otherwise, φ is equivariant with respect to the action of Σ on $\mathbb{R}^2$ defined by $\sigma\cdot(x,y)=(x,-y)$.

Let $\rho\colon\mathbb{R}^2\to\mathbb{H}$ be the differentiable map defined by $\rho(x,y)=(x,y^2)$ for $(x,y)\in\mathbb{R}^2$. Let $V'=\rho(U')$. Then, ρ is the quotient of $\mathbb{R}^2$ by the action of Σ in the category of locally ringed spaces. In particular, its restriction $\rho_{|U'}\colon U'\to V'$ is the quotient of U' by the action of Σ in the category of locally ringed spaces. Since $\pi_{|U}\colon U\to V$ is the quotient of U by the action of Σ in the category of locally ringed spaces, there is an isomorphism of locally ringed spaces $\psi\colon V\to V'$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & U' \\ {\scriptstyle \pi_{|U}}\downarrow & & \downarrow{\scriptstyle \rho_{|U'}} \\ V & \xrightarrow[\psi]{} & V' \end{array}$$

commutes. This means that $\psi\circ\pi\circ\varphi^{-1}(x,y)=(x,y^2)$ for $(x,y)\in U'$. □

Let X be a real algebraic curve. Let $g=g(X)$ be its genus. One calls X *dividing* whenever $X(\mathbb{C})\backslash X(\mathbb{R})$ is not connected. Otherwise, X is *nondividing.*

Observe that the number of connected components of $U=X(\mathbb{C})\backslash X(\mathbb{R})$ is at most 2. Indeed, the quotient U/Σ is diffeomorphic to the interior of the surface with boundary $X(\mathbb{C})/\Sigma$. Since $X(\mathbb{C})/\Sigma$ is connected, U/Σ is connected. Now, Σ acts fixed point-freely on U. Hence, $X(\mathbb{C})\backslash X(\mathbb{R})U$ has at most 2 connected components.

Note also that, since σ acts orientation-reversingly on $X(\mathbb{C})$ (cf. Section11), the quotient U/Σ is orientable if and only if X is dividing. Since U/Σ is diffeomorphic to the interior of $X(\mathbb{C})/\Sigma$, the real algebraic curve X is dividing if and only if $X(\mathbb{C})/\Sigma$ is orientable.

For a real algebraic curve X, let $c=c(X)$ denote the number of connected components of the set $X(\mathbb{R})$ of real points of X.

THEOREM 5.3 (Weichold [**21**]). *Let X be a real algebraic curve. Let $g = g(X)$ and $c = c(X)$. Then,*

$$\begin{cases} 1 \leq c \leq g+1 \text{ and } c \equiv g+1 \mod 2 & \text{if } X \text{ is dividing, and} \\ 0 \leq c \leq g & \text{if } X \text{ is nondividing.} \end{cases}$$

PROOF. Let U be the complement of $X(\mathbb{R})$ in $X(\mathbb{C})$. Let i be the inclusion of $X(\mathbb{R})$ into $X(\mathbb{C})$, and let j the inclusion from U into $X(\mathbb{C})$. Then, we have a short exact sequence of sheaves on $X(\mathbb{C})$ (we refer to [**7**] for notation and facts on sheaves and their cohomology):

$$0 \longrightarrow j_!\mathbb{Z}_U \longrightarrow \mathbb{Z}_{X(\mathbb{C})} \longrightarrow i_\star\mathbb{Z}_{X(\mathbb{R})} \longrightarrow 0.$$

Taking Euler characteristics, one obtains

$$\begin{aligned} \chi(X(\mathbb{C})) &= \chi(j_!\mathbb{Z}_U) + \chi(i_\star\mathbb{Z}_{X(\mathbb{R})}) \\ &= \chi_c(U) + \chi(X(\mathbb{R})) \\ &= \chi_c(U), \end{aligned}$$

where χ_c denotes the Euler characteristic with compact support. Note that the Euler characteristic $\chi(X(\mathbb{R}))$ of $X(\mathbb{R})$ vanishes since $X(\mathbb{R})$ is a compact differentiable curve. Since $X(\mathbb{C})$ admits the structure of a Riemann surface of genus g, one has $\chi(X(\mathbb{C})) = 2 - 2g$. Moreover, the group Σ acts fixed-point freely on U. Therefore,

$$\chi_c(U/\Sigma) = \tfrac{1}{2}\chi_c(U) = \tfrac{1}{2}\chi(X(\mathbb{C})) = 1 - g.$$

Denoting the inclusions of U/Σ and $X(\mathbb{R})$ into $X(\mathbb{C})/\Sigma$ also by j and i, respectively, one has a short exact sequence

$$0 \longrightarrow j_!\mathbb{Z}_{U/\Sigma} \longrightarrow \mathbb{Z}_{X(\mathbb{C})/\Sigma} \longrightarrow i_\star\mathbb{Z}_{X(\mathbb{R})} \longrightarrow 0.$$

Taking again Euler characteristics, one gets

$$\begin{aligned} \chi(X(\mathbb{C})/\Sigma) &= \chi(j_!\mathbb{Z}_{U/\Sigma}) + \chi(i_\star\mathbb{Z}_{X(\mathbb{R})}) \\ &= \chi_c(U/\Sigma) + \chi(X(\mathbb{R})) \\ &= \tfrac{1}{2}\chi_c(U) \\ &= 1 - g. \end{aligned}$$

Now one has to distinguish the case whether the real algebraic curve X is dividing or not.

Suppose that X is dividing. Then, the quotient $X(\mathbb{C})/\Sigma$ is orientable. Hence, $X(\mathbb{C})/\Sigma$ is diffeomorphic to the complement of c disjoint open disc in an orientable compact differentiable surface of genus g', say. Then,

$$\chi(X(\mathbb{C})/\Sigma) = 2 - 2g' - c.$$

It follows that $c = g + 1 - 2g'$. This shows that $c \equiv g + 1 \mod 2$ and $c \leq g + 1$. One has $c \geq 1$, since a dividing real algebraic curve necessarily has real points.

Suppose that X is nondividing. Then, $X(\mathbb{C})/\Sigma$ is not orientable. It follows that $X(\mathbb{C})/\Sigma$ is diffeomorphic to the complement of c disjoint open discs in a nonorientable surface of genus g', say. Then,

$$\chi(X(\mathbb{C})/\Sigma) = 2 - g' - c.$$

Hence, $c = 1 + g - g'$. It follows that $0 \leq c \leq g$, since $g' > 0$. □

I cannot resist mentioning the following famous consequence of Weichold's classification.

COROLLARY 5.4 (Harnack's Inequality). *Let X be a real algebraic curve. Let g be the genus of X and let c be the number of connected components of $X(\mathbb{R})$. Then, $c \leq g+1$.* □

One has the following converse to Theorem 5.3. Given integers g and c satisfying $1 \leq c \leq g+1$ and $c \equiv g+1 \mod 2$, there is a dividing real algebraic curve X such that $g(X) = g$ and $c(X) = c$. Also, given integers g and c satisfying $0 \leq c \leq g$, there is a nondividing real algebraic curve X such that $g(X) = g$ and $c(X) = c$. This statement goes back to Klein [**8**]. A modern proof is due to Alling and Greenleaf [**1**]. Their proof essentially constructs the real algebraic curves as ramified coverings of $\mathbb{P}^1$. In Section 15, we shall show the converse to Theorem 5.3 in another way.

Classically, the *topological type* of a real algebraic curve X of genus g is defined as the data of $c(X)$ and its qualification of being dividing or not. The following well-known proposition provides useful equivalent conditions for two real algebraic curves to be of the same topological type.

PROPOSITION 5.5. *Let X and Y be real algebraic curves of the same genus. Then, the following condition are equivalent.*

1. *The differentiable manifolds $X(\mathbb{C})$ and $Y(\mathbb{C})$ are equivariantly diffeomorphic.*
2. *The quotients $X(\mathbb{C})/\Sigma$ and $Y(\mathbb{C})/\Sigma$ are diffeomorphic.*
3. *X and Y are either both dividing or both nondividing, and $c(X) = c(Y)$.*

PROOF. The implication $1 \Rightarrow 3$ is trivial. We show $3 \Rightarrow 2$ and $2 \Rightarrow 1$. Let g be the genus of X. To simplify notation, let $M = X(\mathbb{C})$ and $N = Y(\mathbb{C})$.

$3 \Rightarrow 2$: Let $g = g(X)$ and $c = c(X)$. We have seen that the differentiable manifold with boundary M/Σ is orientable if X is dividing and is nonorientable if X is nondividing. In the proof of Theorem 5.3, we have seen that, in the former case, M/Σ is diffeomorphic to the complement of c disjoint open discs in an orientable compact surface of genus $g' = \frac{1}{2}(1+g-c)$. In the latter case, M/Σ is diffeomorphic to the complement of c disjoint open discs in a nonorientable compact surface of genus $g' = 1+g-c$. Since $g(Y) = g$, $c(Y) = c$ and Y is dividing if and only if X is dividing, the same holds for N/Σ. Hence, M/Σ and N/Σ are diffeomorphic.

$2 \Rightarrow 1$: We show that the differentiable manifold M, together with the action of Σ, is uniquely determined, up to equivariant diffeomorphism, by its quotient M/Σ. Indeed, let $T(M/\Sigma)$ be the tangent bundle of M/Σ, and $T(M/\Sigma)^\curlyvee$ its dual bundle. Let L be the line bundle

$$L = \bigwedge^2 (T(M/\Sigma)^\curlyvee)$$

on M/Σ. As is true for the tensor square of any line bundle, $L^{\otimes 2}$ is isomorphic to the trivial line bundle $\mathcal{T}$ on M/Σ. Choose an isomorphism $\varphi\colon L^{\otimes 2} \to \mathcal{T}$ such that $\varphi(\omega \otimes \omega) \geq 0$ for all elements ω of any fiber of L. We identify $L^{\otimes 2}$ and $\mathcal{T}$ through φ. Choose a differentiable function f on M/Σ such that $f \geq 0$, $f^{-1}(0) = \partial(M/\Sigma)$ and $df \neq 0$ on $\partial(M/\Sigma)$. Define

$$M' = \{\, (p,\omega) \in L \mid \omega \otimes \omega = f(p) \,\}.$$

Then, a local consideration reveals that M' is a differentiable submanifold of L without boundary. Let $\rho\colon M' \to M/\Sigma$ be the map defined by $\rho(p,\omega) = p$. Then ρ, being the restriction of the projection $L \to M/\Sigma$, is a differentiable map. Observe that the pair (M', ρ) is uniquely determined by M/Σ, up to diffeomorphism. We call the map $\rho\colon M' \to M/\Sigma$ the *double cover of M/Σ ramified along its boundary.*

Define an action of Σ on M' by $\sigma \cdot (p, \omega) = (p, -\omega)$. Then a local consideration shows that ρ is a quotient map for the action of Σ on M'. In particular, ρ induces a diffeomorphism between M'/Σ and M/Σ.

Now, we show that M' is equivariantly diffeomorphic to M. Let $\pi\colon M \to M/\Sigma$ be the quotient map. Let $q \in M$. Since $M = X(\mathbb{C})$, the differentiable manifold M is canonically oriented (see Section 11). If q is not a fixed point for the action of Σ, then there is a unique

$$\omega(q) \in \bigwedge^2 (T_{\pi(q)}(M/\Sigma)^\curlyvee)$$

such that $\omega(q) \otimes \omega(q) = f(\pi(q))$ and such that $\pi^\star \omega(q)$ defines an orientation of $T_q M$ that coincides with its initial orientation. If q is a fixed point of Σ, we define $\omega(q)$ to be 0. Define a map $h\colon M \to M'$ by $h(q) = (\pi(q), \omega(q))$. It follows from the local presentation of π at fixed points of Σ (e.g. Proposition 5.2) that h is a submersive differentiable map. Since σ acts orientation-reversingly on M, h is equivariant, surjective and injective. Hence, h is an equivariant diffeomorphism. □

6. Complex structures

In this section we recall notions related to complex structures on real vector spaces. These will be used when we study almost complex structures on differentiable manifolds.

Let V be a real vector space. A *complex structure* on V is a morphism $\Phi\colon \mathbb{C} \to \mathrm{End}(V)$ of $\mathbb{R}$-algebras. This is equivalent to a structure of a complex vector space on V such that the induced structure of a real vector space coincides with the given real structure of V. In particular, a finite-dimensional real vector space has a complex structure only if it is of even dimension. Since $\mathbb{C} = \mathbb{R}[\sqrt{-1}]$, a complex structure Φ on V is completely determined by the image of $\sqrt{-1}$ in $\mathrm{End}(V)$. Therefore, it also is equivalent to give a complex structure on V or to give an endomorphism J of V such that $J^2 = -\mathrm{id}$.

Let V and V' be real vector spaces equipped with complex structures Φ and Φ', respectively. An $\mathbb{R}$-linear map $L\colon V \to V'$ is *complex* if $L \circ \Phi(\lambda) = \Phi(\lambda) \circ L$ for all $\lambda \in \mathbb{C}$. Of course, this is equivalent to requiring L to be $\mathbb{C}$-linear with respect to the corresponding structures of $\mathbb{C}$-vector spaces. In terms of the endomorphisms $J = \Phi(\sqrt{-1})$ and $J' = \Phi'(\sqrt{-1})$, the condition on L is that $L \circ J = J' \circ L$.

Let $L\colon V \to V'$ be an $\mathbb{R}$-linear isomorphism of real vector spaces. If Φ' is a complex structure on V', then there is a unique complex structure Φ on V such that L is complex with respect to Φ and Φ'. Indeed, one defines Φ by $\Phi(\lambda) = L^{-1} \circ \Phi'(\lambda) \circ L$ for $\lambda \in \mathbb{C}$. This induced complex structure is denoted by $L^\star \Phi'$.

Let V be a finite-dimensional real vector space. A complex structure Φ on V induces an orientation ν_Φ of V. Indeed, let $\{v_1, \ldots, v_n\}$ be a basis of V as a complex vector space. Then, one puts

$$\nu_\Phi = [v_1, Jv_1, \ldots, v_n, Jv_n],$$

where $J = \Phi(\sqrt{-1})$. If V is an oriented finite-dimensional real vector space, then a complex structure Φ on V is said to be *compatible with the orientation* if ν_Φ coincides with the orientation of V.

Let V be a real vector space and let Φ be a complex structure on V. The *complex conjugate structure* $\overline{\Phi}$ is defined by $\overline{\Phi}(\lambda) = \Phi(\overline{\lambda})$ for $\lambda \in \mathbb{C}$. In terms of $J = \Phi(\sqrt{-1})$, the complex conjugate structure is given by the endomorphism $-J$. If V is finite-dimensional, say $2n = \dim V$, then the orientation $\nu_{\overline{\Phi}}$ induced by $\overline{\Phi}$ is opposite to ν_Φ if n is odd. Otherwise, $\nu_{\overline{\Phi}} = \nu_\Phi$.

7. Conformal structures

In this section we recall some notions relative to conformal structures on real vector spaces. These will be useful when we study conformal structures on differentiable manifolds.

Recall that an *inner product* on a real vector space is a positive definite symmetric bilinear form. Let $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ be inner products on a real vector space V. We say that these two inner products are *proportional* if there is a positive real number λ such that

$$\langle v, w\rangle' = \lambda \cdot \langle v, w\rangle$$

for all $v, w \in V$. Obviously, the proportionality relation is an equivalence relation. A *conformal structure* on V is the proportionality class of an inner product on V.

Recall that one has a bijective correspondence between positive definite symmetric bilinear forms and positive definite quadratic forms. Indeed, if $\langle\cdot,\cdot\rangle$ is a positive definite symmetric bilinear form on V, then $q\colon V \to \mathbb{R}$ defined by $q(v) = \langle v, v\rangle$ is the associated positive definite quadratic form on V. The assignment $\langle\cdot,\cdot\rangle \mapsto q$ is the desired bijective correspondence.

Let q and q' be positive definite quadratic forms on V. We say that q and q' are *proportional* if there is a positive real number λ such that $q' = \lambda \cdot q$. This proportionality relation is also an equivalence relation. It then follows from the preceding paragraph that it is equivalent to give a conformal structure on V or to give the proportionality class of a positive definite quadratic form on V.

Let V and V' be real vector spaces. Let $L\colon V \to V'$ be an $\mathbb{R}$-linear isomorphism. If φ' is a conformal structure on V', then one has an induced conformal structure $L^\star\varphi'$ on V. Indeed, let $\langle\cdot,\cdot\rangle'$ be a representing inner product of φ'. Then, the bilinear form $\langle\cdot,\cdot\rangle = L^\star\langle\cdot,\cdot\rangle'$ on V defined by

$$\langle v, w\rangle = \langle Lv, Lw\rangle'$$

is an inner product. Its proportionality class does not depend on the choice of $\langle\cdot,\cdot\rangle'$, and therefore defines a conformal structure $L^\star\varphi'$ on V.

Let V and V' be real vector spaces equipped with conformal structures φ and φ', respectively. An $\mathbb{R}$-linear isomorphism $L\colon V \to V'$ is *conformal* if $L^\star\varphi' = \varphi$.

8. Complex and conformal structures

In this section we recall how complex and conformal structures on 2-dimensional real vector spaces are related.

A complex structure Φ on a 2-dimensional real vector space V induces a conformal structure on V. Indeed, let J be the endomorphism $\Phi(\sqrt{-1})$ of V, and let x be any nonzero element in the dual of V. Put $y = x \circ J$. Define a quadratic form q_x on V by

$$q_x(v) = x(v)^2 + y(v)^2$$

for all $v \in V$. Then, q_x is a positive definite quadratic form. We show that the proportionality class of q_x does not depend on x. Let x' be also a nonzero element of the dual of V. Then, $x' = ax + by$ for some $a, b \in \mathbb{R}$, not both zero. If we put $y' = x' \circ J$, then $y' = -bx + ay$ and

$$\begin{aligned} q_{x'}(v) &= x'(v)^2 + y'(v)^2 = \\ &= a^2x(v)^2 + 2abx(v)y(v) + b^2y(v)^2 + \\ &\quad + b^2x(v)^2 - 2abx(v)y(v) + a^2y(v)^2 = \\ &= (a^2 + b^2) \cdot (x(v)^2 + y(v)^2) = \\ &= (a^2 + b^2) \cdot q_x(v) \end{aligned}$$

for all $v \in V$. We see that the conformal structure φ defined by q_x does not depend on the choice of x. Therefore, a complex structure Φ on a 2-dimensional real vector space determines a conformal structure φ.

Let V be a 2-dimensional real vector space. Let Φ be a complex structure on V and let φ be the induced conformal structure. It follows from its construction that φ is also the conformal structure associated to the conjugate complex structure $\overline{\Phi}$. Therefore, an unordered pair of conjugate complex structures on V determines a conformal structure.

Conversely, a conformal structure on a 2-dimensional vector space V determines an unordered pair of conjugate complex structures on V. Indeed, let φ be a conformal structure on V. Choose a representing inner product $\langle \cdot, \cdot \rangle$ of φ. Let $\mathrm{SO}(V)$ be the special orthogonal group of V with respect to $\langle \cdot, \cdot \rangle$. Then, there are exactly two elements J, J' of $\mathrm{SO}(V)$ satisfying the equation $X^2 = -\mathrm{id}$. Let Φ and Φ' be the associated complex structures on V. Since $J' = -J$, Φ and Φ' are each other's complex conjugate structure. Since the subgroup $\mathrm{SO}(V)$ of $\mathrm{GL}(V)$ only depends on the conformal structure, this unordered pair of conjugate complex structures does not depend on the choice of $\langle \cdot, \cdot \rangle$.

Observe that the preceding constructions give rise to a bijective correspondence between the set of conformal structures on a 2-dimensional real vector space V and the set of unordered pairs of conjugate complex structures on V. If one gives oneself an orientation ν of V, then for each unordered pair of conjugate complex structures, exactly one of them is compatible with the orientation ν. Therefore, given an oriented 2-dimensional real vector space (V, ν), we actually have a bijective correspondence between the set conformal structures on V and the set of complex structures on V compatible with ν.

This bijective correspondence is natural. Indeed, let V and V' be 2-dimensional real vector spaces. Let $L\colon V \to V'$ be an $\mathbb{R}$-linear isomorphism. Let Φ' be a complex structure on V' and let φ' be its associated conformal structure. Then, it follows from their definitions that the conformal structure associated to the complex structure $L^\star\Phi'$ is equal to the conformal structure $L^\star\varphi'$.

As a consequence, an orientation-preserving $\mathbb{R}$-linear isomorphism L is complex if and only if it is conformal. More precisely, let V and V' be oriented 2-dimensional real vector spaces. Let Φ and Φ' be complex structures on V and V' compatible with the orientations. Let φ and φ' be the corresponding conformal structures. An orientation-preserving $\mathbb{R}$-linear isomorphism L from V into V' is complex if and only if it is conformal.

9. Beltrami coefficients

When endowed with some extra structure, conformal structures on a 2-dimensional real vector space are in bijective correspondence with the set of complex numbers of norm less than 1. These associated complex numbers are called Beltrami coefficients. In this section we explain this bijective correspondence.

Let V be a 2-dimensional real vector space. We fix a reference complex structure Φ_0 on V and a $\mathbb{C}$-linear isomorphism $z\colon V \to \mathbb{C}$ with respect to Φ_0. Let φ be any conformal structure on V. Let q be a representing positive definite quadratic form of φ. Put $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2\sqrt{-1}}(z - \overline{z})$. Then, $\{x, y\}$ is a basis for the real dual of V. Therefore, there are $a, b, c \in \mathbb{R}$ such that $q = ax^2 + 2bxy + cy^2$. Since q is positive definite, $a, c > 0$ and $b^2 - ac < 0$. Now, one easily checks that there are unique $\lambda \in \mathbb{R}^+$ and $\mu \in \mathbb{C}$, $|\mu| < 1$, such that

$$q = \lambda \cdot |z + \mu\overline{z}|^2.$$

It follows that the complex number μ of norm less than 1 is uniquely determined by the conformal structure φ. This complex number μ is called the *Beltrami coefficient* of φ.

Conversely, given $\mu \in \mathbb{C}$ with $|\mu| < 1$,

$$q = |z + \mu\overline{z}|^2$$

defines a positive definite quadratic form on V. Thus, the equivalence class of q is a conformal structure φ on V. Clearly, its Beltrami coefficient is equal to μ.

In conclusion, the assignment $\varphi \mapsto \mu$ defined above gives rise to a bijective correspondence between the open unit disc in $\mathbb{C}$ and the set of conformal structures on a 2-dimensional real vector space equipped with a reference complex structure and a $\mathbb{C}$-linear isomorphism onto $\mathbb{C}$.

Here is a geometric interpretation of the Beltrami coefficient of a conformal structure. Let μ be a complex number satisfying $|\mu| < 1$. Let q be the quadratic form $|z + \mu\overline{z}|^2$. Then, the unit circle $|z|^2 = 1$ in V is, with respect to the quadratic form q, an ellipse whose longest semiaxis is of length $1 + |\mu|$, and whose shortest semiaxis is of length $1 - |\mu|$. Moreover, the angle between the longest semiaxis and the positive x-axis is equal to $\frac{1}{2}\arg(\mu)$. Therefore, the Beltrami coefficient μ of a conformal structure φ measures the amount of distortion with respect to the reference complex structure.

It should be noted that the Beltrami coefficient of a conformal structure depends on the choice of the $\mathbb{C}$-linear isomorphism. Indeed, let z and z' be $\mathbb{C}$-linear isomorphisms from V onto $\mathbb{C}$. Let φ be any conformal structure on V. Let μ and μ' be the Beltrami coefficients of φ relative to z and z', respectively. Since z and z' are both $\mathbb{C}$-linear isomorphisms, there is a nonzero $\kappa \in \mathbb{C}$ such that $z = \kappa z'$. Hence,

$$|z + \mu\overline{z}|^2 = |\kappa|^2 \cdot |z' + \mu\tfrac{\overline{\kappa}}{\kappa}\overline{z}'|^2.$$

Therefore, $\mu' = \mu\frac{\overline{\kappa}}{\kappa}$.

Let V be a 2-dimensional real vector space, Φ_0 a complex structure on V and $z\colon V \to \mathbb{C}$ a $\mathbb{C}$-linear isomorphism. Let φ be any conformal structure on V. Let μ be the Beltrami coefficient of φ. It follows from its definition that the Beltrami coefficient of φ relative to the complex conjugate structure $\overline{\Phi_0}$ and the $\mathbb{C}$-linear isomorphism $\overline{z}$ is equal to $\overline{\mu}$.

10. Complex dilation

In this section we recall the definition of the complex and circular dilations of, essentially, an $\mathbb{R}$-linear isomorphism between 2-dimensional real vector spaces equipped with conformal structures.

Let V be a 2-dimensional real vector space equipped with a reference complex structure Φ_0 and a $\mathbb{C}$-linear isomorphism $z\colon V \to \mathbb{C}$. Denote the corresponding conformal structure by φ_0 (see Section 8). Let V' be any 2-dimensional real vector space equipped with a conformal structure φ'. Let $L\colon V \to V'$ be an $\mathbb{R}$-linear isomorphism. The *complex dilation* μ_L of L is by definition the Beltrami coefficient of the induced conformal structure $\varphi = L^\star\varphi'$. Clearly, L is conformal with respect to φ_0 and φ' if and only if $\mu_L = 0$.

The complex dilation of L can be easily determined as follows. Let Φ' be the complex structure on V' inducing the conformal structure φ' such that L is orientation-preserving with respect to the orientations induced by Φ_0 and Φ'. Let $z'\colon V' \to \mathbb{C}$ be a $\mathbb{C}$-linear isomorphism with respect to Φ'. There are $\alpha, \beta \in \mathbb{C}$ such that $L^\star z' = \alpha z + \beta\overline{z}$. Since L is an orientation-preserving isomorphism, $|\alpha| > |\beta|$. Then,

$$L^\star|z'|^2 = |\alpha z + \beta\overline{z}|^2 = |\alpha|^2 \cdot |z + \tfrac{\beta}{\alpha} \cdot \overline{z}|^2,$$

since $\alpha \neq 0$. Hence, $\mu_L = \frac{\beta}{\alpha}$. Once again, one sees that L is conformal if and only if $\mu_L = 0$.

Let μ be the complex dilation of L. Let q' be the quadratic form $|z' + \mu\overline{z}'|^2$ on V'. Then, the image by L of the unit circle $|z|^2 = 1$ is an ellipse whose longest semiaxis is of length $1 + |\mu|$ and whose shortest semiaxis is of length $1 - |\mu|$. One defines the *circular dilation* of L as the fraction

$$\gamma_L = \frac{1 + |\mu|}{1 - |\mu|}.$$

Clearly, γ_L is a real number greater than or equal to 1. Moreover, $\gamma_L = 1$ if and only if L is conformal with respect to φ_0 and φ'.

11. Almost complex manifolds

In this section we recall the definition of almost complex structures on differentiable manifolds and some related notions.

Let M be a differentiable manifold. Let TM be the tangent bundle of M. An *almost complex structure* on M is a complex structure Φ_p on the tangent space T_pM of M at every point p of M such that $\Phi(\lambda) \cdot X$ is a differentiable vector field for every differentiable vector field X on M and for all $\lambda \in \mathbb{C}$. Equivalently, an almost complex structure on M is a morphism $\Phi\colon \mathbb{C} \to \operatorname{End}(TM)$ of $\mathbb{R}$-algebras. This is also equivalent to a structure of a complex vector bundle on TM such that the induced structure of a real vector bundle coincides with the original real structure on TM. In particular, a differentiable manifold has an almost complex structure only if it is of even dimension. Of course, an almost complex structure on M is entirely determined by the image of $\sqrt{-1}$ in $\operatorname{End}(TM)$. Therefore, it is equivalent to give an almost complex structure on M or to give an endomorphism J of TM satisfying $J^2 = -\mathrm{id}$. An *almost complex manifold* is a differentiable manifold equipped with an almost complex structure.

Let M and M' be differentiable manifolds equipped with almost complex structures Φ and Φ', respectively. A differentiable map $f\colon M \to M'$ is said to be *almost* *complex* if the $\mathbb{R}$-linear map $T_p f$ is complex with respect to the complex structures Φ_p and $\Phi'_{f(p)}$ at every point p of M.

Let M and M' be differentiable manifolds. Let $f\colon M \to M'$ be a local diffeomorphism. Let Φ' be an almost complex structure on M'. Then, there is a unique almost complex structure Φ on M such that f is almost complex. Indeed, for any $p \in M$, the tangent map $T_p f$ is an $\mathbb{R}$-linear isomorphism. Therefore, there is a unique complex structure Φ_p on T_pM such that $T_p f$ is complex. It follows from the construction of Φ_p that Φ is an almost complex structure on M. This almost complex structure is denoted by $f^\star\Phi'$.

An almost complex structure Φ on M induces an orientation on M. Indeed, at each point p of M we have a complex structure Φ_p on the fiber T_pM. We have seen in Section 6 that such a structure induces an orientation of T_pM. This orientation depends smoothly on p, and therefore gives an orientation of M. If M is oriented, then Φ is said to be *compatible with the orientation* if Φ_p is compatible with the orientation of T_pM for all $p \in M$.

Let Φ be an almost complex structure on a differentiable manifold M. Then, its *conjugate almost complex structure* $\overline{\Phi}$ is defined by $\overline{\Phi}_p = \overline{(\Phi_p)}$ for all $p \in M$.

12. Complex analytic and almost complex manifolds

In this section we recall the relationship between complex analytic manifolds and almost complex manifolds.

Let N be a complex analytic manifold. We denote by T the complex tangent bundle of N. We write N_{dm} when we want to stress that we consider N as a differentiable manifold, i.e., forgetting about the complex analytic structure. The point we want to make is that N_{dm} has a canonical almost complex structure. This can be seen as follows. The complex vector bundle T is canonically a subbundle of the complexification $\mathbb{C}\otimes_{\mathbb{R}} TN_{\mathrm{dm}}$ of the real tangent bundle TN_{dm}. Denote by $\overline{T}$ its image by complex conjugation. Then,

$$\mathbb{C}\otimes_{\mathbb{R}} TN_{\mathrm{dm}} = T \oplus \overline{T}.$$

But, as a real vector bundle, $\mathbb{C}\otimes_{\mathbb{R}} TN_{\mathrm{dm}}$ is also the direct sum of TN_{dm} and $\sqrt{-1}\cdot TN_{\mathrm{dm}}$. One then has an associated projection from $\mathbb{C}\otimes_{\mathbb{R}} TN_{\mathrm{dm}}$ onto TN_{dm}. It is easy to check that its restriction to the complex subbundle T is a real isomorphism onto TN_{dm}. By transport of structure, one gets a structure of a complex vector bundle on TN_{dm}, i.e., an almost complex structure on N_{dm}.

The assignment $N \mapsto N_{\mathrm{dm}}$ is clearly a functor from the category of complex analytic varieties into the category of almost complex manifolds. Indeed, let N and N' be complex analytic manifolds. If $f\colon N \to N'$ is a holomorphic map then, of course, f is differentiable considered as a map from N_{dm} into N'_{dm}, and moreover, f is almost complex.

This functor is in fact fully faithful, i.e., any almost complex differentiable map $f\colon N_{\mathrm{dm}} \to N'_{\mathrm{dm}}$ is holomorphic [**9**, Proposition 2.3]. In particular, N and N' are biholomorphic if and only if N and N' are diffeomorphic as almost complex manifolds.

In the case of 2-dimensional differentiable manifolds, an almost complex structure on a differentiable manifold always comes from a complex structure. Let M be a 2-dimensional differentiable manifold and let Φ be an almost complex structure on M. Then, M *is* the differentiable structure N_{dm} of a complex analytic manifold N [**9**, Example IX.2.8]. Moreover, the almost complex structure on M induced by this complex analytic structure coincides with Φ.

What this shows is that, given a 2-dimensional differentiable manifold M, there is a bijective correspondence between the set of almost complex structures on M and the set of complex analytic structures on M.

Next, we study the behavior of this bijective correspondence with respect to conjugate complex structures. Let M be a 2-dimensional differentiable manifold. Let N be a complex analytic structure on M, i.e., N is a complex analytic variety with $N_{\mathrm{dm}} = M$. Let Φ be the almost complex structure on M induced by N. Then, the conjugate Riemann surface N^σ is also a complex analytic structure on M. The almost complex structure on M induced by N^σ is equal to the conjugate almost complex structure $\overline{\Phi}$ of Φ (see Section 11).

Let M be a 2-dimensional orientable differentiable manifold endowed with an action of Σ. Let N be a complex analytic structure on M. If the action of Σ on M is an action of Σ on the Riemann surface N, then the almost complex structure Φ on M induced by N satisfies $\sigma^\star\Phi = \overline{\Phi}$. Conversely, let Φ be an almost complex structure on M such that $\sigma^\star\Phi = \overline{\Phi}$. Let N be the associated complex analytic structure on M. Then, the action of Σ on M is an action of Σ on the Riemann surface N.

13. Conformal manifolds

In this section we define the notion of a conformal manifold and some related notions.

Let M be a differentiable manifold. Let $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ be Riemannian metrics on M. One calls $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ *proportional* if the inner products $\langle\cdot,\cdot\rangle_p$ and $\langle\cdot,\cdot\rangle'_p$ are proportional at every point p of M. Equivalently, $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ are proportional if there is a differentiable map $\lambda\colon M \to \mathbb{R}^+$ such that

$$\langle X, Y\rangle' = \lambda \cdot \langle X, Y\rangle$$

for all differentiable vector fields X and Y on M. A *conformal structure* on M is the proportionality class of a Riemannian metric on M. If φ is a conformal structure on M, then φ induces a conformal structure φ_p at T_pM for all $p \in M$. A *conformal manifold* is a pair (M, φ) consisting of a differentiable manifold M and a conformal structure φ on it.

Let M and M' be differentiable manifolds. Let φ' be a conformal structure on M'. If $f\colon M \to M'$ is a local diffeomorphism, then one has an induced conformal structure $f^\star\varphi'$ on M. Indeed, one defines $f^\star\varphi'$ at each point p of M by $(f^\star\varphi')_p = (T_pf)^\star\varphi'_p$.

Let (M, φ) and (M', φ') be conformal manifolds. A local diffeomorphism $f\colon M \to M'$ is *conformal* if $f^\star\varphi' = \varphi$.

As for conformal structures on real vector spaces, one can equivalently define a conformal structure on a differentiable manifold M as the proportionality class of a positive definite quadratic form on its tangent bundle TM.

14. Almost complex and conformal manifolds

In this section we show that, given a 2-dimensional oriented differentiable manifold, there is a bijective correspondence between the set of conformal structures on M and the set of almost complex structures on M that are compatible with the orientation.

Let M be a 2-dimensional orientable differentiable manifold. An almost complex structure Φ on M induces a conformal structure φ on M. Indeed, Φ_p is a complex structure on T_pM, for all $p \in M$. This complex structure induces a conformal structure φ_p on T_pM (see Section 8). It follows from the construction of φ_p, and using a partition of unity, that φ is a conformal structure on M.

Let ν be an orientation of M. Then, a conformal structure φ on M induces an almost complex structure Φ on M compatible with the orientation. Indeed, the conformal structure φ_p of T_pM induces a complex structure Φ_p on T_pM compatible with the orientation (see Section 8). It follows from its construction that Φ is an almost complex structure on M compatible with the orientation.

What we have established is a bijective correspondence between the set of conformal structures on a 2-dimensional oriented differentiable manifold M and the set of almost complex structures on M compatible with the orientation.

This bijective correspondence is natural. This follows from the following property. Let M and M' be 2-dimensional differentiable manifolds and let $f\colon M \to M'$ be a local diffeomorphism. Let Φ' be an almost complex structure on M' and φ' its associated conformal structure. Then, $f^\star\varphi'$ is the conformal structure associated to $f^\star\Phi'$.

We conclude this section with two other properties which are satisfied by the bijective correspondence between the set of conformal structures and the set of complex structures on a given oriented differentiable manifold.

Let (M, Φ) and (M', Φ') be 2-dimensional almost complex manifolds. Let φ and φ' be the induced conformal structures. Let $f\colon M \to M'$ be a local diffeomorphism, orientation-preserving with respect to the orientations induced by the almost complex structures. Then f is almost complex with respect to Φ and Φ' if and only if f is conformal with respect to φ and φ' (see Section 8).

Let M be a 2-dimensional oriented differentiable manifold equipped with an action of Σ. Let Φ be an almost complex structure on M compatible with the orientation and such that $\sigma^\star\Phi = \overline{\Phi}$. Then, the induced conformal structure φ satisfies $\sigma^\star\varphi = \varphi$ (see Section 8). Conversely, if φ is a conformal structure on M such that $\sigma^\star\varphi = \varphi$, then the induced almost complex structure Φ on M compatible with the orientation satisfies $\sigma^\star\Phi = \overline{\Phi}$.

15. Realizing a topological type

With all what we have done so far, we are able to prove the converse of Theorem 5.3.

THEOREM 15.1 (Klein [**8**]). *Let g be a nonnegative integer.*

1. *Let c be an integer satisfying $1 \le c \le g+1$ and $c \equiv g+1 \mod 2$. Then there is a dividing real algebraic curve X of genus g such that $c(X) = c$.*
2. *Let c be an integer satisfying $0 \le c \le g$. Then there is a nondividing real algebraic curve X of genus g such that $c(X) = c$.*

PROOF. We prove both statements at the same time. Let N be the complement of c disjoint open discs in a compact connected orientable differentiable surface of genus $g' = \frac{1}{2}(g+1-c)$ (in a compact connected nonorientable differentiable surface of genus $g' = g+1-c$, respectively). Then, N is a differentiable surface with boundary. Let M be the double cover of N ramified along its boundary (see the proof of Proposition 5.5). Then, M is an orientable compact connected differentiable surface which comes with an action of Σ. It follows from topological considerations as in the proof of Theorem 5.3 that M is of genus g. It is clear that the set of fixed points M^Σ consists of c components and that $M \setminus M^\Sigma$ is nonconnected (connected, respectively). It thus suffices to show that there is a real algebraic curve X such that $X(\mathbb{C})$ is equivariantly diffeomorphic to M.

Choose any Riemannian metric $\langle \cdot, \cdot \rangle$ on M. Let $\langle \cdot, \cdot \rangle'$ be the Riemannian metric $\langle \cdot, \cdot \rangle + \sigma^\star \langle \cdot, \cdot \rangle$. Then, the Riemannian metric $\langle \cdot, \cdot \rangle'$ satisfies $\sigma^\star \langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle'$, i.e., the action of Σ on M is by isometries. In particular, the conformal structure φ induced by $\langle \cdot, \cdot \rangle'$ satisfies $\sigma^\star \varphi = \varphi$. Then, according to Section 14, there is an almost complex structure Φ on M compatible with the orientation and such that $\sigma^\star \Phi = \overline{\Phi}$. It then follows from Section 12 that M carries the structure of a Riemann surface on which Σ acts. Applying Section 3, M is equal to the set $Y(\mathbb{C})$ of complex points of a complex algebraic curve Y on which Σ acts. By Section 4, there is a real algebraic curve X such that $X_\mathbb{C}$ is equivariantly isomorphic to Y. In particular, $X(\mathbb{C})$ is equivariantly diffeomorphic to M. □

16. Connectedness of moduli spaces by elementary techniques

In this section we endow the set $R(X)$ of isomorphism classes of real algebraic curves of the same topological type as X, with a natural topology. This set may then be called a moduli space. We show by elementary techniques, i.e., only using the preceding sections, that the moduli space $R(X)$ is connected.

Let X be a real algebraic curve. Recall that $R(X)$ is the set of isomorphism classes of real algebraic curves Y of the same topological type as X, i.e.,

$$R(X) = \left\{ Y \;\middle|\; \begin{array}{l} g(Y) = g(X),\ c(Y) = c(X), \text{ and } Y \text{ and } X \\ \text{are either both dividing or both nondividing} \end{array} \right\} \Big/ \cong .$$

Often we shall not distinguish carefully between a real algebraic curve Y and its isomorphism class. Let $\Gamma(X)$ be the set of conformal structures on X; i.e., the elements of $\Gamma(X)$ are conformal structures φ on the orientable differentiable surface $X(\mathbb{C})$ satisfying $\sigma^\star \varphi = \varphi$. One has a map

$$\rho \colon \Gamma(X) \longrightarrow R(X)$$

defined as follows. Let φ be a conformal structure on X. Let Φ be the almost complex structure on $X(\mathbb{C})$ induced by φ (see Section 14). Then, according to Section 14, the almost complex structure Φ on $X(\mathbb{C})$ satisfies $\sigma^\star \Phi = \overline{\Phi}$. We have seen in Sections 3 and 4 that, associated to Φ, there is a real algebraic curve Y such that $Y(\mathbb{C})$ and $X(\mathbb{C})$ are equivariantly diffeomorphic. One then defines the map ρ by $\rho(\varphi) = Y$.

We show that the map ρ is surjective. Let Y be any real algebraic curve in $R(X)$. By Proposition 5.5, $Y(\mathbb{C})$ is equivariantly diffeomorphic to $X(\mathbb{C})$. Choose an equivariant diffeomorphism f from $X(\mathbb{C})$ onto $Y(\mathbb{C})$. Let φ' be the conformal structure on $Y(\mathbb{C})$ induced by the complex analytic structure. Then, $\varphi = f^\star \varphi'$ is a conformal structure in $\Gamma(X)$, and $\rho(\varphi) = Y$ in $R(X)$. This shows that ρ is surjective.

The set $\Gamma(X)$ has a natural topology as a quotient of the space of equivariant Riemannian metrics on the differentiable manifold $X(\mathbb{C})$. We endow $R(X)$ with the quotient topology, i.e., the finest topology for which ρ is continuous. The topological space $R(X)$ is called the *moduli space of real algebraic curves of the same topological type as* X.

THEOREM 16.1. *Let X be a real algebraic curve. Then, the moduli space $R(X)$ of real algebraic curves of the same topological type as X, is connected.*

PROOF. We show that the set $\Gamma(X)$ is path connected. Let φ and φ' be conformal structures on X. Choose representing inner Riemannian metrics $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$, respectively. Replacing $\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle+\sigma^\star\langle\cdot,\cdot\rangle$, and similarly for $\langle\cdot,\cdot\rangle'$, we may assume that $\sigma^\star\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle$ and $\sigma^\star\langle\cdot,\cdot\rangle'=\langle\cdot,\cdot\rangle'$. Define a path from $\langle\cdot,\cdot\rangle$ to $\langle\cdot,\cdot\rangle'$ in the space of Riemannian metrics on $X(\mathbb{C})$ by

$$\langle\cdot,\cdot\rangle_t=(1-t)\cdot\langle\cdot,\cdot\rangle+t\cdot\langle\cdot,\cdot\rangle'$$

for $t\in[0,1]$. Then, for all $t\in[0,1]$, the Riemannian metric $\langle\cdot,\cdot\rangle_t$ on $X(\mathbb{C})$ defines a conformal structure φ_t satisfying $\sigma^\star\varphi_t=\varphi_t$. Moreover, $\varphi_0=\varphi$ and $\varphi_1=\varphi'$. This proves that $\Gamma(X)$ is path connected and, hence, that $R(X)$ is path connected. □

Let g be a nonnegative integer. For $i=1,\dots,[\frac{1}{2}(3g+4)]$, let X_i be a real algebraic curve of genus g such that X_i and X_j are of different topological types whenever $i\neq j$. It is natural to endow $M_{g/\mathbb{R}}$ with the topology for which the subsets $R(X_i)$, $i=1,\dots,[\frac{1}{2}(3g+4)]$, are its connected components. The topological space $M_{g/\mathbb{R}}$ is called the *moduli space of real algebraic curves of genus* g. As a consequence of Proposition 5.5 and Theorems 15.1 and 16.1, one has the following statement which is part of Theorem 1.2.

THEOREM 16.2. *Let g be a nonnegative integer. Then, the number of connected components of the moduli space $M_{g/\mathbb{R}}$ of real algebraic curves of genus g is equal to $[\frac{1}{2}(3g+4)]$. Two real algebraic curves X and Y of genus g belong to the same connected component if and only if X and Y are of the same topological type.*

17. Beltrami differentials

In this section we recall the definition of Beltrami differentials and some related notions [**13**].

Let (M,Φ_0) be a compact 2-dimensional almost complex manifold. According to Section 12, M is the almost complex differentiable structure of a Riemann surface, which we denote again by M. Denote by Ω the complex line bundle of complex differential forms on M. Denote by $\overline{\Omega}$ its conjugate complex line bundle. Observe that the line bundle $\Omega^{-1}\otimes\overline{\Omega}$ is a normed line bundle if we put

$$\left\|\frac{d\bar z}{dz}\right\|=1$$

for every local analytic coordinate z on M. One then considers the sup-norm, denoted by $\|\cdot\|_\infty$, on the global sections of $\Omega^{-1}\otimes\overline{\Omega}$. A *differentiable Beltrami differential* on M is a differentiable global section μ of the vector bundle $\Omega^{-1}\otimes\overline{\Omega}$ of norm $\|\mu\|_\infty$ less than 1. The set of differentiable Beltrami differentials on M is denoted by $C^\infty_{(-1,1)}(M)_1$.

Let φ be any conformal structure on the differentiable manifold M. One associates with φ a differentiable Beltrami differential μ as follows. For each local analytic coordinate z, the differential d_pz at a point p of M is a $\mathbb{C}$-linear isomorphism from T_pM onto $\mathbb{C}$. According to Section 9, one gets a Beltrami coefficient $\mu_{z,p}$ of φ_p relative to $(\Phi_0)_p$ and d_pz. It is clear from its construction that $\mu_{z,p}$ varies differentiably in p. It follows from Section 9 that the local section $\mu_z \cdot \frac{d\bar{z}}{dz}$ of $\Omega^{-1}\otimes\overline{\Omega}$ does not depend on the local analytic coordinate z. Since M is compact, it defines a differentiable global section μ of $\Omega^{-1}\otimes\overline{\Omega}$ of norm $\|\mu\|_\infty$ strictly less than 1; i.e., μ is a differentiable Beltrami differential. One calls μ the *Beltrami differential* of the conformal structure φ.

Conversely, if μ is a differentiable Beltrami differential on M, then one associates to μ a conformal structure φ whose differentiable Beltrami differential is equal to μ. Indeed, in a local analytic coordinate z, $\mu = \mu_z \cdot \frac{d\bar{z}}{dz}$ for some differentiable function μ_z. Then, locally, one defines the conformal structure φ as the equivalence class of $|dz + \mu_z d\bar{z}|^2$. It is easily checked that φ does not depend on the choice of the local analytic coordinate z.

We conclude that the assignment $\varphi \mapsto \mu$ from the set of conformal structures on M into the set of differentiable Beltrami differentials on M is a bijection.

Let (M, Φ_0) be a compact 2-dimensional almost complex manifold equipped with an action of Σ satisfying $\sigma^\star\Phi = \overline{\Phi}$. Then, one has an action of Σ on the line bundle Ω, which, in turn, induces an action of Σ on the set of Beltrami differentials $C^\infty_{(-1,1)}(M)_1$ of M. Locally, this action is as follows. One can cover M by equivariant local complex analytic coordinates $z\colon U \to \mathbb{C}$. Let $\mu_z \cdot \frac{d\bar{z}}{dz}$ be a Beltrami differential over the open subset U of M. Then,

$$\sigma \cdot \left(\mu_z \cdot \frac{d\bar{z}}{dz}\right) = \overline{\mu_z} \cdot \frac{d\bar{z}}{dz}.$$

It follows from the definition of the Beltrami coefficient associated to a conformal structure (see Section 9) that the map that associates to a conformal structure on M its Beltrami differential, is equivariant.

Let (M, Φ_0) be a compact 2-dimensional almost complex differentiable manifold. Let M' be any differentiable manifold equipped with a conformal structure φ'. Let $f\colon M \to M'$ be a local diffeomorphism. Then, one has an induced conformal structure $\varphi = f^\star\varphi'$ on M. Its Beltrami differential is denoted by μ_f and is called the *complex dilation* of f. Of course, f is conformal if and only if its complex dilation μ_f vanishes.

If g is a conformal local diffeomorphism of M' into a conformal manifold (M'', φ''), then the maps f and $g\circ f$ have the same complex dilation. Indeed, $(g\circ f)^\star\varphi'' = f^\star(g^\star\varphi'') = f^\star\varphi'$, since g is conformal. It follows that $\mu_{g\circ f} = \mu_f$.

One can easily determine the Beltrami differential of the local diffeomorphism f. Suppose Φ' is an almost complex structure on M' such that Φ' induces the conformal structure φ' and such that f is orientation-preserving with respect to the orientations induced by Φ_0 and Φ'. Of course, such an almost complex structure always exists at least locally. Let z' be a local analytic coordinate for M'. Since $f^\star dz' = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$, it follows from Section 10 that, locally,

$$\mu_f = \frac{\frac{\partial f}{\partial \bar{z}}}{\frac{\partial f}{\partial z}} \cdot \frac{d\bar{z}}{dz}. \tag{1}$$

Once again one sees that f is conformal if and only if its complex dilation vanishes.

Let μ be the complex dilation of the local diffeomorphism f. One defines the (global) *circular dilation* of f as the real number

$$\gamma_f = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}.$$

18. Quasiconformal homeomorphisms

In order to have a good Teichmüller theory, one relaxes the requirement that the maps between Riemann surfaces be differentiable. If one still wants these maps to have complex dilations, one is led to the notion of quasiconformal orientation-preserving homeomorphisms. For the precise definition of these homeomorphisms one is referred to [**13**].

For our purposes it will suffice to know that a quasiconformal orientation-preserving homeomorphism f from an open subset U of $\mathbb{C}$ onto an open subset V of $\mathbb{C}$ has complex derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ almost everywhere in U, i.e., outside a set of measure 0. The complex derivative $\frac{\partial f}{\partial z}$ is then nonzero almost everywhere. Moreover, the quotient

$$\mu = \frac{\frac{\partial f}{\partial \bar{z}}}{\frac{\partial f}{\partial z}}$$

is a measurable function on U with $|\mu| \leq k < 1$, almost everywhere, for certain k; i.e., μ is an element of the complex Banach space $L^\infty(U)$ of norm $\|\mu\|_\infty$ less than 1. The composition of two quasiconformal orientation-preserving homeomorphisms is again quasiconformal. The inverse of a quasiconformal orientation-preserving homeomorphism is again quasiconformal. Furthermore, if $\mu = 0$ in $L^\infty(U)$, then f is holomorphic.

That f be quasiconformal does not depend on the choice of the complex analytic coordinates on U and V. Therefore, it makes sense to speak of quasiconformal orientation-preserving local homeomorphisms between Riemann surfaces. It follows that the set of quasiconformal orientation-preserving self-homeomorphisms of a Riemann surface M is a group. This group is denoted by $Q^+(M)$. Observe that orientation-preserving local diffeomorphisms between Riemann surfaces are quasiconformal.

Let M be a Riemann surface. Consider the measurable essentially bounded global sections of the line bundle $\Omega^{-1} \otimes \overline{\Omega}$. These constitute a complex Banach space $L^\infty(\Omega^{-1} \otimes \overline{\Omega})$ which is denoted by $L^\infty_{(-1,1)}(M)$. The open unit ball in this Banach space is the set of *Beltrami differentials* on M. This set is denoted by $L^\infty_{(-1,1)}(M)_1$. It contains the set $C^\infty_{(-1,1)}(M)_1$ of differentiable Beltrami differentials on M. If X is a complex algebraic curve, the set of Beltrami differentials on X is denoted by $L^\infty_{(-1,1)}(X)_1$.

Let (M, Φ_0) and (M', Φ') be compact 2-dimensional almost complex manifolds. We denote again by M and M' the unique structures of a Riemann surface on M and M'. Let f be a quasiconformal orientation-preserving local homeomorphism from M into M'. Then, Equation 1 defines a Beltrami differential μ_f on M. This associated Beltrami differential is called the *complex dilation* of f. Clearly, if f is differentiable, this complex dilation coincides with the formerly defined one (see Section 17). It follows from what is said above that a quasiconformal orientation-preserving local homeomorphism between Riemann surfaces is holomorphic if its complex dilation vanishes.

Let μ be the complex dilation of the quasiconformal orientation-preserving local homeomorphism f. One defines the *circular dilation* of f as the real number

$$\gamma_f = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}.$$

Obviously, if f is a local diffeomorphism, this circular dilation coincides with the previously defined one (see Section 17). Moreover, a quasiconformal orientation-preserving local homeomorphism between Riemann surfaces is holomorphic if its circular dilation is equal to 1.

Let M be a Riemann surface equipped with an action of Σ. Then, one has an action of Σ on the set $L^\infty_{(-1,1)}(M)_1$ of Beltrami differentials. Indeed, the action of Σ on M induces an action of Σ on the line bundle Ω. This action, in turn, induces an action of Σ by isometries on the complex Banach space $L^\infty(\Omega^{-1}\otimes\overline{\Omega})$ of L^∞-sections of $\Omega^{-1}\otimes\overline{\Omega}$ (compare Section 17). It follows that Σ acts on the set of Beltrami differentials on M. The action of σ on a Beltrami differential μ is denoted by μ^σ.

Let M be a Riemann surface equipped with an action of Σ. Let $f\colon M\to M'$ be a quasiconformal orientation-preserving local homeomorphism of Riemann surfaces. Consider the conjugate map f^σ from M^σ into $(M')^\sigma$. Since M is equipped with an action of Σ, one canonically identifies M^σ with M. Then, the complex dilation of f^σ is equal to the conjugate of the complex dilation of f, i.e.,

$$\mu_{f^\sigma}=\mu_f^\sigma. \tag{2}$$

19. Complex Teichmüller spaces

In this section we recall definitions and some facts about complex Teichmüller spaces (see [**13**] for more details).

Let g be a nonnegative integer, and let X be a reference complex algebraic curve of genus g. Then, one defines the moduli space of X—which is, for the moment, only a set—as the set of isomorphism classes of complex algebraic curves Y such that there is a homeomorphism between $Y(\mathbb{C})$ and $X(\mathbb{C})$. In fact, $R(X)$ is nothing but the set of isomorphism classes of complex algebraic curves of genus g, and is also denoted by R_g or M_g, in the literature. Here, it will be convenient, however, to denote this set by $R(X)$.

A *marked complex algebraic curve modeled on* X is a pair (Y,f) consisting of a complex algebraic curve Y and a quasiconformal orientation-preserving homeomorphism $f\colon X(\mathbb{C})\to Y(\mathbb{C})$. Let $\widehat{M}(X)$ be the set of marked complex algebraic curves modeled on X. One has a canonical map

$$\widehat{\rho}\colon \widehat{M}(X)\longrightarrow R(X)$$

which associates to a marked curve (Y,f) the isomorphism class of Y. Since for every complex algebraic curve Y of genus g, the oriented differentiable manifold $Y(\mathbb{C})$ is orientation-preservingly diffeomorphic to $X(\mathbb{C})$, the map ρ is surjective.

Let (Y,f) and (Z,h) be marked complex algebraic curves modeled on X. Then, (Y,f) and (Z,h) are said to be *equivalent* as marked curves if the map $h^{-1}\circ f\colon Y(\mathbb{C})\to Z(\mathbb{C})$ is conformal. This is denoted by $(Y,f)\sim(Z,h)$. Obviously, the relation $\sim$ on $\widehat{M}(X)$ is an equivalence relation. The quotient $\widehat{M}(X)/{\sim}$ is denoted by $M(X)$. The quotient map is denoted by π. Of course, the map $\widehat{\rho}$ induces a map

$$\rho\colon M(X)\to R(X).$$

Let $Q^+(X)$ be the group of quasiconformal orientation-preserving self-homeomorphisms of $X(\mathbb{C})$. The group $Q^+(X)$ acts on the set $\widehat{M}(X)$ of marked curves. Indeed, if $(Y,f)\in\widehat{M}(X)$ and $\alpha\in Q^+(X)$, then one defines $(Y,f)\cdot\alpha$ by $(Y,f\circ\alpha)$. From the definition of the equivalence relation $\sim$ on $\widehat{M}(X)$ it readily follows that one has an induced action of $Q^+(X)$ on $M(X)$. The map ρ then is the quotient of $M(X)$ by the action of $Q^+(X)$.

Let $Q_0^+(X)$ be the subgroup of $Q^+(X)$ consisting of those self-homeomorphisms of $X(\mathbb{C})$ that are homotopic to the identity. The *Teichmüller space* of the complex algebraic curve X—which is also, for the moment, only a set—is the quotient $M(X)/Q_0^+(X)$, and is denoted by $T(X)$. One has a quotient map

$$\tau\colon M(X) \longrightarrow T(X).$$

By definition, two marked curves (Y, f) and (Z, h) in $M(X)$ have the same image in $T(X)$ if and only if there is a self-homeomorphism α of $X(\mathbb{C})$ in $Q_0^+(X)$ such that $h \circ (f \circ \alpha)^{-1}$ is conformal. The latter condition is equivalent to the existence of an orientation-preserving conformal homeomorphism $\varphi\colon Y(\mathbb{C}) \to Z(\mathbb{C})$ such that the maps h and $\varphi \circ f$ are homotopic.

Since the map ρ is the quotient of $M(X)$ by the full group $Q^+(X)$, ρ factorizes through τ and gives a map

$$\rho'\colon T(X) \longrightarrow R(X).$$

One has the following commutative diagram.

$$\begin{array}{ccc} \widehat{M}(X) & \xrightarrow{\ \widehat{\rho}\ } & R(X) \\ {\scriptstyle\pi}\downarrow & \overset{\rho}{\nearrow} & \uparrow{\scriptstyle\rho'} \\ M(X) & \xrightarrow{\ \tau\ } & T(X) \end{array}$$

Observe that $Q_0^+(X)$ is a normal subgroup of the group $Q^+(X)$. The quotient $Q^+(X)/Q_0^+(X)$ is the *modular group* of X, denoted by $\mathrm{Mod}(X)$. Of course, the action of $Q^+(X)$ on $M(X)$ induces an action of the modular group $\mathrm{Mod}(X)$ on the Teichmüller space $T(X)$. Clearly, the map ρ' is the quotient of $T(X)$ by the action of the group $\mathrm{Mod}(X)$.

In order to construct a structure of a complex analytic manifold on $T(X)$, one considers Beltrami differentials on X, i.e., Beltrami differentials on $X(\mathbb{C})$. Recall that the set of Beltrami differentials on X is the open unit ball $L^\infty_{(-1,1)}(X)_1$ in the Banach space of L^∞-sections of the line bundle $\Omega^{-1} \otimes \overline{\Omega}$. It follows from the preceding sections that the map μ that associates to an element (Y, f) of $M(X)$ the complex dilation μ_f of f is a bijection from $M(X)$ onto $L^\infty_{(-1,1)}(X)_1$. Hence, one has induced maps $\tau' = \tau \circ \mu^{-1}$ and $\rho'' = \rho' \circ \tau'$, so that one has the following commutative diagram:

$$\begin{array}{ccccc} \widehat{M}(X) & \xrightarrow{\ \widehat{\rho}\ } & R(X) & & \\ {\scriptstyle\pi}\downarrow & \overset{\rho}{\nearrow} & \uparrow{\scriptstyle\rho'} & \overset{\rho''}{\nwarrow} & \\ M(X) & \xrightarrow{\ \tau\ } & T(X) & \xleftarrow{\ \tau'\ } & L^\infty_{(-1,1)}(X)_1 \end{array}$$

$$\mu\colon M(X) \longrightarrow L^\infty_{(-1,1)}(X)_1$$

Now, the set $L^\infty_{(-1,1)}(X)_1$ of Beltrami differentials, being an open subset of a complex Banach space, is naturally a complex Banach manifold. Moreover, μ being a bijection, one has an induced action of the group $Q^+(X)$ on $L^\infty_{(-1,1)}(X)_1$. Obviously, the map τ' is the quotient map for the action of the subgroup $Q_0^+(X)$. The action of $Q^+(X)$ on $L^\infty_{(-1,1)}(X)_1$ turns out to be holomorphic, from which one concludes that the quotient, i.e., $T(X)$, is a complex Banach manifold, and that the modular group $\mathrm{Mod}(X)$ acts holomorphically on $T(X)$. In fact, $T(X)$ is finite-dimensional, i.e., $T(X)$ is a complex analytic manifold, and the modular group acts properly discontinuously on $T(X)$. Then, the quotient $R(X) = T(X)/\mathrm{Mod}(X)$ gets the structure of a normal complex analytic variety [**2**].

Observe that, if we have an action of Σ on $X(\mathbb{C})$, then Σ acts on the whole diagram above. Indeed, the action of σ on $R(X)$ sends a complex algebraic curve Y to its complex conjugate Y^σ. It sends a marked complex algebraic curve (Y,f) modeled on X onto the marked curve (Y^σ, f^σ) which is also modeled on X, since $X^\sigma = X$, canonically. This action is easily checked to induce actions on $M(X)$ and $T(X)$. The maps $\widehat{\rho}$, ρ, ρ', π and τ are Σ-equivariant. According to Equation 2, the map μ is Σ-equivariant with respect to the canonical action of Σ on the set of Beltrami differentials. Then, the maps τ' and ρ'' are Σ-equivariant, too.

20. Real Teichmüller spaces

In this section we recall the definition of the real Teichmüller space of a real algebraic curve and related notions.

Let g be a nonnegative integer, and let X be a reference real algebraic curve of genus g. Recall from Section 16 that the set $R(X)$ is the set of isomorphism classes of real algebraic curves Y having the same topological type as X. Equivalently, one can define $R(X)$ as the set of isomorphism classes of real algebraic curves Y such that there is an equivariant quasiconformal homeomorphism between $Y(\mathbb{C})$ and $X(\mathbb{C})$ (cf. Proposition 5.5). We consider $R(X)$, for the moment, only as a set. The moduli problem is to find a "natural" geometric structure on $R(X)$.

A *marked real algebraic curve modeled on* X is a pair (Y,f) consisting of a real algebraic curve Y and an equivariant quasiconformal orientation-preserving homeomorphism $f\colon X(\mathbb{C}) \to Y(\mathbb{C})$. Let $\widehat{M}(X)$ be the set of marked real algebraic curves modeled on X. One has a canonical map

$$\widehat{\rho}\colon \widehat{M}(X) \longrightarrow R(X)$$

which associates to a marked curve (Y,f) the isomorphism class of Y. Obviously, $\widehat{\rho}$ is surjective.

Let (Y,f) and (Z,h) be marked real algebraic curves modeled on X. Then, (Y,f) and (Z,h) are said to be *equivalent* as marked curves if the map $h^{-1} \circ f\colon Y(\mathbb{C}) \to Z(\mathbb{C})$ is conformal. This is denoted by $(Y,f) \sim (Z,h)$. Obviously, the relation $\sim$ on $\widehat{M}(X)$ is an equivalence relation. The quotient $\widehat{M}(X)/{\sim}$ is denoted by $M(X)$. The quotient map is denoted by π. It is clear that the map $\widehat{\rho}$ induces a map

$$\rho\colon M(X) \longrightarrow R(X).$$

Consider the canonical map from $M(X)$ into $M(X_\mathbb{C})$. It is clear that this map is injective. Moreover, its image is the set of fixed points $M(X_\mathbb{C})^\Sigma$ for the action of Σ on $M(X_\mathbb{C})$. Indeed, any marked real algebraic curve gives rise to a fixed point in $M(X_\mathbb{C})$ for the action of Σ. Conversely, let (Y,f) be the equivalence class of a marked complex algebraic curve modeled on $X_\mathbb{C}$ satisfying $(Y,f)^\sigma = (Y,f)$ in $M(X_\mathbb{C})$. This means that the homeomorphism $f^\sigma \circ f^{-1}$ from $Y(\mathbb{C})$ onto $Y^\sigma(\mathbb{C})$ is conformal. But then,

$$(f^\sigma \circ f^{-1})^\sigma \circ (f^\sigma \circ f^{-1}) = f \circ (f^\sigma)^{-1} \circ f^\sigma \circ f^{-1} = \mathrm{id}.$$

That is, $f^\sigma \circ f^{-1}$ is an isomorphism of Y onto Y^σ defining a real structure on Y. With respect to this real structure, the homeomorphism f is Σ-equivariant, for

$$(f^\sigma \circ f^{-1}) \circ f = f^\sigma.$$

This shows that the image of $M(X)$ is equal to $M(X_{\mathbb{C}})^\Sigma$.

Another way to see that the image of $M(X)$ in $M(X_{\mathbb{C}})$ is equal to the set of fixed points $M(X_{\mathbb{C}})^\Sigma$ is by using the equivariance of the complex dilation map

$$\mu\colon M(X_{\mathbb{C}}) \longrightarrow L^\infty_{(-1,1)}(X_{\mathbb{C}})_1.$$

Indeed, if (Y, f) is a fixed point for the action of Σ on $M(X_{\mathbb{C}})$, then its complex dilation $\mu = \mu_f$ is equivariant. According to [**13**, Section 1.3.2], there is then an almost complex structure on the topological space $X(\mathbb{C})$, denoted by $X(\mathbb{C})_\mu$, such that the identity $\mathrm{id}\colon X(\mathbb{C}) \to X(\mathbb{C})_\mu$ has complex dilation μ. It follows that f considered as a map from $X(\mathbb{C})_\mu$ into $Y(\mathbb{C})$ is holomorphic. Hence, denoting by X_μ the complex algebraic curve whose underlying Riemann surface is $X(\mathbb{C})_\mu$, the marked curve (X_μ, id) represents the same element of $M(X_{\mathbb{C}})$. Since μ is a fixed point for the action of Σ, the action of Σ on $X(\mathbb{C})_\mu$ is an action of Σ on a Riemann surface. Therefore, X_μ comes, in fact, from a real algebraic curve X'_μ, and the identity map id from $X(\mathbb{C})$ onto $X'_\mu(\mathbb{C})$ is Σ-equivariant, i.e., (X'_μ, id) is a marked curve modeled on the real curve X whose image in $M(X_{\mathbb{C}})$ is equal to (Y, f).

Let $Q^+(X)$ be the group of Σ-equivariant quasiconformal orientation-preserving self-homeomorphisms of $X(\mathbb{C})$. The group $Q^+(X)$ acts on the set $\widehat{M}(X)$ of marked curves. Indeed, if $(Y, f) \in \widehat{M}(X)$ and $\alpha \in Q^+(X)$, then one defines $(Y, f) \cdot \alpha$ by $(Y, f \circ \alpha)$. It follows immediately from the definition of the equivalence relation $\sim$ on $\widehat{M}(X)$ that one has an induced action of $Q^+(X)$ on $M(X)$. The map ρ then is the quotient of $M(X)$ by the action of $Q^+(X)$.

Let $Q_0^+(X)$ be the subgroup of $Q^+(X)$ consisting of those self-homeomorphisms of $X(\mathbb{C})$ that are homotopic to the identity. The *Teichmüller space* of the real algebraic curve X—which is, for the time being, merely a set—is the quotient $M(X)/Q_0^+(X)$, and is denoted by $T(X)$. One has a quotient map

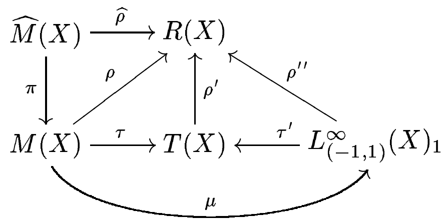
$$\tau\colon M(X) \longrightarrow T(X).$$

By definition, two marked curves (Y, f) and (Z, h) in $M(X)$ have the same image in $T(X)$ if and only if there is a self-homeomorphism α of $X(\mathbb{C})$ in $Q_0^+(X)$ such that $h \circ (f \circ \alpha)^{-1}$ is conformal. The latter condition is equivalent to the existence of a Σ-equivariant conformal orientation-preserving homeomorphism $\varphi\colon Y(\mathbb{C}) \to Z(\mathbb{C})$ such that the maps h and $\varphi \circ f$ are homotopic.

Since ρ is the quotient of $M(X)$ by the full group $Q^+(X)$, the map ρ factorizes through τ and gives a map

$$\rho'\colon T(X) \longrightarrow R(X).$$

Denote by $L^\infty_{(-1,1)}(X)_1$ the set of Beltrami differentials of $L^\infty_{(-1,1)}(X_{\mathbb{C}})$ that are Σ-invariant. Since the complex dilation map from $M(X_{\mathbb{C}})$ into $L^\infty_{(-1,1)}(X_{\mathbb{C}})_1$ is a Σ-equivariant bijection, and since $M(X)$ can naturally be identified with $M(X_{\mathbb{C}})^\Sigma$, one has an induced bijection, again denoted by μ, from $M(X)$ onto $L^\infty_{(-1,1)}(X)_1$. As in the complex case, one then puts $\tau' = \tau \circ \mu^{-1}$ and $\rho'' = \rho' \circ \tau'$, and one has the following commutative diagram:



Observe that $Q_0^+(X)$ is a normal subgroup of the group $Q^+(X)$. The quotient $Q^+(X)/Q_0^+(X)$ is the *modular group* of the real algebraic curve X, denoted by $\mathrm{Mod}(X)$. Of course, the action of $Q^+(X)$ on $M(X)$ induces an action of the modular group $\mathrm{Mod}(X)$ on the Teichmüller space $T(X)$. Clearly, the map ρ' is the quotient of $T(X)$ by the action of the group $\mathrm{Mod}(X)$.

Since $L^\infty_{(-1,1)}(X)_1$ is a real Banach manifold and $Q_0^+(X)$ acts analytically, one might show directly that the quotient, the real Teichmüller space $T(X)$, is a real Banach manifold of finite dimension, i.e., a real analytic manifold. However, we shall prove the latter fact in a more or less indirect way using Teichmüller's Theorem.

21. Teichmüller's Theorem

We recall Teichmüller's Theorem [**13**, Theorem 2.6.4], and apply it to derive results on real Teichmüller spaces.

Let X be a complex algebraic curve. Let (Y, f) be a marked complex algebraic curve modeled on X. Teichmüller's Theorem states that there is a unique extremal quasiconformal orientation-preserving homeomorphism f_T from $X(\mathbb{C})$ onto $Y(\mathbb{C})$ such that f_T is homotopic to f, i.e., (Y, f) and (Y, f_T) represent the same point in $T(X)$. By "extremal" is meant that the circular dilation $\gamma(f_T)$ of f_T is minimal, i.e.,

$$\gamma(f_T) = \inf\{\, \gamma(h) \mid h\colon X(\mathbb{C}) \to Y(\mathbb{C}),\ \text{homotopic to } f\,\}.$$

In fact, the statement of Teichmüller's Theorem is more precise [**13**, Section 2.6.4].

Let $A^2(X)$ be the complex vector space of global quadratic differentials on X, i.e.,

$$A^2(X) = \Gamma(X, \Omega^{\otimes 2}).$$

One makes $A^2(X)$ into a normed complex vector space by defining

$$\|\varphi\| = \int_{X(\mathbb{C})} |\varphi|$$

for $\varphi \in A^2(X)$. Denote by $A^2(X)_1$ the open unit ball in $A^2(X)$ with respect to this norm. Let $\varphi \in A^2(X)_1$. Then, one defines a Beltrami differential $\mu_T(\varphi)$ on X. With respect to a local analytic coordinate system z, φ is of the form $\varphi(z)\,dz^2$. Then, in this same local coordinate system,

$$\mu_T(\varphi) = \|\varphi\| \cdot \frac{\overline{\varphi(z)}}{|\varphi(z)|} \frac{d\overline{z}}{dz}$$

if $\varphi \neq 0$; otherwise $\mu_T(\varphi) = 0$. Let

$$H_{T/\mathbb{C}}\colon A^2(X)_1 \longrightarrow T(X)$$

be the map defined by $H_{T/\mathbb{C}}(\varphi) = \tau'(\mu_T(\varphi))$. Then, it essentially follows from Teichmüller's Theorem that $H_{T/\mathbb{C}}$ is a homeomorphism [**13**, Section 2.6.6]. In particular, if (Y, f) represents an element of $T(X)$, there is a unique $\varphi \in A^2(X)_1$ such that $H_{T/\mathbb{C}}(\varphi) = (Y, f)$ in $T(X)$. In fact, φ is such that $\mu_T(\varphi)$ is equal to the complex dilation of the extremal Teichmüller map f_T in the homotopy class of f.

Now, the idea of the construction of the structure of a real analytic manifold on $T(X)$ is to show that $T(X)$ can be identified with $T(X_{\mathbb{C}})^\Sigma$ for a real algebraic curve X. This is the statement of the following result.

THEOREM 21.1 (Earle [**4**]). *Let X be a real algebraic curve. Then the canonical map from $T(X)$ into $T(X_{\mathbb{C}})$ is a bijection onto $T(X_{\mathbb{C}})^\Sigma$.*

PROOF (SEPPÄLÄ [**16**]). Denote by $A^2(X)$ the real vector space of global quadratic differentials on X, i.e.,

$$A^2(X) = \Gamma(X, \Omega^{\otimes 2}).$$

In fact, $A^2(X)$ is the set of fixed points for the action of Σ on $A^2(X_{\mathbb{C}})$. Hence, $A^2(X)$ is a normed real vector space. Denote its open unit ball by $A^2(X)_1$. It is clear from its construction that the Beltrami differential $\mu_T(\varphi)$ is Σ-invariant for $\varphi \in A^2(X)_1$, i.e., $\mu_T(\varphi)$ is an element of $L^\infty_{(-1,1)}(X)_1$. Denote again by μ_T the map from $A^2(X)_1$ into $L^\infty_{(-1,1)}(X)_1$ which associates $\mu_T(\varphi)$ to φ. Denote by $H_{T/\mathbb{R}}$ the composition $\tau' \circ \mu_T$ from $A^2(X)_1$ into $T(X)$. Then, one has the following commutative diagram:

$$\begin{array}{ccccc}
 & & \overset{H_{T/\mathbb{C}}}{\frown} & & \\
A^2(X_{\mathbb{C}})_1 & \xrightarrow[\mu_T]{} & L^\infty_{(-1,1)}(X_{\mathbb{C}})_1 & \xrightarrow[\tau']{} & T(X_{\mathbb{C}}) \\
\uparrow & & \uparrow & & \uparrow \\
A^2(X)_1 & \xrightarrow[\mu_T]{} & L^\infty_{(-1,1)}(X)_1 & \xrightarrow[\tau']{} & T(X) \\
 & & \underset{H_{T/\mathbb{R}}}{\smile} & &
\end{array}$$

By Teichmüller's Theorem, $H_{T/\mathbb{C}}$ is a Σ-equivariant homeomorphism. Hence, its restriction to the set of fixed points $A^2(X_{\mathbb{C}})_1^\Sigma$ is a homeomorphism onto $T(X_{\mathbb{C}})^\Sigma$. Since $A^2(X_{\mathbb{C}})_1^\Sigma = A^2(X)_1$, one has that the canonical map from $T(X)$ into $T(X_{\mathbb{C}})$ has as image $T(X_{\mathbb{C}})^\Sigma$. In order to see that this map is injective, it suffices to show that $H_{T/\mathbb{R}}$ is surjective.

Let (Y, f) represent an element of $T(X)$. Let f_T be the extremal Teichmüller map in the homotopy class of f. Then, there is a unique φ in $A^2(X_{\mathbb{C}})_1$ such that the complex dilation of f_T is equal to $\mu_T(\varphi)$. By uniqueness of f_T, the map f_T is Σ-equivariant. Therefore, its complex dilation is Σ-invariant, i.e., φ is in $A^2(X)_1$ and $H_{T/\mathbb{R}}(\varphi) = (Y, f)$ in $T(X)$. This proves that $H_{T/\mathbb{R}}$ is surjective. □

Since $T(X_{\mathbb{C}})^\Sigma$ is a real analytic manifold, one obtains, by transport of structure, the structure of a real analytic manifold on $T(X)$. In particular, $T(X)$ is a topological space. It then follows that the map $H_{T/\mathbb{R}}$ is a homeomorphism. Therefore, one has the following real version of Teichmüller's Theorem, which is essentially due to Kravetz [**10**].

THEOREM 21.2. *Let X be a real algebraic curve. Then, the map $H_{T/\mathbb{R}}$ from $A^2(X)_1$ into $T(X)$ is a homeomorphism.* □

Let X be a real algebraic curve and let g be its genus. By Riemann-Roch, the dimension of the real vector space $A^2(X)$ is equal to 0 if $g = 0$; to 1 if $g = 1$; and to $3g-3$ if $g \geq 2$. Therefore, the real Teichmüller space $T(X)$ is a real analytic manifold of dimension 0 if $g=0$; 1 if $g=1$; and $3g-3$ if $g \geq 2$.

22. Moduli of real algebraic curves

Recall the following well known fact on quotients of real analytic manifolds by discontinuous group actions. Let M be a real analytic manifold. Let G be a group acting properly discontinuously on M. Then, the quotient M/G is a semianalytic variety (cf. [**6**] for details).

Let X be a real algebraic curve and let g be its genus. Since the modular group $\mathrm{Mod}(X_{\mathbb{C}})$ of the complex algebraic curve $X_{\mathbb{C}}$ acts properly discontinuously on $T(X_{\mathbb{C}})$, its subgroup $\mathrm{Mod}(X)$ also acts properly discontinuously on $T(X_{\mathbb{C}})$. In particular, $\mathrm{Mod}(X)$ acts properly discontinuously on $T(X)$. Therefore, the quotient $T(X)/\mathrm{Mod}(X)$ is a connected semianalytic variety. But this quotient is nothing but the moduli space $R(X)$ of real algebraic curves having the same topological type as X. Therefore, one has the following result.

THEOREM 22.1. *Let X be a real algebraic curve. Then, the moduli space $R(X)$ of real algebraic curves having the same topological type as X is a connected semianalytic variety.*

One easily verifies that the topology on $R(X)$ induced by its semianalytic structure coincides with the topology on $R(X)$ that was defined in Section 16.

For $i = 1, \dots, [\frac{1}{2}(3g+4)]$, let X_i be a real algebraic curve of genus g such that X_i and X_j are of different topological type whenever $i \neq j$. The set $M_{g/\mathbb{R}}$ of isomorphism classes of all real algebraic curves of genus g acquires the structure of a semianalytic variety for which the semianalytic varieties $R(X_i)$, $i = 1, \dots, [\frac{1}{2}(3g+4)]$, are its connected components.

We show that the structure of a semianalytic variety on $R(X)$ is natural. First, we need to explain what is meant by natural. For this we need to introduce the notion of an analytic family of real algebraic curves.

Let M be a real analytic manifold. Denote by $\mathcal{R}$ the sheaf of real analytic functions on M. A *family of real algebraic curves of genus g* over M is a triple $(\mathcal{C}, \mathcal{U}, \varphi)$, often simply denoted by $\mathcal{C}$, where

1. $\mathcal{U}$ is an open covering of M,
2. $\mathcal{C}(U)$ is a curve over the ring $\mathcal{R}(U)$ of real analytic functions on U for all $U \in \mathcal{U}$; i.e., $\mathcal{C}(U)$ is a proper and flat scheme over $\mathcal{R}(U)$ of finite presentation whose geometric fibers are algebraic curves of genus g (see [**5**] for details), and
3. φ constitutes gluing data for the curves $\mathcal{C}(U)$, i.e., for all $U, V \in \mathcal{U}$ we have isomorphisms
$$\varphi_{V,U} \colon \mathcal{C}(U) \otimes_{\mathcal{R}(U)} \mathcal{R}(U \cap V) \longrightarrow \mathcal{C}(V) \otimes_{\mathcal{R}(V)} \mathcal{R}(U \cap V)$$
of curves over the ring $\mathcal{R}(U \cap V)$ such that $\varphi_{U,U} = \mathrm{id}_{\mathcal{C}(U)}$ and $\varphi_{W,V} \circ \varphi_{V,U} = \varphi_{W,U}$ over $U \cap V \cap W$, for all $U, V, W \in \mathcal{U}$.

If $\mathcal{C}$ is a family of real algebraic curves of genus g over a real analytic manifold M, then for every $p \in M$, the fiber $\mathcal{C}_p$ is a real algebraic curve defined up to a specified isomorphism. Indeed, choose an open subset $U \in \mathcal{U}$ such that $p \in U$; then $\mathcal{C}_p$ is the real algebraic curve $\mathcal{C}(U) \otimes_{\mathcal{R}(U)} \mathbb{R}$, where $\mathbb{R}$ is considered as an $\mathcal{R}(U)$-algebra via the map of evaluation at p. One should think of the family $\mathcal{C}$ as a collection of real algebraic curves parametrized analytically by the real analytic manifold M. We say that the family of real algebraic curves $\mathcal{C}$ over M of genus g is a family of real algebraic curves having the same topological type as X, if for every $p \in M$ the fiber $\mathcal{C}_p$ is a real algebraic curve of the same topological type as X.

To say that the structure of a semianalytic variety on $R(X)$ is natural is to say the following.

THEOREM 22.2. *Let X be a real algebraic curve and let g be its genus. Let M be a real analytic manifold and let $\mathcal{C}$ be an analytic family of real algebraic curves of genus g over M having the same topological type as X. Then, the map $f\colon M \to R(X)$ defined by $f(p) = \mathcal{C}_p$ is analytic.*

In order to prove this result, one shows that a similar universal property holds for the real Teichmüller space $T(X)$. Let M be a real analytic manifold. An *analytic family of marked real algebraic curves modeled on X* over M is an analytic family $(\mathcal{C}, \mathcal{U}, \varphi)$ of real algebraic curves of genus g over M equipped with continuous families of real marking maps

$$f_U\colon \mathcal{C}(U)(\mathbb{C}) \to U \times X(\mathbb{C});$$

i.e., f_U is a Σ-equivariant homeomorphism and its restriction to each fiber $\mathcal{C}_p(\mathbb{C})$ is a quasiconformal orientation-preserving homeomorphism onto $\{p\} \times X(\mathbb{C})$. Of course, on overlaps $U \cap V$, the maps f_U and f_V are supposed to coincide.

One has the following important example of an analytic family of marked real algebraic curves modeled on X. Recall [**13**, Section 5.4.3] that one has a universal analytic family of complex algebraic curves $\mathcal{X}_{\mathbb{C}}$ over the complex analytic Teichmüller space $T(X_{\mathbb{C}})$. We show that the restriction of $\mathcal{X}_{\mathbb{C}}$ to $T(X)$ is an analytic family of marked real algebraic curves. Indeed, it follows from the universal property of $\mathcal{X}_{\mathbb{C}}$ that there is an action of Σ on the total space of $\mathcal{X}_{\mathbb{C}}$. The restriction $\mathcal{X}$ of $\mathcal{X}_{\mathbb{C}}$ to $T(X)$ then is an analytic family of marked real algebraic curves modeled on X. In fact, this family is the universal family of marked real algebraic curves modeled on X.

THEOREM 22.3. *Let X be a real algebraic curve. Let $\mathcal{C}$ be an analytic family of marked real algebraic curves modeled on X over a real manifold M. Let $f\colon M \to T(X)$ be the map defined by letting $f(p)$ be the element of $T(X)$ that is represented by the real algebraic curve $\mathcal{C}_p$ modeled on X. Then, f is analytic.*

PROOF. It suffices to prove the statement locally, i.e., we may suppose that the covering relative to which the family $\mathcal{C}$ is defined is trivial. Since the curve $\mathcal{C}(M)$ is of finite presentation, there are

1. a complex analytic manifold N endowed with an action of Σ, and
2. a family of marked complex algebraic curves $\mathcal{D}$ over N modeled on $X_{\mathbb{C}}$, endowed with an action of Σ over the action of Σ on N,

such that N^{Σ} is isomorphic to M and the restriction of $\mathcal{D}$ to N^{Σ} is isomorphic to $\mathcal{C}$. One then applies the universal property of $\mathcal{X}_{\mathbb{C}}$ to conclude the proof. □

The preceding result states that the real Teichmüller space is a coarse moduli space. In fact, one can prove that the real Teichmüller space $T(X)$ is even a fine moduli space if the genus of X is at least 2, but that is of no use in this paper.

PROOF OF THEOREM 22.2. Theorem 22.2 follows from Theorem 22.3, since every analytic family of real algebraic curves over a real analytic manifold is locally a family of marked real algebraic curves. □

As is explained in the Introduction, the property of $R(X)$ as stated in Theorem 22.2 does not uniquely determine the semianalytic structure on $R(X)$. The following result characterizes the semianalytic structure among all semianalytic structures on $R(X)$ satisfying the statement of Theorem 22.2.

THEOREM 22.4. *Let X be a real algebraic curve. Let $R(X)'$ be any semianalytic structure on the set $R(X)$ having the following property. If M is a real analytic manifold and $\mathcal{C}$ is an analytic family of real algebraic curves of the same topological type as X, then the map $f\colon M \to R(X)'$ defined by $f(p) = \mathcal{C}_p$ is analytic. Then, the identity map* $\mathrm{id}\colon R(X) \to R(X)'$ *is analytic.*

PROOF. We apply the hypothesis to the universal family of marked real algebraic curves $\mathcal{X}$ over $T(X)$ modeled on X. The map $f\colon T(X) \to R(X)'$ defined by letting $f(p)$ be the isomorphism class of the real algebraic curve $\mathcal{X}_p$ is analytic. Since $f \circ \alpha = f$ for any $\alpha \in \mathrm{Mod}(X)$, the map f factorizes through the quotient map $\rho'\colon T(X) \to R(X)$. This resulting map is, obviously, equal to the identity map and is analytic by definition of the quotient $T(X)/\mathrm{Mod}(X)$. □

PROOF OF THEOREM 1.2. For $i = 1, \dots, [\frac{1}{2}(3g+4)]$, let X_i be a real algebraic curve of genus g such that X_i and X_j are of different topological types whenever $i \neq j$. We have endowed $M_{g/\mathbb{R}}$ with the semianalytic structure for which the semianalytic varieties $R(X_i)$ are its connected components.

Let M be a real analytic manifold and let $\mathcal{C}$ be an analytic family of real algebraic curves of genus g over M. Let $f\colon M \to M_{g/\mathbb{R}}$ be the map defined by letting $f(p)$ be the isomorphism class of the fiber $\mathcal{C}_p$ for any $p \in M$. We have to show that f is analytic. For $i = 1, \dots, [\frac{1}{2}(3g+4)]$, let M_i be the subset of M consisting of those $p \in M$ for which the fiber $\mathcal{C}_p$ is of the same topological type as X_i. Then, each subset M_i is open and closed in M. Therefore, each M_i is a real analytic submanifold of M and M is the disjoint union of all M_i. It follows from Theorem 22.2 that the restriction of f to M_i is an analytic map into $R(X_i)$. Hence, the map f is analytic. This proves statement 1.

Let $M'_{g/\mathbb{R}}$ be another semianalytic structure on the set $M_{g/\mathbb{R}}$ satisfying statement 1. We have to show that the identity map $\mathrm{id}\colon M_{g/R} \to M'_{g/\mathbb{R}}$ is analytic. Let T be the disjoint union of the real analytic manifolds $T(X_i)$, $i = 1, \dots, [\frac{1}{2}(3g+4)]$. Let $\mathcal{X}$ be the universal real algebraic curve over T. Since $M'_{g/\mathbb{R}}$ satisfies statement 1, the map $f\colon T \to M'_{g/\mathbb{R}}$ defined by $f(p) = \mathcal{X}_p$, for $p \in M$, is analytic. As in the proof of Theorem 22.4, the map f induces an analytic map from $M_{g/\mathbb{R}}$ into $M'_{g/\mathbb{R}}$, and this map is the identity. This proves statement 2.

The other statements were proved in Section 16. □

References

[1] Alling, N. L., Greenleaf, N.: *Foundations of the theory of Klein surfaces.* Lect. Notes in Math. 219, Springer Verlag, 1971

[2] Cartan, H.: Quotient d'un espace analytique par un groupe d'automorphismes. *Algebraic geometry and topology, A symposium in honor of S. Lefschetz,* Princeton University Press, 90–102, 1957. Also in: *Oeuvres-Collected works/Henri Cartan,* R. Remmert, J.-P. Serre (eds.), Springer Verlag, 1979, Vol. 2, 687–699

[3] Gross, B. H., Harris, J.: Real algebraic curves. *Ann. Sci. Ec. Norm. Sup.* 4^e série, t. 14 (1981), 157–182
[4] Earle, C. J.: Moduli of surfaces with symmetries. *Advances in the theory of Riemann surfaces.* Ahlfors, L.V. et. al. (eds) Princeton University Press and University of Tokyo Press, 1971
[5] Hartshorne, R.: *Algebraic geometry.* Grad. Texts in Math., Vol. 52, Springer Verlag, 1977
[6] Huisman, J.: Real quotient singularities and nonsingular real algebraic curves in the boundary of the moduli space. *Compositio Math.* 118 (1999), 43–60
[7] Iversen, B.: *Cohomology of sheaves.* Universitext, Springer Verlag, 1986
[8] Klein, F.: *On Riemann's theory of algebraic functions and their integrals.* Dover, 1963
[9] Kobayashi, S., Nomizu, K.: *Foundations of differential geometry,* Vol. II, Interscience Publishers, 1969
[10] Kravetz, S.: On the geometry of Teichmüller spaces and the structure of their modular groups. *Ann. Acad. Sci. Fenn.* Ser. A, I. Math., 278 (1959), 1–35
[11] Marin, A.: Sur un théorème de Cheponkus. *Real analytic and algebraic geometry.* M. Galbiati, A. Tognoli (eds). Lect. Notes in Math. 1420, Springer Verlag, 1990, 191–193
[12] Mumford, D., Fogarty, J.: *Geometric invariant theory (second enlarged edition).* Ergeb. Math. 34, Springer Verlag, 1982
[13] Nag, S.: *The complex analytic theory of Teichmüller spaces.* John Wiley & Sons, 1988
[14] Natanzon, S. M.: Klein surfaces. *Uspekhi Mat. Nauk* 45:6 (1990), 47–90
[15] Rauch, H. E.: A transcendental view of the space of algebraic Riemann surfaces. *Bull. Amer. Math. Soc.* 71 (1965), 1–39
[16] Seppälä, M.: Teichmüller spaces of Klein surfaces. *Ann. Acad. Sci. Fen.* Ser. A, I Math. Diss. 15 (1978), 1–37
[17] Seppälä, M.: Moduli spaces of stable real algebraic curves. *Ann. Sci. Ec. Norm. Sup.* 4^e série t. 24 (1991), 519–544
[18] Seppälä, M., Silhol, R.: Moduli spaces for real algebraic curves and real abelian varieties. *Math. Z.* 201 (1989), 151–165
[19] Serre, J.-P.: Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier* 6 (1956), 1–42
[20] Silhol, R.: Compactifications of moduli spaces in real algebraic geometry. *Invent. math.* 107 (1992) 151–202
[21] Weichold, G.: Über symmetrische Riemann'sche Flächen und die Periodicitätsmoduln der zugehörigen Abel'schen Normalintegrale erster Gattung, *Zeit. f. Math. u. Phys.* 28 (1883), 321–351

Institut Mathématique, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France
E-mail address: huisman@univ-rennes1.fr