Real quotient singularities and nonsingular real algebraic curves in the boundary of the moduli space

J. Huisman

Abstract

The quotient of a real analytic manifold by a properly discontinuous group action is, in general, only a semianalytic variety. We study the boundary of such a quotient, i.e., the set of points at which the quotient is not analytic. We apply the results to the moduli space $M_{g/\mathbb{R}}$ of nonsingular real algebraic curves of genus g ($g \geq 2$). This moduli space has a natural structure of a semianalytic variety. We determine the dimension of the boundary of any connected component of $M_{g/\mathbb{R}}$. It turns out that every connected component has a nonempty boundary. In particular, no connected component of $M_{g/\mathbb{R}}$ is real analytic. We conclude that $M_{g/\mathbb{R}}$ is not a real analytic variety.

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1 Introduction

Let $M_{g/\mathbb{R}}$ be the moduli space of nonsingular real algebraic curves of genus g, where g is an integer greater than or equal to 2. It is well known that $M_{g/\mathbb{R}}$ has $g + 1 + [\frac{1}{2}(g + 2)]$ connected components [11]. These connected components correspond to the different topological types a nonsingular real algebraic curve can have.

Any connected component of $M_{g/\mathbb{R}}$ has a natural structure of a semianalytic variety. This can be seen by Teichmüller theory: Let X be a nonsingular real algebraic curve of genus g. Then, there is a connected real analytic manifold $T(X)$, called the real Teichmüller space of X of marked real algebraic curves modeled on X (cf. [4, 6, 11] or Section 5). Also, there is a group $\mathrm{Mod}(X)$, called the real modular group of X, which acts properly discontinuously on $T(X)$. The quotient $R(X) = T(X)/\mathrm{Mod}(X)$ is the moduli space of the real algebraic curve X. In fact, $R(X)$ consists of the isomorphism classes of all nonsingular real algebraic curves Y of genus g having the same topological type as X. Hence, $R(X)$ is the connected component of the moduli space $M_{g/\mathbb{R}}$ that contains X.

Since the action of $\mathrm{Mod}(X)$ on the $(3g-3)$-dimensional real analytic manifold $T(X)$ is properly discontinuous, the moduli space $R(X)$ of the real algebraic curve X is a semianalytic variety of the same dimension. Therefore, any connected component of $M_{g/\mathbb{R}}$—hence, also $M_{g/\mathbb{R}}$ itself—acquires a natural structure of a semianalytic variety of dimension $3g-3$. In fact, equipped with this structure, $M_{g/\mathbb{R}}$ is the coarse moduli space of nonsingular real algebraic curves of genus g [6].

In general, one can define, for a semianalytic variety N, its boundary ∂N as the subset of points at which the germ of N is not real analytic. It is known that the boundary ∂N is itself semianalytic [7], Proposition 16.1, so that it makes sense to speak of its dimension.

We concentrate on the following situation. Let M be a real analytic manifold and G a group acting properly discontinuously on M. Let N be the quotient M/G. Then, N is semianalytic. This is the statement of Proposition 3.1. We determine the boundary ∂N in terms of the action of G on M (see Proposition 3.2). As an example, in the case that M is connected and G acts faithfully, the image in N of an element x of M belongs to the boundary of N if and only if there is an element α of order 2 in G such that $\alpha \cdot x = x$.

We also determine the dimension of the boundary ∂N in terms of the action of G on M. When applied to the moduli space $R(X)$ of any nonsingular real algebraic curve X of genus $g \geq 2$, it will follow that the boundary $\partial R(X)$ is of positive dimension (cf. Theorems 5.2 and 5.4). Consequently,

Theorem 1.1. *Let $g \geq 2$. No connected component of the moduli space $M_{g/\mathbb{R}}$ of nonsingular real algebraic curves of genus g is real analytic. In particular, the semianalytic variety $M_{g/\mathbb{R}}$ is not real analytic.*

This result refutes [11], Theorem 2.2, to the effect that all connected components of $M_{g/\mathbb{R}}$ would be real analytic (cf. Remark 3.4).

The paper is organized as follows. In Section 2 we address the question when real quotient singularities are analytic. In Section 3 we apply the results of Section 2 to quotients of real analytic manifolds by properly discontinuous group actions. The main result there determines the boundary of such quotients, and expresses its dimension in terms of the group action on the manifold. In order to apply the results of Section 3 to real Teichmüller spaces, we need to study automorphisms of real algebraic curves of order 2, or what amounts to the same, morphisms of real algebraic curves of degree 2. This is done in Section 4. Section 5 is then devoted to our main result concerning the dimension of the boundary of any connected component of the moduli space $M_{g/\mathbb{R}}$.

Conventions and notation. An analytic variety is not necessarily nonsingular. An analytic manifold is a nonsingular analytic variety. Algebraic curves will always be nonsingular complete and geometrically irreducible. If G is a cyclic group generated by g then by an action of g we will mean an action of G. If g acts on a set X then X^g is the subset of fixed points X^G for the action of G. Furthermore, X/g will be the quotient X/G. The Galois group of $\mathbb{C}/\mathbb{R}$ will be denoted by Σ. Of course, $\Sigma = \{1, \sigma\}$ where σ is complex conjugation. When Σ is said to act on a complex analytic variety then it is understood that σ acts antiholomorphically. For x a real number, the greatest integer less than or equal to x is denoted by $[x]$.

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2 Real quotient singularities

Let G be a finite group acting linearly on a finite-dimensional real vector space V. We show that the quotient V/G is a semianalytic variety, and we study the question when this quotient is a real analytic variety. First, we need to introduce some notation and to establish a preliminary lemma.

Let W be the complexification $\mathbb{C} \otimes_{\mathbb{R}} V$ of V. Then, there is an induced action of G on W, and the canonical map $V \to W$ is G-equivariant.

Since W is the complexification of V, there is also a canonical action of the Galois group Σ on W. We consider V as a real subspace of W by means of the canonical map $V \to W$. Obviously, the set of fixed points W^Σ is equal to V.

The actions of Σ and G on W commute with each other. Therefore, there is an induced action of Σ on the set-theoretical quotient W/G. The inclusion of V into W induces an injective map

$$V/G \longrightarrow (W/G)^\Sigma.$$

We will identify V/G with its image in $(W/G)^\Sigma$.

Lemma 2.1. *Let V be a finite-dimensional real vector space. Let G be a finite group acting linearly and faithfully on V. Denote by W the complexification of V. Then, the quotient V/G is equal to $(W/G)^\Sigma$ if and only if the order of G is odd.*

Proof. Suppose that the order of G is even. Then, there is an element $\alpha \in G$ of order 2. Since G acts faithfully on V, α acts nontrivially on V. Hence, there is an element $v \in V$, $v \neq 0$ such that $\alpha v = -v$. But then, $w = \sqrt{-1} \cdot v$ is in W and satisfies $\alpha w = \sigma w$. Therefore, the image of w in W/G is a fixed point for the action of Σ. This fixed point is clearly not an element of V/G.

Suppose that the order of G is odd. Let $w \in W$ be such that its image in the quotient W/G is a fixed point for the action of Σ. That is, the orbit Gw of w in W is Σ-stable. Since the order of G is odd, the cardinality of Gw is odd too. Therefore, Σ, being a group of order 2, has a fixed point in Gw. But then, $Gw \subseteq V$. In particular, $w \in V$. □

Remark 2.2. The hypothesis of faithfulness of the action of G on V in the statement of Lemma 2.1 is only made to simplify the exposition. In fact, the general case of a not necessarily faithful action is a consequence of Lemma 2.1. Indeed, suppose a finite group G acts linearly, but not necessarily faithfully on a finite-dimensional real vector space V. Let again W be the complexification of V. Let K be the kernel of the representation morphism $G \to \mathrm{GL}(V)$. Then, the quotient G/K acts linearly and faithfully on V. Applying Lemma 2.1 to the action of G/K, one concludes that V/G is equal to $(W/G)^\Sigma$ if and only if the index $[G:K]$ of K in G is odd.

Let again V be a finite-dimensional real vector space, G a finite group acting linearly on V, and W the complexification of V.

We now consider the real vector space V with the Euclidean topology. Endow V/G with the quotient topology. Let $\pi\colon V \to V/G$ be the quotient map. Let $\mathcal{R}$ be the sheaf of real analytic functions on V. The group G acts on the real analytic variety $(V, \mathcal{R})$. Therefore, we get a G-action on the sheaf $\pi_\star\mathcal{R}$ on V/G. Let $\mathcal{R}' = (\pi_\star\mathcal{R})^G$. Then, $(V/G, \mathcal{R}')$ is the quotient of $(V, \mathcal{R})$ in the category of locally ringed spaces. We simply denote this space by V/G. We will see that V/G is, in fact, a semianalytic variety (cf. Lemma 2.4).

Similarly, we endow the complex vector space W with its Euclidean topology and its sheaf $\mathcal{O}$ of complex analytic functions. Then, G acts on W, and the quotient W/G of W is a complex analytic variety [2], Théorème 4. Let $\rho'\colon W \to W/G$ be the quotient map. Denote by $\mathcal{O}'$ the structure sheaf of W/G. Then, in fact, $\mathcal{O}' = (\rho'_\star\mathcal{O})^G$.

Clearly, we have an induced action of the Galois group Σ on the complex analytic variety W/G. In particular, we have a real analytic action of Σ on

W/G considered as a real analytic variety. Therefore, the subset $(W/G)^\Sigma$ of W/G is a real analytic subset of W/G. Hence, $(W/G)^\Sigma$ acquires the structure of a real analytic variety. Denote its structure sheaf by $\mathcal{R}''$. This sheaf is, in a natural way, the surjective image of the sheaf $(\mathcal{O}'_{|(W/G)^\Sigma})^\Sigma$ on $(W/G)^\Sigma$. This natural surjective morphism

$$(\mathcal{O}'_{|(W/G)^\Sigma})^\Sigma \longrightarrow \mathcal{R}''$$

is not injective at the stalks over the points x of $(W/G)^\Sigma$ at which the local dimension $\dim_x (W/G)^\Sigma$ is strictly smaller than the global dimension $\dim (W/G)^\Sigma$. At the other stalks, the map is an isomorphism.

Remark 2.3. The inclusion $i\colon V/G \to (W/G)^\Sigma$ is a map of locally ringed spaces. Since $\dim_x (W/G)^\Sigma$ at any point x of V/G is equal to $\dim (W/G)^\Sigma$, the morphism of sheaves $i^\sharp\colon \mathcal{R}'' \to i_\star \mathcal{R}'$ is an isomorphism at the stalks over the points of V/G. To put it differently, i is a closed embedding of locally ringed spaces.

Lemma 2.4. *Let V be a finite-dimensional vector space and let G be a finite group acting linearly on V. Then, the locally ringed space V/G is a semianalytic variety.*

Proof. By Remark 2.3, it suffices to show that V/G is a semianalytic subset of $(W/G)^\Sigma$. It is well known that W/G is actually a complex algebraic variety [2], Proposition 4, on which Σ acts. The set of fixed points $(W/G)^\Sigma$ is then a real algebraic variety, and the map $V \to (W/G)^\Sigma$ a morphism of real algebraic varieties. Therefore, its image V/G is a semialgebraic subset of $(W/G)^\Sigma$. In fact, V/G is basic closed, i.e., a finite intersection of subsets of the form $f \geq 0$, where f is a polynomial map on $(W/G)^\Sigma$ (cf. [9] and also [1]). In any case, V/G is a fortiori a semianalytic subset of $(W/G)^\Sigma$. $\square$

The following proposition is the main result of this section as it describes exactly when the real semianalytic quotient singularity V/G is analytic.

Proposition 2.5. *Let V be a finite-dimensional real vector space. Let G be a finite group acting linearly and faithfully on V. Denote by W the complexification of V. Then, the following conditions are equivalent:*

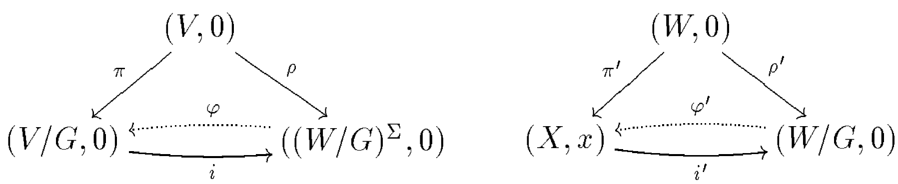
1. *V/G is a real analytic variety;*
2. *the germ $(V/G, 0)$ is real analytic;*
3. *$V/G = (W/G)^\Sigma$;*
4. *$\#G$ is odd.*

Proof. By Lemma 2.1, (3) and (4) are equivalent. The implication (3) $\Rightarrow$ (1) is clear since, under assumption of (3), the morphism i is an isomorphism from the semianalytic variety V/G onto the real analytic variety $(W/G)^\Sigma$ by Remark 2.3. The implication (1) $\Rightarrow$ (2) is obvious. We only need to show the implication (2) $\Rightarrow$ (3).

The morphisms π, ρ', i induce maps of germs at 0 which will be denoted by the same symbols:

$$\begin{aligned} \pi\colon (V,0) &\longrightarrow (V/G,0), \\ \rho'\colon (W,0) &\longrightarrow (W/G,0), \\ i\colon (V/G,0) &\longrightarrow ((W/G)^\Sigma,0). \end{aligned}$$

Suppose $(V/G,0)$ is a real analytic germ. Then, there is a complexification (X,x) of $(V/G,0)$. Since $(W,0)$ is a complexification of $(V,0)$, the map π induces a map of complex analytic germs π' from $(W,0)$ into (X,x). Similarly, since (X,x) is a complexification of $(V/G,0)$, the map i induces a map of complex analytic germs i' from (X,x) into $(W/G,0)$. Moreover, since ρ' is Σ-equivariant, ρ' induces a morphism of real analytic germs ρ from $(V,0)$ into $((W/G)^\Sigma,0)$. Consider the following two diagrams.



The two diagrams of solid arrows are commutative. Indeed, the one to the left clearly is commutative. It follows that the map of germs from $(V,0)$ into $((W/G)^\Sigma,0)$ induced by $i'\circ\pi'$ is equal to ρ. Since ρ' is the unique Σ-equivariant map of complex analytic germs from $(W,0)$ into $(W/G,0)$ inducing ρ, the diagram of solid arrows to the right is commutative too.

The map of germs φ' is obtained as follows. Letting G act trivially on (X,x), the map of germs π' is G-equivariant. Since ρ' is the quotient of $(W,0)$ by G, there is a unique map of germs φ' from $(W/G,0)$ into (X,x) such that $\varphi'\circ\rho'=\pi'$. Since φ' is automatically Σ-equivariant, φ' induces a map of germs φ from $((W/G)^\Sigma,0)$ into $(V/G,0)$.

Now, one has $i'\circ\varphi'\circ\rho' = i'\circ\pi' = \rho'$. Since ρ' is the quotient of $(W,0)$ by G, this implies $i'\circ\varphi' = \mathrm{id}$. Which, in turn, implies $i\circ\varphi=\mathrm{id}$. Therefore, V/G contains an open neighborhood of 0 in $(W/G)^\Sigma$. Using the induced $\mathbb{R}^\star$-action on $(W/G)^\Sigma$, it follows that $V/G=(W/G)^\Sigma$. □

Remark 2.6. Of course, the equivalences between the conditions 1, 2 and 3 of Proposition 2.5 do also hold when the action of G on V is not faithful.

Remark 2.7. Proposition 2.5 also holds when every occurrence of "analytic" is replaced by "algebraic."

3 Quotients of real analytic manifolds

In this section we show that the quotient of a real analytic manifold by a properly discontinuous group action is a semianalytic variety. We study the boundary of such a quotient and, in particular, its dimension.

Proposition 3.1. *Let M be a real analytic manifold and let G be a group acting properly discontinuously on M. Then, the quotient of M by G as a locally ringed space is a semianalytic variety.*

Proof. Denote by N the quotient of M by G, and denote by π the quotient map $M \to N$.

Let x be an element of M. Since G acts properly discontinuously, there is an open neighborhood U of x in M such that

1. U is G_x-stable, i.e., for all $\alpha \in G_x$, we have $\alpha \cdot U \subseteq U$, and
2. for all $\alpha \in G \backslash G_x$, we have $(\alpha \cdot U) \cap U = \emptyset$.

Here, G_x denotes the stabilizer of x, i.e., the subgroup of all elements α of G such that $\alpha \cdot x = x$. Then, the image $\pi(U)$ of U is an open neighborhood of $\pi(x)$ in N, and is isomorphic to the quotient U/G_x. Hence, in order to prove that N is a semianalytic variety, it suffices to prove that U/G_x is semianalytic.

Replacing U, if necessary, by a smaller open neighborhood of x satisfying the two conditions above, the action of G_x on U can be linearized (see [2], §4). This means that there is a linear action of G_x on a finite-dimensional real vector space V having the following property. There is an open neighborhood U' of the origin in V which is stable for the action of G_x, such that U and U' are G_x-equivariantly isomorphic as real analytic manifolds. Under this isomorphism the origin of V and the point x are supposed to correspond.

Observe that the group G_x is finite since the action of G on M is properly discontinuous. Therefore, by Lemma 2.4, the quotient V/G_x is semianalytic, in particular, U'/G_x—and hence U/G_x—is semianalytic. This shows that the quotient N is semianalytic. □

In general, one can intrinsically define the boundary ∂N of any semianalytic variety N as

$$\partial N = \{x \in N \mid \text{the germ } (N,x) \text{ is not real analytic}\}.$$

Our main result in this section concerns the boundary of the quotient of a real analytic manifold by a faithful and properly discontinuous group action.

Proposition 3.2. *Let M be a connected real analytic manifold and let G be a group acting faithfully and properly discontinuously on M. Denote the quotient by N, and the quotient map $M \to N$ by π.*

1. *The image $\pi(x)$ of an element x of M belongs to the boundary ∂N of N if and only if the stabilizer G_x of G at x is of even order. Equivalently,*

$$\partial N = \bigcup_{\substack{\alpha \in G \\ \mathrm{ord}(\alpha)=2}} \pi(M^\alpha).$$

   In particular, the semianalytic variety N is real analytic if and only if all elements of order 2 of G act fixed point-free on M.

2. *The local dimension of the boundary ∂N of N at the image $\pi(x)$ of an element x of M satisfies*

$$\dim_{\pi(x)} \partial N = \sup_{\substack{\alpha \in G_x \\ \mathrm{ord}(\alpha)=2}} \dim(T_x M)^\alpha,$$

   where $T_x M$ is the tangent space to M at x. In particular, the dimension of the boundary ∂N of N satisfies

$$\dim \partial N = \sup_{x \in M} \dim_{\pi(x)} \partial N = \sup_{x \in M} \sup_{\substack{\alpha \in G_x \\ \mathrm{ord}(\alpha)=2}} \dim(T_x M)^\alpha.$$

Proof. Let x be an element of M. Since G acts properly discontinuously, there is an open neighborhood U of x in M such that U is G_x-stable and $(\alpha \cdot U) \cap U = \emptyset$ for all $\alpha \in G \backslash G_x$. Since M is connected and G acts faithfully on M by real analytic automorphisms, the group G_x acts faithfully on U.

As in the proof of Proposition 3.1, one can linearize the action of G_x on U, replacing U by a smaller neighborhood if necessary. Hence, one reduces to the case that M is an open neighborhood of the origin in a finite-dimensional real vector space V, that x is equal to the origin of V, and that G is a finite group whose action on M extends to a linear and faithful action on V. In fact, one might as well suppose that M is all of V.

Statement 1 then follows immediately from Proposition 2.5.

In order to prove statement 2 of the proposition, one applies what we have proven so far to the quotient V/G, to conclude that its boundary is equal to the image in V/G of the union

$$\bigcup_{\substack{\alpha \in G \\ \mathrm{ord}(\alpha)=2}} V^\alpha.$$

Then, statement 2 of the proposition follows from the natural identification of V with its tangent space T_0V at the origin. □

Remark 3.3. The hypotheses of Proposition 3.2 that M is connected and that the action of G on M is faithful, are only made to simplify the exposition. In fact, the general case of a not necessarily connected real analytic manifold and a not necessarily faithful action follow from Proposition 3.2. Indeed, let M be such a manifold and let G be a group acting properly discontinuously, but not necessarily faithfully on M. Denote again the quotient M/G by N, and the quotient map $M \to N$ by π. Let $x \in M$ and let $C \subseteq M$ be the connected component of M containing x. Let G_C be the stabilizer of the connected component C, i.e., G_C is the subgroup of $\alpha \in G$ such that $\alpha \cdot C \subseteq C$. Let K be the kernel of the representation morphism $G_C \to \mathrm{Aut}(C)$. Then, the quotient G_C/K acts properly discontinuously and faithfully on the connected real analytic manifold C. The stabilizer $(G_C/K)_x$ of x is equal to G_x/K. Applying Proposition 3.2, one concludes that $\pi(x)$ is in the boundary of N if and only if the index $[G_x : K]$ of K in G_x is even. To put it differently, $\pi(x)$ is in the boundary of N if and only if there is an element $\alpha \in G$ having x as a fixed point, such that α does not act trivially in a neighborhood of x, whereas α^2 does act trivially in a neighborhood of x.

One can similarly generalize the other statements of Proposition 3.2.

Remark 3.4. The argument that made one conclude that the moduli space $M_{g/\mathbb{R}}$ would be real analytic was that the quotient M/G would be real analytic under an additional hypothesis [10], Theorem 1. This additional hypothesis consisted of M being a real analytic subset of a complex analytic manifold X such that the action of G on M is the restriction of a properly discontinuous action of G on X. However, it is false that this implies that the quotient M/G is real analytic, as shows the staggeringly simple counterexample of $M = \mathbb{R}$, $X = \mathbb{C}$ and G the multiplicative group $\{\pm 1\}$ acting linearly on M in the natural way.

The flaw in the "proof" of [10], Theorem 1, is of an interesting subtlety: Denote by X_{ran} the induced real analytic structure on a complex analytic variety X. Let G be a group acting properly discontinuously on X. This induces an action of G on X_{ran}. Denote this action by G_{ran}. Then, in general, $(X_{\mathrm{ran}})/(G_{\mathrm{ran}})$ is not isomorphic to $(X/G)_{\mathrm{ran}}$! Indeed, let us take again $X = \mathbb{C}$ and $G = \{\pm 1\}$. Then, $X/G \cong \mathbb{C}$, hence $(X/G)_{\mathrm{ran}} \cong \mathbb{R}^2$ and is real analytic. However, $X_{\mathrm{ran}} = \mathbb{R}^2$ and the quotient $\mathbb{R}^2/\{\pm 1\}$ is not real analytic according to Proposition 2.5. In fact, this quotient is a semicone (see Figure 1).

The example shows that X_{ran}, as defined above, is not the right thing to consider. One should rather consider restriction of scalars à la Weil of the complex analytic variety X with respect to the field extension $\mathbb{C}/\mathbb{R}$.

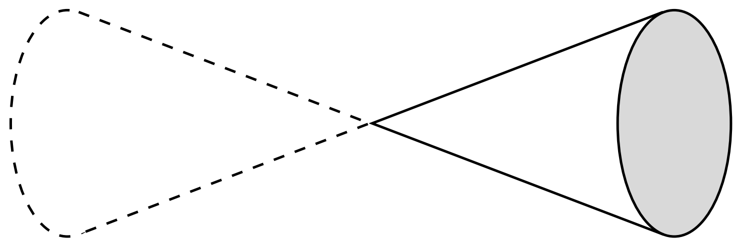


Figure 1: In solid lines: the quotient of the real analytic manifold $\mathbb{R}^2$ by the linear group action of $\{\pm 1\}$, embedded in $\mathbb{R}^3$ as the semicone $w^2 = uv$, $u \geq 0$, $v \geq 0$. The embedding is given by $(u, v, w) = (x^2, y^2, xy)$.

This means that one should define X_{ran} as the complex analytic variety $X \times \bar{X}$ endowed with its canonical Σ-action, where $\bar{X}$ is the complex conjugate variety. One gets an induced action of G_{ran}, now defined as $G_{\text{ran}} = G \times G$, on X_{ran}, and, this time, one has, indeed, a canonical Σ-equivariant isomorphism

$$(X_{\text{ran}})/(G_{\text{ran}}) \cong (X/G)_{\text{ran}}.$$

4 Morphisms of degree 2

We start this section by recalling some facts concerning the topology of real algebraic curves.

Let X be a real algebraic curve. The *topological type* of X is the homeomorphism class of the pair $(X(\mathbb{C}), X(\mathbb{R}))$.

Denote by $g = g(X)$ the genus of X, and denote by $c = c(X)$ the number of connected components of the set of real points $X(\mathbb{R})$ of X. The real algebraic curve X is said to be *dividing* if $X(\mathbb{C}) \backslash X(\mathbb{R})$ is not connected. It is well-known that these data, i.e., the genus of X, the number of connected components of $X(\mathbb{R})$, and whether or not X is dividing, determine completely the topological type of X. In other words, if X and Y are real algebraic curves, then the topological pairs $(X(\mathbb{C}), X(\mathbb{R}))$ and $(Y(\mathbb{C}), Y(\mathbb{R}))$ are homeomorphic if and only if $g(X) = g(Y)$, $c(X) = c(Y)$, and X and Y are either both dividing or both nondividing.

Yet another way to formulate that the real algebraic curves X and Y have the same topological type is to say that the topological surfaces $X(\mathbb{C})$ and $Y(\mathbb{C})$ are Σ-equivariantly homeomorphic.

It is also well-known that the integers $g = g(X)$ and $c = c(X)$, for a real algebraic curve X, satisfy the following relations. If X is dividing then $c \equiv g + 1 \mod 2$ and $1 \leq c \leq g + 1$. If X is nondividing then $0 \leq c \leq g$.

In fact these are the only relations satisfied by c and g. More precisely, given a nonnegative integer g and an integer c satisfying $c \equiv g+1 \mod 2$ and $1 \le c \le g+1$, then there is a dividing real algebraic curve X such that $g(X) = g$ and $c(X) = c$. Similarly, given a nonnegative integer g and an integer c satisfying $0 \le c \le g$, then there is a nondividing real algebraic curve X such that $g(X) = g$ and $c(X) = c$.

Recall also that a real algebraic curve X is called *hyperelliptic* if the genus of X is greater than or equal to 2 and X admits a morphism onto $\mathbb{P}^1$ of degree 2.

It is known [5], Proposition 6.3, that if X is a hyperelliptic real algebraic curve then either X is nondividing, or X is dividing and c is equal to 1, 2 or $g+1$. Thus, one suspects a relation between the topological type of a real algebraic curve X and the least integer i such that there is a morphism of degree 2 from X onto a real algebraic curve Y of genus i.

Let us put

$$i(X) = \inf\{g(Y) \mid \exists f\colon X \to Y \text{ with } \deg(f) = 2\}.$$

Of course, if there are no morphisms of degree 2 from X to any real algebraic curve, then $i(X) = \infty$. For a real algebraic curve X of genus less than 2, the index $i(X)$ is equal to 0. For a curve X of genus greater than or equal to 2, the index $i(X)$ measures how nonhyperelliptic X is, i.e., $i(X) = 0$ if and only if X is hyperelliptic.

Proposition 4.1. *Let g be a nonnegative integer.*

1. *Let X be a dividing real algebraic curve of genus g. Let $c = c(X)$ be the number of connected components of $X(\mathbb{R})$. Then,*

$$i(X) \ge \min\{\tfrac{1}{2}(g+1-c), [\tfrac{1}{2}(c+1)] - 1\}.$$

2. *Let c be an integer satisfying $1 \le c \le g+1$ and $c \equiv g+1 \mod 2$. Then, there is a dividing real algebraic curve X of genus g such that $c(X) = c$ and*

$$i(X) = \min\{\tfrac{1}{2}(g+1-c), [\tfrac{1}{2}(c+1)] - 1\}.$$

Note that this proposition generalizes [5], Proposition 6.3. Indeed, if X is hyperelliptic then, by definition, $i(X) = 0$. The real curve X is either nondividing or dividing. In the latter case, according to Proposition 4.1.1, one has either $\frac{1}{2}(g+1-c) = 0$ or $[\frac{1}{2}(c+1)] - 1 = 0$, i.e., $c = 1, 2$ or $g+1$.

For the proof of Proposition 4.1 and also for the applications we have in mind, it will be convenient to change slightly our point of view.

To give a morphism of degree 2 from X onto a curve Y is equivalent to to give an automorphism α of X of order 2. Then, by Riemann-Hurwitz, $\#X(\mathbb{C})^\alpha = 2g+2-4g(Y)$. Putting

$$\lambda(X) = \sup\{\#X(\mathbb{C})^\alpha \mid \alpha \in \mathrm{Aut}(X), \mathrm{ord}(\alpha) = 2\},$$

one has $\lambda(X) = 2g+2-4i(X)$. Therefore, Proposition 4.1 is equivalent to the following one.

Proposition 4.2. *Let g be a nonnegative integer.*

1. *Let X be a dividing real algebraic curve of genus g. Let $c = c(X)$ be the number of connected components of $X(\mathbb{R})$. Then,*

$$\lambda(X) \le \max\{2c, 2g+6-4[\tfrac{1}{2}(c+1)]\}.$$

2. *Let c be an integer satisfying $1 \le c \le g+1$ and $c \equiv g+1 \mod 2$. Then, there is a dividing real algebraic curve X of genus g such that $c(X) = c$ and*

$$\lambda(X) = \max\{2c, 2g+6-4[\tfrac{1}{2}(c+1)]\}.$$

Proof. 1. Let α be an automorphism of X of order 2. Put $Y = X/\alpha$. Let h be the genus of Y. We show that either $h \ge \frac{1}{2}(g+1-c)$ or $h \ge [\frac{1}{2}(c+1)]-1$, so that, in fact, we are showing Proposition 4.1.1.

Let $F \subseteq X(\mathbb{C})$ be the closure of a connected component of $X(\mathbb{C})\backslash X(\mathbb{R})$. Then, F is an orientable manifold with boundary and its double is homeomorphic to $X(\mathbb{C})$. Let g' be its genus. Then, $2g'+c-1 = g$. Since α, considered as an automorphism of $X(\mathbb{C})$, commutes with the action of Σ, we have either $\alpha(F) \cap F = \partial F$ or $\alpha(F) \cap F = F$.

If $\alpha(F) \cap F = F$ then all fixed points of α are in the interior $F^0 = F - \partial F$ of F. The induced map on the real points $X(\mathbb{R}) \to Y(\mathbb{R})$ is then a topological covering and each fiber consists of exactly 2 points. The number of connected components of $Y(\mathbb{R})$ is then necessarily greater than or equal to $[\frac{1}{2}(c+1)]$. The genus h of Y then satisfies $h \ge [\frac{1}{2}(c+1)] - 1$.

If $\alpha(F) \cap F = \partial F$, then α has all its fixed points on ∂F. That is, the map $F^0 \to Y(\mathbb{C})$ is a diffeomorphism onto an open subset of $Y(\mathbb{C})$. The complement of this open subset of $Y(\mathbb{C})$ is the image of ∂F. Therefore, the genus h of Y then satisfies $h \ge g' = \frac{1}{2}(g+1-c)$.

2. Let S be an orientable connected compact C^∞-surface of genus g. Let Σ act on S such that complex conjugation $\sigma \in \Sigma$ acts orientation-reversingly. Moreover, this action is such that $S \backslash S^\Sigma$ is not connected, and the number

of connected components of S^Σ is equal to c. We are going to construct two Σ-equivariant orientation-preserving automorphisms of S of order 2, say α and β, such that the number of fixed points of α is equal to $2c$, and the number of fixed points of β is equal to $2g+6-4[\frac{1}{2}(c+1)]$.

Let us first show how this proves Proposition 4.2.2. Let S/α be the quotient of S in the category of C^∞-manifolds. It is easily seen, after locally linearizing the action of α, that such a quotient exists. Then, S/α is an orientable connected compact C^∞-surface. Since α is Σ-equivariant, we have an action of Σ on S/α such that the quotient map $\pi\colon S \to S/\alpha$ is Σ-equivariant. Clearly, σ acts orientation-reversingly on S/α.

There is a complex structure on S/α such that σ acts antiholomorphically on S/α. This can be easily seen as follows. Let μ' be any Riemannian metric on S/α. Then, $\mu = \mu' + \sigma^\star\mu'$ is a Σ-equivariant Riemannian metric on S/α. The action of σ on S is then antiholomorphic with respect to any of the two complex structures on S/α that are compatible with μ.

By local considerations, there is a complex structure on S such that the map $\pi\colon S \to S/\alpha$ is holomorphic. Then, the diffeomorphism α of S is biholomorphic and the action of σ on S is antiholomorphic. Indeed, let J be the endomorphism of the real tangent bundle TS of S corresponding to multiplication by $\sqrt{-1}$. Let J' be the one for S/α. Then,

$$T\pi \circ T\alpha \circ J = T\pi \circ J = J' \circ T\pi = J' \circ T\pi \circ T\alpha = T\pi \circ J \circ T\alpha.$$

Hence, $T\alpha \circ J = J \circ T\alpha$, i.e., the diffeomorphism α of S is biholomorphic. Similarly, $T\sigma \circ J = -J \circ T\sigma$ on TS. Hence, the action of σ is antiholomorphic.

Let X' be the complex algebraic curve such that the Riemann surface $X'(\mathbb{C})$ is equal to S. The action of Σ on $X'(\mathbb{C})$ is then algebraic, i.e., this action is induced by an action of Σ on X'. Let X be the quotient X'/Σ. Then, X is a real algebraic curve satisfying $X \otimes_{\mathbb{R}} \mathbb{C} = X'$. In particular, $X(\mathbb{C}) = S$.

There is an automorphism γ' of X' such that the biholomorphic automorphism of $X'(\mathbb{C})$ it induces is equal to α. Since α is Σ-equivariant, γ' is Σ-equivariant too. It follows that there is an automorphism γ of X such that the automorphism of X' it induces is equal to γ'. In particular, α is equal to the biholomorphic automorphism of $X(\mathbb{C})$ induced by γ. Since γ is an automorphism of X of order 2 and $\#X(\mathbb{C})^\gamma = 2c$, one has $\lambda(X) \geq 2c$. In particular, there is a dividing real algebraic curve X of genus g with $c(X) = c$ and $\lambda(X) \geq 2c$.

One similarly shows that the automorphism β of S gives rise to a real algebraic curve X equipped with an automorphism γ of order 2 such that $\#X(\mathbb{C})^\gamma = 2g+6-4[\frac{1}{2}(c+1)]$. In particular, there is a dividing real algebraic curve X of genus g with $c(X) = c$ and $\lambda(X) \geq 2g+6-4[\frac{1}{2}(c+1)]$.

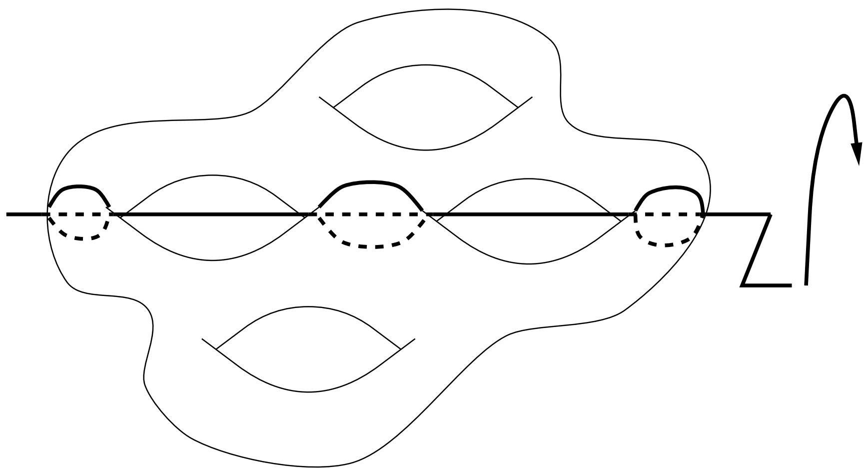


Figure 2: The arrow designates the action of the automorphism α. The topological circles form the set of fixed points S^Σ.

It then follows that there is a dividing real algebraic curve X of genus g with $c(X) = c$ such that

$$\lambda(X) \geq \max\{2c, 2g + 6 - 4[\tfrac{1}{2}(c+1)]\}.$$

Hence, by statement 1 of the proposition, this inequality is, in fact, an equality. Therefore, in order to prove Proposition 4.2.2, it suffices, indeed, to construct the automorphisms α and β having the required properties.

We construct the automorphism α as follows. Choose on every connected component of S^Σ two different points. Then, there is an Σ-equivariant orientation-preserving automorphism α of S of order 2 which has precisely the chosen points as fixed points (see Figure 2). Hence, the number of fixed points of α is equal to $2c$.

Next, we construct the automorphism β. Let F be the closure of a connected component of $S \backslash S^\Sigma$. Then, F is an oriented connected compact C^∞ surface with boundary of genus $g' = \frac{1}{2}(g+1-c)$. The boundary of F consists of c connected components.

If c is odd (resp. even) we can choose $2g'+1$ (resp. $2g'+2$) points on F such that there is an orientation-preserving automorphism β' of F of order 2 having precisely these points as its fixed points. Then, define $\beta\colon S \to S$ by

$$\beta(x) = \begin{cases} \beta'(x), & \text{if } x \in F, \\ \sigma \cdot \beta'(\sigma \cdot x), & \text{if } x \in \sigma \cdot F. \end{cases}$$

(See Figure 3 for the case c is odd, the case c is even is similar.) It is clear

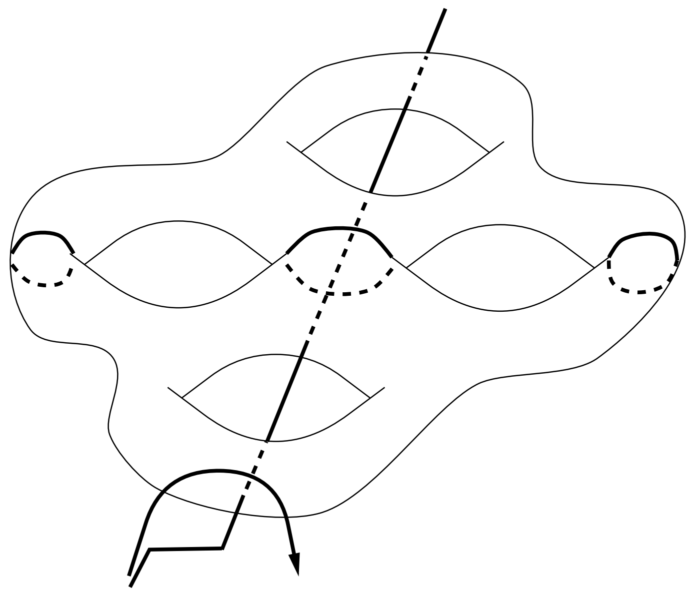


Figure 3: The arrow designates the action of the automorphism β. The topological circles form the set of fixed points S^Σ.

that β' can be chosen in such a way that β is of class C^∞. Then, β is a Σ-equivariant orientation-preserving automorphism of S of order 2 such that the number of its fixed points is equal to $2(2g'+1)$ (resp. $2(2g'+2)$). Since $2(2g'+1) = 2g+4-2c = 2g+6-4[\frac{1}{2}(c+1)]$ if c is odd, and $2(2g'+2) = 2g+6-2c = 2g+6-4[\frac{1}{2}(c+1)]$ if c is even, the automorphism β has the required number of fixed points. □

The situation for nondividing real algebraic curves is more simple.

Proposition 4.3. *Let g be a nonnegative integer.*

1. *Let X be a nondividing real algebraic curve of genus g. Then, $i(X) \geq 0$ and $\lambda(X) \leq 2g+2$.*

2. *Let c be an integer satisfying $0 \leq c \leq g$. Then, there is a nondividing real algebraic curve X of genus g such that $c(X) = c$, $i(X) = 0$ and $\lambda(X) = 2g+2$.*

Proof. 1. This follows immediately from the definition of $i(X)$ and $\lambda(X)$.

2. Let $P \in \mathbb{R}[T]$ be a separable polynomial of degree $2g+2$ having exactly $2c$ real roots and having a negative dominant coefficient. Let X be the real algebraic curve defined by the affine equation $S^2 = P(T)$. Then, there is

an obvious morphism $f\colon X \to \mathbb{P}^1$ of degree 2. Since the zeros of P are precisely the points over which f is ramified, the curve X is of genus g, by Riemann-Hurwitz. Hence, $i(X) = 0$ and $\lambda(X) = 2g + 2$.

Let $P_1 < P_2 < \cdots < P_{2c}$ be the real roots of P. Since P has a negative dominant coefficient, the image by f of the set of real points of X is a union of intervals:

$$f(X(\mathbb{R})) = \bigcup_{i=1}^{c} [P_{2i-1}, P_{2i}].$$

Since the cardinality of any fiber of $f_{|X(\mathbb{R}))}$ is at most 2, and since the connected components of $X(\mathbb{R})$ are topological circles, the number of connected components of $X(\mathbb{R})$ is equal to c.

Finally, we show that X is nondividing. Suppose, to the contrary, that X is dividing. Let C_1 and C_2 be the connected components of $X(\mathbb{C})\backslash X(\mathbb{R})$. The restriction of f to $C_1 \cup C_2$ is a closed and open map onto its image. In fact, this image is

$$f(C_1 \cup C_2) = \mathbb{P}^1(\mathbb{C})\backslash \bigcup_{i=1}^{c} [P_{2i-1}, P_{2i}].$$

In particular, $f(C_1 \cup C_2)$ is connected. Therefore, $f(C_1) = f(C_1 \cup C_2) = f(C_2)$. Hence, all fibers of f over $f(C_1 \cup C_2)$ are of cardinality 2. But, since $2c < 2g + 2$, the polynomial P has a nonreal root whose fiber is necessarily a singleton. Contradiction. □

5 Moduli of real algebraic curves

We need to recall some facts on real Teichmüller theory. They can be easily obtained from usual, i.e., complex Teichmüller theory (see [8] for complex Teichmüller theory, and [4, 6, 11] for real Teichmüller theory).

Let $g \geq 2$ and let X be a real algebraic curve of genus g. A *marked real algebraic curve modeled on* X is a pair (Y, f), where Y is a real algebraic curve and $f\colon X(\mathbb{C}) \to Y(\mathbb{C})$ is a Σ-equivariant orientation-preserving quasiconformal homeomorphism. Two such pairs (Y, f) and (Z, g) are *real Teichmüller equivalent* when there is an isomorphism $\varphi\colon Y \to Z$ such that $g^{-1} \circ \varphi \circ f\colon X(\mathbb{C}) \to X(\mathbb{C})$ is homotopic to the identity. The *real Teichmüller space* $T(X)$ of X is the set of real Teichmüller equivalence classes of marked real algebraic curves modeled on X.

The set $T(X)$ has a natural structure of a connected real analytic manifold of dimension $3g-3$. The tangent space to $T(X)$ at a point (Y, f) is naturally isomorphic to the real vector space $H^1(Y, \Omega^\curlyvee)$, where $\Omega^\curlyvee$ is the dual of the sheaf Ω of differentials on Y. Recall that by Serre duality, $H^1(Y, \Omega^{\vee})$ is canonically isomorphic to the dual of $H^0(Y, \Omega^{\otimes 2})$.

Let $\mathrm{Mod}(X)$ be the *real modular group* of X, i.e., $\mathrm{Mod}(X)$ is the group of Σ-equivariant orientation-preserving quasiconformal self-homeomorphisms of $X(\mathbb{C})$, modulo the subgroup of those self-homeomorphisms that are homotopic to the identity. The group $\mathrm{Mod}(X)$ acts on $T(X)$ by letting $(Y, f) \cdot \alpha = (Y, f \circ \alpha)$, for $(Y, f) \in T(X)$ and $\alpha \in \mathrm{Mod}(X)$. This action is properly discontinuous. It is faithful if $g > 2$. It is not faithful if $g = 2$. In fact, if $g = 2$ then X is hyperelliptic. Let ι be its hyperelliptic involution. Then, ι acts trivially on $T(X)$ and the induced action of $\mathrm{Mod}(X)/ <\iota>$ is faithful.

The quotient $R(X) = T(X)/\mathrm{Mod}(X)$ is the moduli space of the real algebraic curve X, i.e., $R(X)$ is the set of isomorphism classes of all real algebraic curves Y such that $Y(\mathbb{C})$ is Σ-equivariantly homeomorphic to $X(\mathbb{C})$. Or, to put it differently, $R(X)$ is the set of isomorphism classes of all real algebraic curves Y having the same topological type as X.

Since $T(X)$ is a connected real analytic manifold of dimension $3g - 3$, and since $\mathrm{Mod}(X)$ (resp. $\mathrm{Mod}(X)/ <\iota>$) acts properly discontinuously and faithfully if $g > 2$ (resp. $g = 2$), the moduli space $R(X)$ is a semianalytic variety of dimension $3g - 3$, by Proposition 3.1.

The stabilizer of a point (Y, f) in $T(X)$ for the action of $\mathrm{Mod}(X)$ is canonically isomorphic to the group of automorphisms $\mathrm{Aut}(Y)$ of Y. Therefore, we have the following consequence of Proposition 3.2.

Theorem 5.1. *Let $g \geq 2$ and let X be a real algebraic curve of genus g. Let Y be in the moduli space $R(X)$ of real algebraic curves of genus g having the same topological type as X.*

1. *If $g = 2$ then Y is in the boundary of $R(X)$ if and only if*

$$\#\mathrm{Aut}(Y) \equiv 0 \mod 4.$$

2. *If $g > 2$ then Y is in the boundary of $R(X)$ if and only if*

$$\#\mathrm{Aut}(Y) \equiv 0 \mod 2.$$

It follows already from Theorem 5.1 that the whole moduli space $M_{g/\mathbb{R}}$ is not a real analytic variety since there are, for any $g \geq 2$, real algebraic curves Y of genus g satisfying the conditions of the theorem (e.g., hyperelliptic curves if $g > 2$, and bielliptic curves if $g = 2$; see below). However, we will push our study of the semianalytic structure of $M_{g/\mathbb{R}}$ a little further by determining the dimension of the boundary of any connected component of

$M_{g/\mathbb{R}}$. It then will easily follow that not only $M_{g/\mathbb{R}}$ is not real analytic, but even every connected component is not real analytic.

In order to determine the dimension of the boundary of $R(X)$, we need to recall a result on the representation of a group of automorphisms of an algebraic curve on quadratic differentials. Let Y be a real algebraic curve of genus g. Let α be an automorphism of Y of order 2. Then, α acts on the real vector space $H^0(Y,\Omega^{\otimes 2})$ of quadratic differentials. The decomposition of the representation of α on $H^0(Y,\Omega^{\otimes 2})$ into irreducible representations is known [3]. In particular, one can determine the number of irreducible trivial representations, i.e., the dimension of $H^0(Y,\Omega^{\otimes 2})^\alpha$. According to [3],

$$\dim H^0(Y,\Omega^{\otimes 2})^\alpha = \tfrac{1}{2}(3g-3) + \tfrac{1}{4}\ell,$$

where ℓ is the number of complex ramification points of the quotient map $Y \to Y/\alpha$, i.e., $\ell = \#Y(\mathbb{C})^\alpha$.

Theorem 5.2. *Let $g > 2$ and let X be a real algebraic curve of genus g. Let $c = c(X)$ be the number of connected components of $X(\mathbb{R})$. Then,*

$$\dim \partial R(X) = \begin{cases} \max\{\frac{1}{2}(3g-3+c), 2g - [\frac{1}{2}(c+1)]\}, & \textit{if } X \textit{ is dividing,} \\ 2g-1, & \textit{if } X \textit{ is nondividing.} \end{cases}$$

In particular, the boundary $\partial R(X)$ of the moduli space $R(X)$ is nonempty.

Proof. Let (Y,f) be an element of $T(X)$. According to Proposition 3.2 and the preceding observations, the local dimension of the boundary $\partial R(X)$ of $R(X)$ at Y is equal to

$$\begin{aligned} \dim_Y \partial R(X) &= \sup_{\substack{\alpha \in \mathrm{Mod}(X)_{(Y,f)} \\ \mathrm{ord}(\alpha)=2}} \dim(T_{(Y,f)}T(X))^\alpha = \\ &= \sup_{\substack{\alpha \in \mathrm{Aut}(Y) \\ \mathrm{ord}(\alpha)=2}} \dim H^0(Y,\Omega^{\otimes 2})^\alpha = \\ &= \sup_{\substack{\alpha \in \mathrm{Aut}(Y) \\ \mathrm{ord}(\alpha)=2}} \tfrac{1}{2}(3g-3) + \tfrac{1}{4}\#Y(\mathbb{C})^\alpha. \end{aligned}$$

By Propositions 4.2.2 and 4.3.2 there is a real algebraic curve Y in $R(X)$ admitting an automorphism α of order 2 such that

$$\#Y(\mathbb{C})^\alpha = \begin{cases} \max\{2c, 2g+6-4[\frac{1}{2}(c+1)]\}, & \text{if } X \text{ is dividing,} \\ 2g+2, & \text{if } X \text{ is nondividing.} \end{cases}$$

Then, it follows from Propositions 4.2.1 and 4.3.1 that

$$\dim \partial R(X) = \dim_Y \partial R(X) = \tfrac{1}{2}(3g-3) + \tfrac{1}{4}(2g+2) = 2g-1$$

if X is nondividing, and

$$\begin{aligned}\dim \partial R(X) = \dim_Y \partial R(X) = \\ &= \tfrac{1}{2}(3g-3) + \tfrac{1}{4}\max\{2c, 2g+6-4[\tfrac{1}{2}(c+1)]\} = \\ &= \max\{\tfrac{1}{2}(3g-3+c), 2g-[\tfrac{1}{2}(c+1)]\}\end{aligned}$$

if X is dividing. □

Next, we turn our attention to the moduli space of real algebraic curves of genus 2. Recall that a real algebraic curve Y is *bielliptic* if Y is of genus greater than or equal to 2, and there is a morphism of degree 2 from Y onto a real algebraic curve of genus 1.

Lemma 5.3. *Let X be a real algebraic curve of genus 2. Then, there is a bielliptic real algebraic curve Y having the same topological type as X.*

Proof. We construct a bielliptic real algebraic curve Y as follows. Let $P \in \mathbb{R}[T]$ be a separable polynomial of degree 6 such that $P(-T) = P(T)$. Then, the affine equation $S^2 = P(T)$ defines a real algebraic curve $Y = Y_P$.

It is easy to see that Y is a bielliptic real algebraic curve of genus 2. Indeed, applying Riemann-Hurwitz to the obvious degree 2 map $f\colon Y \to \mathbb{P}^1$, one concludes that the curve Y is of genus 2. Let $Q \in R[T]$ be such that $Q(T^2) = P(T)$. Then, the mapping $(S,T) \mapsto (S,T^2)$ defines a morphism from the curve Y onto the real algebraic curve defined by the affine equation $S^2 = Q(T)$. The latter curve is of genus 1 since Q is a separable polynomial of degree 3. The morphism is obviously of degree 2. This shows that Y is a bielliptic real algebraic curve of genus 2.

Now, one chooses the polynomial P as above such that $Y = Y_P$ has the same topological type as X. We need to distinguish the 5 different topological types that X can have. Let again $c = c(X)$ be the number of connected components of $X(\mathbb{R})$.

If X is dividing and $c = 3$, then one chooses P to have only real roots. Since P has 6 real roots, $Y(\mathbb{R})$ has 3 connected components. Since Y is of genus 2, the curve Y is necessarily dividing.

If X is dividing and $c = 1$, then one chooses P to have no real roots and to have a positive dominant coefficient. Then, the image by $f\colon Y \to \mathbb{P}^1$ of $Y(\mathbb{C})\backslash Y(\mathbb{R})$ is equal to $\mathbb{P}^1(\mathbb{C})\backslash\mathbb{P}^1(\mathbb{R})$. The latter topological space being nonconnected, $Y(\mathbb{C})\backslash Y(\mathbb{R})$ is not connected. Hence Y is dividing. Clearly, $c(Y)$ is either 1 or 2. Since $c(Y) \equiv g(Y)+1 \mod 2$, one has $c(Y) = 1$.

If X is nondividing then one chooses P with a negative dominant coefficient and to have exactly $2c$ real roots. Arguing as in the proof of Proposition 4.3.2, Y is nondividing and $c(Y) = c$.

In all cases, the bielliptic real algebraic curve Y that is constructed has the same topological type as X. $\square$

Theorem 5.4. *Let X be a real algebraic curve of genus 2. The boundary $\partial R(X)$ of the moduli space $R(X)$ of X is of dimension 2.*

Proof. Let X be a real algebraic curve of genus 2. Then, by Lemma 5.3, there is a bielliptic real algebraic curve Y belonging to $R(X)$. Let $\alpha \in \mathrm{Aut}(Y)$ be the bielliptic involution. Then, by Riemann-Hurwitz, $\#Y(\mathbb{C})^\alpha = 2$. As in the proof of Theorem 5.2, the local dimension of the boundary $\partial R(X)$ of $R(X)$ at Y is at least equal to $\frac{1}{2}(3 \cdot 2 - 3) + \frac{1}{4} \cdot 2 = 2$. Since the moduli space $R(X)$ is itself of dimension 3, we necessarily have $\dim_Y \partial R(X) = 2$, i.e., $\dim \partial R(X) = 2$. $\square$

Remark 5.5. Let X be a real algebraic curve of genus g, where g is an integer greater than or equal to 2. It follows from Theorems 5.2 and 5.4, that the boundary of the moduli space $R(X)$ of X is of codimension 1 if g is equal to 2 or 3. If $g > 3$ then the boundary of $R(X)$ is of codimension strictly greater than 1.

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Institut Mathématique de Rennes, Université de Rennes 1,
Campus de Beaulieu, 35042 Rennes Cedex, France.
E-mail: huisman@univ-rennes1.fr

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