Pencils on real curves

Marc Coppens, Johannes Huisman

Abstract

We consider coverings of real algebraic curves to real rational algebraic curves. We show the existence of such coverings having prescribed topological degree on the real locus. From those existence results we prove some results on Brill-Noether Theory for pencils on real curves. For coverings having topological degree $\underline{0}$ we introduce the covering number k and we prove the existence of coverings of degree 4 with prescribed covering number.

2000 Mathematics Subject Classification. 14H51; 14P99

Keywords. real curve, linear pencil, topological degree, morphism, Brill-Noether Theory, divisor

Introduction

In this paper we show the existence of coverings $\pi : X \to \mathbb{P}^1$ of $\mathbb{P}^1$ by real curves satisfying some specific properties related to the real locus $X(\mathbb{R})$. Part of those constructions is related to recent results on real pencils (one-dimensional linear systems) on real curves.

We write $X(\mathbb{C})$ to denote the associated Riemann surface, $g(X)$ to denote its genus and $a(X) \in \{0, 1\}$ such that $2 - a(X)$ is the number of connected components of $X(\mathbb{C}) \setminus X(\mathbb{R})$. Consider a covering $\pi : X \to \mathbb{P}^1$ and choose an orientation on $\mathbb{P}^1(\mathbb{R})$. Let C be a connected component of $X(\mathbb{R})$ and consider the restriction $\pi|_C : C \to \mathbb{P}^1(\mathbb{R})$. Fixing an orientation on C this restriction has a topological degree $\delta_C(\pi)$ and we assume the orientations are chosen such that $\delta_C(\pi) \geq 0$. We consider the existence of coverings having prescribed values for those topological degrees. There are trivial natural restrictions between $d = \deg(\pi)$, $g = g(X)$, $a = a(X)$ and those topological degrees and we prove that there exists such a covering π for all values d, g, a and δ_C satisfying those restrictions.

In particular, for each $s \geq 2$, $g \equiv s - 1 \pmod 2$ with $g \geq s - 1$ there exists a real curve X of genus g such that $a(X) = 0$ and $X(\mathbb{R})$ has exactly s connected components and a morphism $\pi : X \to \mathbb{P}^1$ of degree s such that $\delta_C(\pi) = 1$ for each component C of $X(\mathbb{R})$. In particular $\pi(\mathbb{R}) : X(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ is the union of s homeomorphisms. In [21] one introduces very special linear systems on real curves satisfying strong Clifford-type properties for real curves and in

Proposition 2.1 of [21] it is proved that the only very special pencils are exactly those defined by such coverings. So we obtain a proof for the existence of very special pencils of all possible types.

In [9] one studies some Brill-Noether properties for pencils on real curves. We give another proof of part of the results in [9] making intensive use of results coming from the theory of complex curves. In this way it is enough to prove the existence of one real curve having a pencil of suited degree. Because of our existence results we obtain Brill-Noether properties for pencils having prescribed topological degree on the connected components of the real locus. While the arguments in [9] are restricted to real curves having real points, our arguments also give Brill-Noether properties for pencils on real curves without real points.

The existence and Brill-Noether results are also considered in [5], [6] and [7] without considering the topological degrees. In order to make the arguments more careful with respect to dimension arguments on the real locus of a variety defined over $\mathbb{R}$ we use universal spaces representing morphisms that are known to be globally smooth because of Horikawa deformation theory.

A real divisor D on a real curve is called totally non-real in case the support of D contains no real point of X. It is known that for a real linear system on a real curve the parity of the restriction of a divisor to a connected component of $X(\mathbb{R})$ is constant. Hence if this parity is odd for some component then the linear system does not contain a totally non-real divisor. Restricting to linear systems such that the parity is even for all components of $X(\mathbb{R})$, there is a sharp result in [11] concerning the existence of totally non-real divisors in such a linear system. For linear systems of large dimension, it is not easy to find non-trivial examples of such linear systems having no totally non-real divisors. In the case of pencils our constructions give a lot of such examples coming from coverings having only even topological degrees and at least one of them being non-zero. One can ask for the existence of such examples coming from coverings with all topological degrees equal to zero. We prove the existence of all types of such coverings of the smallest possible degree 4.

In Section 1 we give the constructions that are the base for the existence of coverings with prescribed topological degrees. In Section 2 we prove the existence of coverings with prescribed topological degrees. In Section 3 we discuss Brill-Noether problems for pencils on real curves. Finally in Section 4 we prove the existence of coverings of degee 4 with all topological degrees equal to 0 and having no non-real divisor.

Terminology.

A smooth real curve X of genus $g = g(X)$ is a scheme defined over $\mathbb{R}$ such that the base change $X \times_{\mathbb{R}} \mathbb{C}$, denoted by $X_{\mathbb{C}}$, is a complete connected smooth complex curve of genus g. We also write $X(\mathbb{C})$ to denote the set of closed points of $X_{\mathbb{C}}$ and we consider it as a Riemann surface of genus g. As usual the non-trivial element of $Gal(\mathbb{C}/\mathbb{R})$ is denoted by $z \to \overline{z}$ and it is called (complex) conjugation. For a real curve X it induces an $\mathbb{R}$-involution $\sigma : X_{\mathbb{C}} \to X_{\mathbb{C}}$. For $P \in X(\mathbb{C})$ we write $\overline{P}$ instead of $\sigma(P)$ and we call it the *conjugated point* of P. In case $P = \overline{P}$ then P is called a *real point* of X. The set of real points is called the real locus $X(\mathbb{R}) \subset X(\mathbb{C})$. This locus is a union of $s(X)$ connected components, each one of them homeomorphic to a circle. The invariant $a(X)$ is defined by $a(X) = 1$ in case $X(\mathbb{C}) \setminus X(\mathbb{R})$ is connected and otherwise $a(X) = 0$ (in that case $X(\mathbb{C}) \setminus X(\mathbb{R})$ has two connected components). We say a smooth real curve X has *topological type* (g, s, a) if $g(X) = g$, $s(X) = s$ and $a(X) = a$. By a theorem of Weichold such real curve exists if and only if either $a = 1$ and $0 \leq s \leq g$ else $a = 0$, $s \equiv g + 1 \pmod 2$ and $1 \leq s \leq g + 1$ (see [25], see also [19, Theorem 5.3]). A triple (g, s, a) satisfying those restrictions is called an *admissable topological type* for real curves. In case $P \neq \overline{P}$ then we call $P + \overline{P}$ a *non-real point* of X.

For a divisor $D = \sum_{i=1}^{n} m_i P_i$ on $X_{\mathbb{C}}$ we write $\overline{D} = \sum_{i=1}^{n} m_i \overline{P_i}$ and we call it the conjugated divisor of D. As usual $\sum_{i=1}^{n} m_i$ is called the degree of D, denoted by $\deg(D)$. We say D is a real divisor if $\overline{D} = D$. If we say D is a divisor on X then we mean it is a *real divisor* on $X_{\mathbb{C}}$.

Let L be an invertible sheaf on $X_{\mathbb{C}}$. It can be described by means of trivializations on an open covering $(U_i)_{i \in I}$ of $X(\mathbb{C})$ and transition functions $a_{i,j}$ (those are regular functions $U_i \cap U_j \to \mathbb{C}^*$). The conjugated invertible sheaf $\overline{L}$ is defined by means of trivializations on the open covering $(\sigma(U_i))_{i \in I}$ and transition functions $\overline{a_{i,j}}$ (for $P \in \sigma(U_i \cap U_j)$ one has $\overline{a_{i,j}}(P) = \overline{a_{i,j}(\overline{P})}$). Each invertible sheaf L on $X_{\mathbb{C}}$ is isomorphic to $\mathcal{O}_{X_{\mathbb{C}}}(D)$ for some divisor D on $X_{\mathbb{C}}$. In case $k = \deg(D)$ then we say L is an invertible sheaf of degree k and we write $\deg(L) = k$. In case $L \cong \mathcal{O}_X(D)$ then $\overline{L} \cong \mathcal{O}_X(\overline{D})$.

We say L is a *real invertible sheaf* if $L \cong \mathcal{O}_X(D)$ for some real divisor D. We say L is *invariant under conjugation* if $L \cong \overline{L}$. In particular a real invertible sheaf is invariant under conjugation. Let $\mathrm{Pic}(X)$ be the Picard scheme of X defined over $\mathbb{R}$ (see [15]). Then $\mathrm{Pic}(X)_{\mathbb{C}}$ represents the Picard functor on $X_{\mathbb{C}}$, in particular $\mathrm{Pic}(X)(\mathbb{C})$ parameterizes invertible sheaves on $X_{\mathbb{C}}$. For $k \in \mathbb{Z}$ one has natural subschemes $\mathrm{Pic}^k(X)$ such that $\mathrm{Pic}^k(X)(\mathbb{C})$ parameterizes invertible sheaves of degree k on $X_{\mathbb{C}}$. The real locus $\mathrm{Pic}^k(X)(\mathbb{R})$ parameterizes invertible sheaves of degree k on $X_{\mathbb{C}}$ invariant under conjugation. Let $\mathrm{Pic}^k(X)(\mathbb{R})^+$ be the sublocus parameterizing real invertible sheaves. In case $X(\mathbb{R}) \neq \emptyset$ then $\mathrm{Pic}(X)$ also represents the Picard functor on X and therefore $\mathrm{Pic}^k(X)(\mathbb{R})^+ = \mathrm{Pic}^k(X)(\mathbb{R})$ in that case. In case $X(\mathbb{R}) = \emptyset$ then $\mathrm{Pic}^k(X)(\mathbb{R})^+$ is a subgroup of $\mathrm{Pic}^k(X)(\mathbb{R})$ with quotient $\mathbb{Z}/2\mathbb{Z}$. For $L \in \mathrm{Pic}^k(X)(\mathbb{R})$ with $L \notin \mathrm{Pic}(X)(\mathbb{R})^+$ one has $\deg(L) \equiv g - 1 \pmod 2$ (see [14, Proposition 2.2]).

We write $|L|$ to denote the complete linear system of an invertible sheaf L on $X_{\mathbb{C}}$. In case L is a real invertible sheaf then $|L|(\mathbb{R})$ is the space of real divisors contained in $|L|$. One has $|L| = |L|(\mathbb{R}) \otimes \mathbb{C}$ in a natural way (meaning $|L|(\mathbb{R})$ can be considered as a projective space $\mathbb{P}^r$ and then $|L| = \mathbb{P}^r_{\mathbb{C}}$). As usual we write g^r_d to denote a linear system of dimension r and degree k on $X_{\mathbb{C}}$. In case L is a real invertible sheaf of degree k then we say a linear subsystem g^r_k of $|L|$ is defined over $\mathbb{R}$ if there exists a k-dimensional linear subsystem $g^r_k(\mathbb{R}) \subset |L|(\mathbb{R})$ such that $g^r_k = g^r_k(\mathbb{R}) \otimes \mathbb{C}$. In such case we say g^r_k is a linear system on X. In particular if L is a real invertible sheaf then we say $|L|$ is a linear system on X.

We write $\mathbb{P}^1$ to denote the projective line defined over $\mathbb{R}$ and we write R_0

to denote the smooth real curve of genus 0 without real points (defined by the equation $X^2 + Y^2 + Z^2 = 0$ in $\mathbb{P}^2$).

Let X be a smooth real curve. A covering $\pi : X \to \mathbb{P}^1$ or $\pi : X \to R_0$ is a finite morphism defined over $\mathbb{R}$. We say π has degree d in case the associated covering $\pi_{\mathbb{C}} : X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ has degree d. Such covering corresponds to a base point free pencil g^1_d on $X_{\mathbb{C}}$. In case of $\pi : X \to \mathbb{P}^1$ this is called a *real pencil* on X. In case of $\pi : X \to R_0$ this is called a *non-real invariant pencil* on X. This case only can occur if $X(\mathbb{R}) = \emptyset$ (X has no real points) and moreover, in this case the pencil has no real divisor but it is invariant under complex conjugation. We write $\pi(\mathbb{C}) : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ to denote the covering of Riemann surfaces. In case $X(\mathbb{R})$ is not empty we write $\pi(\mathbb{R}) : X(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ to denote the restriction of $\pi(\mathbb{C})$ to the real locus $X(\mathbb{R})$. For a connected component C of $X(\mathbb{R})$ we write $\pi_C : C \to \mathbb{P}^1(\mathbb{R})$ to denote the restriction of $\pi(\mathbb{R})$ to C. On $\mathbb{P}^1(\mathbb{R})$, homeomorphic to a circle S^1, we choose an orientation and then for C, also homeomorphic to S^1, we choose the orientation such that the degree of π_C, denoted by $\delta_C(\pi)$, is nonnegative. If $P \in C$ and π_C is not ramified at P, then the local degree $\delta_P(\pi)$ is equal to 1 (resp. -1) if π_C preserves (resp. reverses) the orientation locally at P. Let $C_1, \cdots, C_s$ be the connected components of $X(\mathbb{R})$ and let $\delta_i = \delta_{C_i}(\pi)$ for $1 \leq i \leq s$. We can always assume $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_s$ and then we say π has *topological degree* $(\delta_1, \delta_2, \cdots, \delta_s)$ (shortly denoted by $\underline{\delta}$).

1 Constructions

The proof of the existence of coverings with prescribed topological degrees uses an induction argument. To make that argument we start with a smooth real curve Y and a suited covering $\pi_Y : Y \to \mathbb{P}^1$. Using this covering we construct a covering $\pi_0 : X_0 \to \mathbb{P}^1$ with X_0 being a real singular nodal curve and we use a real smoothing $\pi_t : X_t \to \mathbb{P}^1$ of π_0. In this part we describe those constructions giving relations between topological degrees of π_Y and of π_t. This will be the base for the induction argument in the next section.

In those constructions we start by taking local smoothings of the nodes of $X_0(\mathbb{C})$ having a natural antiholomorphic involution. Those local smoothings glue with the complement V of a neighborhood of the nodes of $X_0(\mathbb{C})$ giving rise to a deformation of compact Riemann surfaces $X_t(\mathbb{C})$ and holomorphic coverings $\pi_t(\mathbb{C}) : X_t(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. This complement V can be taken to be invariant under complex conjugation on $X_0(\mathbb{C})$ and this fits with the antiholomorphic involution of the local smoothings under the gluing. Hence we obtain an antiholomorphic involution σ_t on $X_t(\mathbb{C})$. It is well-known that this defines a smooth real curve X_t inducing the Riemann surface $X_t(\mathbb{C})$ such that σ_t corresponds to complex conjugation (see e.g. [19, Section 4]). Moreover the morphism $\pi_t(\mathbb{C}) : X_t(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is invariant under complex conjugation hence it comes from a morphism $\pi_t : X_t \to \mathbb{P}^1$ (indeed, the graph of $\pi_t(\mathbb{C})$ is a closed subspace of $(X_t \times \mathbb{P}^1)(\mathbb{C})$ invariant under complex conjugation on $X_t \times \mathbb{P}^1$).

1.1 Construction I

Let Y be a real curve of genus g and let $\pi : Y \to \mathbb{P}^1$ be a morphism of degree k defined over $\mathbb{R}$. Assume C is a connected component of $Y(\mathbb{R})$ and let P be a point of C such that $\delta_P(\pi) = 1$ (such point P does exist because $\delta_C(\pi) \geq 0$). Take a copy of $\mathbb{P}^1$ and consider the singular curve $X_0 = Y \cup_P \mathbb{P}^1$ obtained by identifying P on X with $\pi(P)$ on $\mathbb{P}^1$. This singular curve has a natural morphism π_0 defined over $\mathbb{R}$ of degree $k+1$ to $\mathbb{P}^1$ having restriction π to Y and the identity to $\mathbb{P}^1$ (see Figure 1).

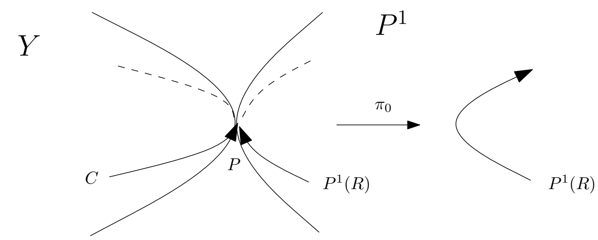

Figure 1: Construction I, the curve X_0

Locally at P this situation can be described inside $\mathbb{C}^2$ such that the curve X_0 has equation $x^2 - y^2 = 0$ and the morphism is locally defined by $(x, y) \to x$ and we can assume the coordinates are compatible with the real structure on X_0 (this means complex conjugation on X_0 corresponds to complex conjugation in $\mathbb{C}^2$). Let U be a small neighborhood of $(0,0)$ in $\mathbb{C}^2$ and $V \subset U$ a much smaller one and let $U_0 = U \cap X_0$, $V_0 = V \cap X_0$. For $Q \neq P$ in U_0 we can use x to define a holomorphic coordinate at Q. Consider a local deformation of X_0 at P defined by $x^2 - y^2 = t$ with $t \in \mathbb{R}$ ($|t|$ very small) and let U_t (resp. V_t) be the intersection with U (resp. V). We use a gluing of U_t and $X_0(\mathbb{C}) \setminus V_0$ as follows. For $z_0 \in \mathbb{C}^*$ let ${}_{z_0}\sqrt{z}$ be the locally defined holomorphic square root function such that ${}_{z_0}\sqrt{z_0^2} = z_0$. A point $Q \in U_0 \setminus V_0$ has coordinates (x, x) or $(x, -x)$. We identify (x, x) with $(x, {}_x\sqrt{x^2 - t})$ and $(x, -x)$ with $(x, -{}_x\sqrt{x^2 - t})$. (One should adapt the description of V_t and U_t to this identification.) This defines a Riemann surface $X_t(\mathbb{C})$. On U_t we define $\sigma_t(x, y) = (\overline{x}, \overline{y})$. We need to show this behaves well under the previous identification. Since ${}_x\sqrt{x^2 - t}$ is close to x one has $\overline{{}_x\sqrt{x^2 - t}}$ is close $\overline{x}$. Moreover $\left(\overline{{}_x\sqrt{x^2 - t}}\right)^2 = \overline{\left({}_x\sqrt{x^2 - t}\right)^2} = \overline{x^2 - t} = \overline{x}^2 - t$ hence $\overline{{}_x\sqrt{x^2 - t}} = {}_{\overline{x}}\sqrt{\overline{x}^2 - t}$. Hence under the identification $(\overline{x}, \overline{x})$ is identified with $(\overline{x}, \overline{{}_x\sqrt{x^2 - t}})$ and similarly $(\overline{x}, -\overline{x})$ is identified with $(\overline{x}, -\overline{{}_x\sqrt{x^2 - t}})$. As mentioned at the beginning of this section we obtain a real smooth curve X_t and a covering $\pi_t : X_t \to \mathbb{P}^1$ of degree $k+1$. Under this deformation the union $C \cup_P \mathbb{P}^1(\mathbb{R})$ deforms to a connected component C_t of $X_t(\mathbb{R})$

In case $t > 0$ the morphism π_t has ramification on C_t above $x = \pm\sqrt{t}$. We call this *the deformation with real ramification*. In order to have an orientation on C_t we have to change the orientation on the attached $\mathbb{P}^1(\mathbb{R})$. This implies that $\delta_{C_t}(\pi_t) = \delta_C(\pi) - 1$. Of course, if $\delta_C(\pi) = 0$ we also change the orientation of C_t in order to obtain $\delta_{C_t}(\pi_t) = 1$ (see Figure 2).

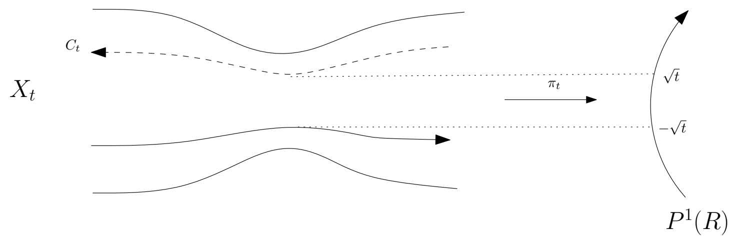

Figure 2: Construction I, deformation with real ramification

In case $t < 0$ the morphism π_t has no ramification on C_t close to P. We call this *the deformation without real ramification.* We use the orientation on C_t obtained from both the orientation on C and the attached $\mathbb{P}^1(\mathbb{R})$. This implies $\delta_{C_t}(\pi_t) = \delta_C(\pi) + 1$ (see Figure 3).

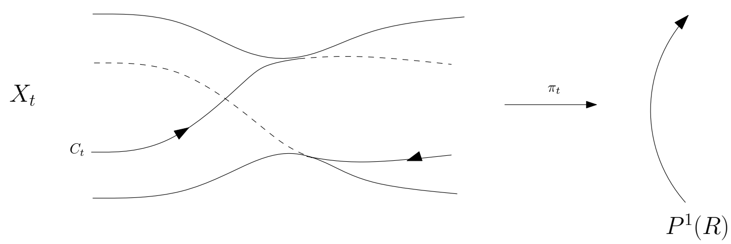

Figure 3: Construction I, deformation without real ramification

In both cases $X_t(\mathbb{C})\backslash X_t(\mathbb{R})$ is connected if and only if $X(\mathbb{C})\backslash X(\mathbb{R})$ is connected.

1.2 Construction II

Let Y be a real curve of genus g and let $\pi : Y \to \mathbb{P}^1$ be a morphism of degree k defined over $\mathbb{R}$. Let $Q \in \mathbb{P}^1(\mathbb{R})$ not a branch point of π and assume $P + \overline{P} \subset \pi^{-1}(Q)$, a non-real point on Y (in particular we assume such non-real point exists). Let X_0 be the singular curve obtained from X by identifying P with $\overline{P}$. This singular curve has a natural morphism of degree k to $\mathbb{P}^1$ defined over $\mathbb{R}$ obtained from π (see Figure 4).

On X_0 the singularity (again denoted by P) is an isolated real point. Locally at P this situation can be described inside $\mathbb{C}^2$ such that the curve has equation $x^2 + y^2 = 0$ and the morphism is locally defined by $(x, y) \to x$ with coordinates compatible with the real structure on X_0. We use U and V as in Construction I. Consider the local deformation of X_0 at P defined by $x^2 + y^2 = t$ with $t \in \mathbb{R}$ ($|t|$

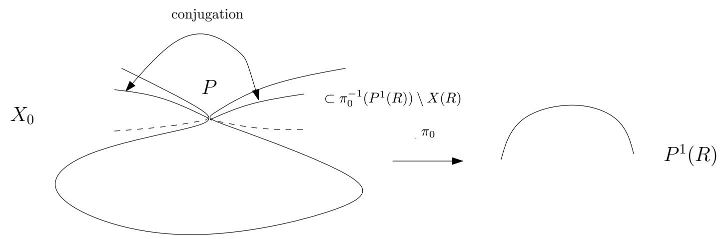

Figure 4: Construction II, the curve X_0

very small). A point $Q \in U_0 \setminus V_0$ has coordinates (x, ix) or $(x, -ix)$ and $\sigma(Q)$ corresponds to resp. $(\overline{x}, -i\overline{x})$ or $(\overline{x}, i\overline{x})$. We identify (x, ix) (resp. $(x, -ix)$)) on $U_0 \setminus V_0$ with $(x, {}_{ix}\sqrt{-x^2+t})$ (resp. $(x, -{}_{ix}\sqrt{-x^2+t})$) on U_t. This defines a Riemann surface $X_t(\mathbb{C})$. On U_t we define $\sigma_t(x, y) = (\overline{x}, \overline{y})$ and we check that it behaves well under the identification. Consider $Q = (x, {}_{ix}\sqrt{-x^2+t})$ on $U_t \setminus V_t$. Since ${}_{ix}\sqrt{-x^2+t}$ is close to ix one has $\overline{{}_{ix}\sqrt{-x^2+t}}$ is close to $-i\overline{x}$, hence $\overline{{}_{ix}\sqrt{-x^2+t}} = -{}_{i\overline{x}}\sqrt{-\overline{x}^2+t}$. Hence $\sigma_t(Q)$ is identified with $(\overline{x}, -i\overline{x}) = \sigma_0(x, ix)$. We obtain a smooth real curve X_t of genus $g+1$ together with a morphism $\pi_t : X_t \to \mathbb{P}^1$ of degree k defined over $\mathbb{R}$.

In case $t > 0$ there is a new component C_t of $X_t(\mathbb{R})$ close to P and under π_t this maps to the interval $[-\sqrt{t}, \sqrt{t}]$. In particular $\delta_{C_t}(\pi_t) = 0$. We call this *the deformation with real ramification.* In this case $X_t(\mathbb{C}) \setminus X_t(\mathbb{R})$ is connected if and only if $X(\mathbb{C}) \setminus X(\mathbb{R})$ is connected (see Figure 5).

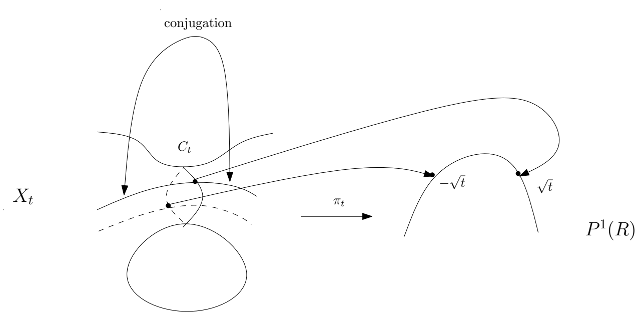

Figure 5: Construction II, deformation with real ramification

In case $t < 0$ then $X_t(\mathbb{R})$ contains no real point close to P. We call this the deformation without real ramification. In this case $X_t(\mathbb{C}) \setminus X_t(\mathbb{R})$ is always connected (see Figure 6).

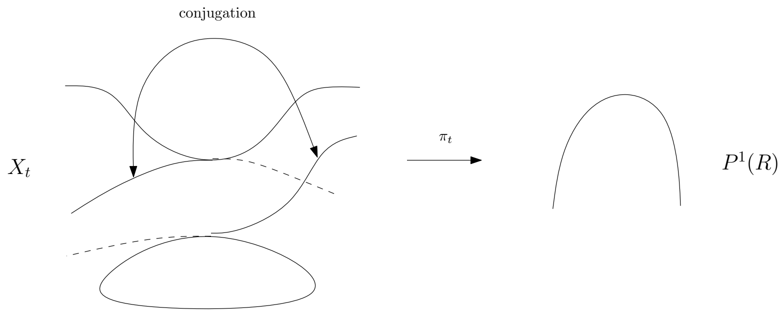

Figure 6: Construction II, deformation without real ramification

1.3 Construction III

Let Y be a real curve of genus g and let $\pi : Y \to \mathbb{P}^1$ be a morphism of degree k defined over $\mathbb{R}$. Assume $P + \overline{P}$ is a non-real point of Y such that $\pi(P) \neq \pi(\overline{P})$ and P, $\overline{P}$ are not ramification points of $\pi_{\mathbb{C}}$. Now we let $X_0 = Y \cup_{P+\overline{P}} \mathbb{P}^1$ be the union of Y and $\mathbb{P}^1$ identifying P with $\pi(P)$ (still denoted by P) and $\overline{P}$ with $\pi(\overline{P})$ (still denoted by $\overline{P}$). This singular curve has a natural morphism π_0 defined over $\mathbb{R}$ of degree $k+1$ to $\mathbb{P}^1$ having restriction π to Y and the identity to $\mathbb{P}^1$ (see Figure 7).

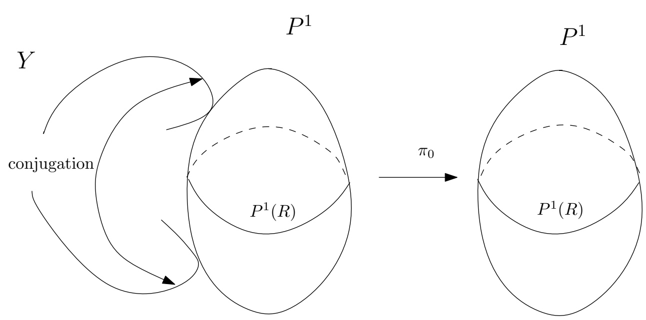

Figure 7: Construction III, the curve X_0

Choose a local deformation over $\mathbb{C}$ making it smooth at P and take the complex conjugated local deformation at $\overline{P}$ and glue it with the remaining part of X_0. We obtain a smooth real curve X_t of genus $g+1$ together with a morphism $\pi_t : X_t \to \mathbb{P}^1$ of degree $k+1$ defined over $\mathbb{R}$. One has $g(X_t) = g+1$ and $\mathbb{P}^1(\mathbb{R})$ on $X \cup_{P+\overline{P}} \mathbb{P}^1$ deforms to a connected component C_t of $X_t(\mathbb{R})$ such that $\delta_{C_t}(\pi_t) = 1$. For this construction $X_t(\mathbb{C}) \backslash X_t(\mathbb{R})$ is connected if and only if $X(\mathbb{C}) \backslash X(\mathbb{R})$ is connected (see Figure 8).

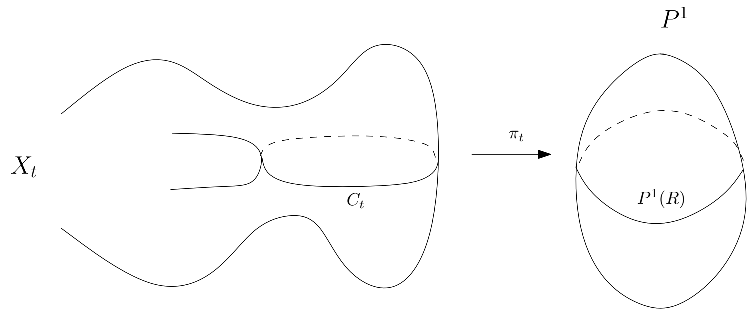

Figure 8: Construction III, deformation

1.4 Construction IV

Consider a double covering $\tau_0 : R_0 \to \mathbb{P}^1$ defined over $\mathbb{R}$. Assume Y is a real curve without real points and $\pi : Y \to \mathbb{P}^1$ is a morphism of degree k defined over $\mathbb{R}$ (in particular k is even). Choose $P + \overline{P}$ on $\mathbb{P}^1$ (a non-real point) such that P, $\overline{P}$ are not branch points of $\pi_{\mathbb{C}}$ and choose non-real points $P_0 + \overline{P_0}$ on R_0 and $Q + \overline{Q}$ on Y such that $\tau_0(\mathbb{C})(P_0) = \pi(\mathbb{C})(Q) = P$. Consider the nodal curve $X_0 = Y \cup_{P+\overline{P}} R_0$ obtained by identifying P_0 to Q (still denoted by Q) and $\overline{P_0}$ to $\overline{Q}$ (still denoted by $\overline{Q}$). This is a singular curve defined over $\mathbb{R}$ without real points and π together with τ_0 induce a morphism of that singular curve to $\mathbb{P}^1$ defined over $\mathbb{R}$. Take a local deformation at Q over $\mathbb{C}$ smoothing the singularity, take the complex conjugated deformation at $\overline{Q}$ and glue it with the remaining part of X_0 to obtain a smooth real curve X_t of genus $g + 1$ together with a morphism $\pi_t : X_t \to \mathbb{P}^1$ of degree $k + 2$. This smooth real curve does not have any real point.

1.5 Construction V

Assume Y is a real curve of genus g without real points and assume $\pi : Y \to R_0$ is a morphism of degree k defined over $\mathbb{R}$. Choose a non-real point $P_0 + \overline{P_0}$ on R_0 such that P_0, $\overline{P_0}$ are not branch points of $\pi_{\mathbb{C}}$ and a non-real point $P + \overline{P}$ on Y with $\pi(P) = P_0$. Consider the nodal curve $X_0 = Y \cup_{P+P_0} R_0$ obtained by identifying P_0 to P (still denoted by P) and $\overline{P_0}$ to $\overline{P}$ (still denoted by $\overline{P}$). This is a singular curve defined over $\mathbb{R}$ without real points and π together with the identity 1_{R_0} induces a morphism of degree $k + 1$ of that singular curve to R_0 defined over $\mathbb{R}$. Take a local deformation at P over $\mathbb{C}$ smoothing the singularity, take the complex conjugated deformation at $\overline{P}$ and glue it with the remaining part of X_0 to obtain a smooth real curve X_t of genus $g + 1$ together with a morphism $\pi_t : X_t \to R_0$ of degree $k + 1$.

2 Existence of real curves having pencils with prescribed topological properties

Let X be a real curve of topological type (g, s, a) and let $\pi : X \to \mathbb{P}^1$ be a morphism of degree k defined over $\mathbb{R}$. Assume $s \geq 1$ and let $\underline{\delta} = (\delta_1, \cdots, \delta_s)$ be the topological degree of π. Of course we need $\sum_{i=1}^{s} \delta_i \leq k$. For each $s \in \mathbb{P}^1(\mathbb{R})$ the number of real points in $\pi^{-1}(s)$ counted with multiplicity has the same parity as $\sum_{i=1}^{s} \delta_i$. Since the non-real points of X are pairs of conjugated points on $X(\mathbb{C})$ it follows $k - \sum_{i=1}^{s} \delta_i \equiv 0 \pmod 2$. In case $\sum_{i=1}^{s} \delta_i = k$ then for each $s \in \mathbb{P}^1(\mathbb{R})$ the fiber $\pi^{-1}(s)$ has exactly δ_i different points on C_i. Since some fiber needs to contain points of C_s one has $\delta_s \geq 1$ in that case, hence $\sum_{i=1}^{s} \delta_i \leq k-2$ in case $\delta_s = 0$ (see also [8]).

Definition 1. We say $\delta_1 \geq \cdots \geq \delta_s \geq 0$ is an *admissible topological degree for morphisms of degree k* if

- $\sum_{i=1}^{s} \delta_i \leq k$,
- $k - \sum_{i=1}^{s} \delta_i \equiv 0 \pmod 2$,
- $\sum_{i=1}^{s} \delta_i \leq k - 2$ in case $\delta_s = 0$.

For the next theorem we need some more restriction. In case there is a morphism $\pi : X \to \mathbb{P}^1$ of degree k with $\sum_{i=1}^{s} \delta_i = k$ then by definition $\pi^{-1}\left(\mathbb{P}^1(\mathbb{R})\right) = X(\mathbb{R})$, hence $X(\mathbb{C}) \setminus X(\mathbb{R}) = \pi^{-1}\left(\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})\right)$. Since $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is disconnected it follows $X(\mathbb{C}) \setminus X(\mathbb{R})$ is disconnected too, hence $a(X) = 0$. Therefore, in case $a(X) = 1$ one always has $\sum_{i=1}^{s} \delta_i \leq k - 2$.

Theorem 2. *Let $k \geq 3$, let (g, s, a) be an admissable topological type of real curves and let $\underline{\delta}$ be an admissible topological degree for morphisms of degree k. Assume $\sum_{i=1}^{s} \delta_i \leq k-2$ in case $a = 1$. There exists a real curve X of topological type (g, s, a) having a morphism $\pi : X \to \mathbb{P}^1$ of degree k of topological degree $\underline{\delta}$.*

Remark 3. The case $k = 2$ is contained in e.g. [14, Section 6]. In that case there are more restrictions in the case $a = 0$. Nevertheless, the existence of real hyperelliptic cruves is the starting point in a lot of cases of the proof of the theorem, even in Case 5 which is excluded for real hyperelliptic curves unless $s = g + 1$.

Section 2.1 is the proof of Theorem 2 in case $a = 1$ and Section 2.2 is the proof of Theorem 2 in case $a = 0$. In Section 2.3 we prove the corresponding theorem for morphisms to R_0.

2.1 The case $a = 1$

First we consider the case $s \neq 0$. Let s' be the number of $\delta_i \neq 0$ and let $\widetilde{X}$ be a real hyperelliptic curve of topological type $(g - s', s - s', 1)$ (such curves do exist, see [14, Section 6]). Let $\widetilde{\pi}$ be the associated double covering of $\mathbb{P}^1$ then it follows that $\delta_{\widetilde{C}_i}(\widetilde{\pi}) = 0$ for $1 \leq i \leq s - s'$. Applying Construction III s' times we obtain $\pi_1 : X_1 \to \mathbb{P}^1$ with $\deg(\pi_1) = 2 + s'$; $g(X_1) = g$ and $X_1(\mathbb{R})$ has s components $C_1^1, \cdots, C_s^1$ such that $\delta_{C_i^1}(\pi_1) = 1$ for $1 \leq i \leq s'$ and $\delta_{C_i^1}(\pi_1) = 0$ for $s' + 1 \leq i \leq s$, in particular X_1 has topological type $(g, s, 1)$. For each $1 \leq i \leq s'$ we use Construction I $\delta_i - 1$ times on C_i^1 and using the deformation without real ramification. We obtain $\pi_2 : X_2 \to \mathbb{P}^1$ with $\deg(\pi_2) = 2 + \sum_{i=1}^{s'} \delta_i$, $g(X_2) = g$ and $X_2(\mathbb{R})$ has s components $C_1^2, \cdots, C_s^2$ such that $\delta_{C_i^2}(\pi_2) = \delta_i$ for $1 \leq i \leq s'$ and $\delta_{C_i^2}(\pi_2) = 0$ for $s' + 1 \leq i \leq s$ hence π_2 has topological degrees $\underline{\delta}$ and X_2 has topological type $(g, s, 1)$. In case $s \neq s'$ we apply Construction I $k - 2 - \sum_{i=1}^{s'} \delta_i$ times on $C_{s'+1}^2$ using the deformation with real ramification. In case $s = s'$ and $s' \neq 0$ we apply Construction I $k - 2 - \sum_{i=1}^{s'} \delta_i$ times on C_1^2 alternating the deformation with and without real ramification. We obtain a real curve X of topological type $(g, s, 1)$ and a real morphism $\pi : X \to \mathbb{P}^1$ having degree k and topological degree $\underline{\delta}$. Indeed, in this final step the degrees of the restrictions of π do not change since $k - 2 - \sum_{i=1}^{s'} \delta_i$ is even.

Now assume $s = 0$ (the case of no real points). In this case k is even. In case $g < k$ a general divisor D on a real curve X of degree k gives rise to a base point free linear system $\mid D \mid$. A general pencil in it defined over $\mathbb{R}$ gives rise to a real base point free g_k^1 on X, hence to a real morphism $\pi : X \to \mathbb{P}^1$ of degree k. So we can assume $g \geq k$. Start with a hyperelliptic curve $\widetilde{X}$ of genus $g - (k/2) + 1$ with $\widetilde{X}(\mathbb{R}) = \emptyset$. We apply Construction IV $(k/2) - 1$ times. Again we obtain a real curve X of genus g with $X(\mathbb{R}) = \emptyset$ and having a real morphism $\pi : X \to \mathbb{P}^1$ of degree k.

2.2 The case $a = 0$

Again, let s' be the number of $\delta_i \neq 0$.

2.2.1 Case 1

Assume $s' = s = 1$ and $\delta_1 = k$. In particular $g \equiv 0 \pmod 2$. Let X_0 be a real hyperelliptic curve of genus g having $a(X_0) = 0$, $s(X_0) = 1$, hence for the hyperelliptic real morphism $\pi_0 : X_0 \to \mathbb{P}^1$ the unique component C_1 of $X_0(\mathbb{R})$ has topological degree 2 (see [14, Section 6]). Apply $k - 2$ times Construction I using the deformation without real ramification. We obtain a real curve X of topological type $(g, 1, 0)$ such that X has a covering $\pi : X \to \mathbb{P}^1$ defined over $\mathbb{R}$ of degree k such that $\delta_{C_1}(\pi) = k$.

2.2.2 Case 2

Assume $s > 1$ and $\sum_{i=1}^{s} \delta_i = k$ in which case $s' = s$. In case all $\delta_i = 1$ then $s = k$ and therefore $g \equiv k + 1 \pmod 2$. In particular $g - k + 2$ is odd. Since $k = s$ and $s \leq g + 1$ also $g - k + 2 \geq 1$. Let X_0 be a real hyperelliptic curve of genus $g_0 = g - k + 2$ having $a(X_0) = 0$, $s(X_0) = 2$ (see [14, Section 6]).

Hence, for the real hyperelliptic morphism $\pi_0 : X_0 \to \mathbb{P}^1$ the 2 components C of $X_0(\mathbb{R})$ satisfy $\delta_C(\pi_0) = 1$. Apply $k-2$ times Construction III. We obtain a real curve X of genus g having $a(X) = 0$, $s(X) = k$ such that X has a covering $\pi : X \to \mathbb{P}^1$ defined over $\mathbb{R}$ of degree k such that each component C of $X(\mathbb{R})$ satisfies $\delta_C(\pi) = 1$.

Assume not all $\delta_i = 1$, hence $\delta_1 \geq 2$. Let X_0 be a real hyperelliptic curve of genus $g_0 = g - s + 1$ (in particular g_0 is even) as in Case 1. Apply $s-1$ times Construction III. We obtain a real curve X_1 of genus g having $a(X) = 0$, $s(X) = s$ such that X_1 has a covering $\pi_1 : X_1 \to \mathbb{P}^1$ defined over $\mathbb{R}$ of degree $s+1$ such that exactly one component C_1 of $X_1(\mathbb{R})$ satisfies $\delta_{C_1}(\pi_1) = 2$ while the other components $C_2, \cdots, C_s$ satisfy $\delta_{C_i}(\pi_1) = 1$. We apply $\delta_1 - 2$ times Construction I without real ramification starting with C_1 and for $2 \leq i \leq s$ we apply $\delta_i - 1$ times Construction I without real ramification starting with C_i. We obtain a real curve X of genus g satisfying $a(X) = 0$, $s(X) = s$ such that X has a covering $\pi : X \to \mathbb{P}^1$ defined over $\mathbb{R}$ of degree $s + 1 + (\delta_1 - 2) + \sum_{i=2}^{s}(\delta_i - 1) = \sum_{i=1}^{s} \delta_i = k$ having topological degree $\underline{\delta}$.

2.2.3 Case 3

Assume $\sum_{i=1}^{s'} \delta_i < k$ and $s' = s$. It follows that $s' \geq 1$ and $g \equiv s' + 1 \pmod 2$. In particular $g - s' + 1$ is even. Let X_0 be a real hyperelliptic curve of genus $g - s' + 1$ such that $a(X_0) = 0$, $s(X_0) = 1$ and for the real hyperelliptic morphism $\pi_0 : X_0 \to \mathbb{P}^1$ one has that the unique component C_1 of $X_0(\mathbb{R})$ satisfies $\delta_{C_1}(\pi_0) = 2$ (see [14, Section 6]). Apply Construction I using the deformation with real ramification to C_1 and apply $s' - 1$ times Construction III. We obtain a real curve X_1 of genus g having $a(X_1) = 0$, $s(X_1) = s' = s$ such that there is a morphism $\pi_1 : X_1 \to \mathbb{P}^1$ of degree $s' + 2$ defined over $\mathbb{R}$ such that each component C_i of $X_1(\mathbb{R})$ satisfies $\delta_{C_i}(\pi_1) = 1$.

For $1 \leq i \leq s'$ apply $\delta_i - 1$ times Construction I using the deformation without real ramification starting with C_i. We obtain a real curve X_2 of genus g having $a(X_2) = 0$, $s(X_2) = s' = s$ and such that there is a morphism $\pi_2 : X_2 \to \mathbb{P}^1$ of degree $s' + 2 + (\sum_{i=1}^{s} \delta_i) - s' = (\sum_{i=1}^{s} \delta_i) + 2$ of topological degree $\underline{\delta}$.

By assumption $k - (\sum_{i=1}^{s} \delta_i) - 2$ is an even non-negative integer. Apply $k - (\sum_{i=1}^{s} \delta_i) - 2$ times Construction I starting with C_1 alternating deformation with and without real ramification (starting with real ramification) we end up with a real curve X of genus g satisfying $a(X) = 0$, $s(X) = s' = s$ and such that there is a morphism $\pi : X \to \mathbb{P}^1$ of degree k of topological degree $\underline{\delta}$.

2.2.4 Case 4

Assume $\sum_{i=1}^{s'} \delta_i < k$, $s > s'$ and $s' \geq 1$. We start using the construction of Case 3 making a real curve X' of genus $g - s + s'$ satisfying $a(X') = 0$, $s(X') = s'$ and such that there is a real morphism $\pi' : X' \to \mathbb{P}^1$ of degree k such that, for the components $C_1, \cdots, C_{s'}$ one has $\delta_{C_i}(\pi') = \delta_i$. Notice that, since $g - s \equiv 1 \pmod 2$ one has $g - s + s' \equiv s' + 1 \pmod 2$. Since $\sum_{i=1}^{s'} \delta_i < k$ the set

$\pi^{-1}(\mathbb{P}^1(\mathbb{R})) \setminus X'(\mathbb{R})$ is not empty. Therefore we can apply Construction II $s-s'$ times using a deformation with real ramification. We obtain a real curve X of genus g such that $a(X)=0$, $s(X)=s$ and X has a real covering $\pi : X \to \mathbb{P}^1$ of degree k of topological degree $\underline{\delta}$.

2.2.5 Case 5

Assume $s'=0$. In this case k is even. Let X_0 be a hyperelliptic curve of genus $g-s+1$ (hence of even genus) such that $a(X_0)=0$, $s(X_0)=1$ and if $\pi_0 : X_0 \to \mathbb{P}^1$ is the hyperelliptic morphism then the unique component C_1 of $X_0(\mathbb{R})$ satisfies $\delta_{C_1}(\pi_0)=2$ (see [14, Section 6]). Apply $k-2$ times Construction I using the deformation with real ramification. One obtains a real curve X_1 of genus $g-s+1$ such that $a(X_1)=0$ and $s(X_1)=1$ and a morphism $\pi_1 : X_1 \to \mathbb{P}^1$ of degree k such that for the unique component C_1 of $X_1(\mathbb{R})$ one has $\delta_{C_1}(\pi_1)=0$. It follows that $(\pi_1)^{-1}(\mathbb{P}^1(\mathbb{R}))\setminus X_1(\mathbb{R}) \neq \emptyset$ and so we apply $s-1$ times Construction II with real ramification. We obtain a real curve X of genus g such that $a(X)=0$, $s(X)=s$ and there exists a real morphism $\pi : X \to \mathbb{P}^1$ of degree k such that for each component C of $X(\mathbb{R})$ one has $\delta_C(\pi)=0$.

2.3 Morphisms to R_0

Assume X is a real curve of genus g with $X(\mathbb{R})=\emptyset$ and assume $\pi : X \to R_0$ is a morphism of degree k defined over $\mathbb{R}$. Consider the associated morphism of Riemann surfaces $\pi(\mathbb{C}) : X(\mathbb{C}) \to R_0(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$ and let $|D|$ be the associated complete linear system on $X(\mathbb{C})$. We are going to show that $|D|$ is not induced by a linear system on X (i.e. $|D|$ does not contain a real divisor).

Notice that complex conjugation of divisors on $X(\mathbb{C})$ acts on $|D|$. Indeed, choose a non-real point $Q+\overline{Q}$ on R_0 and let $\pi(\mathbb{C})^{-1}(Q)=D_1$. Since $\pi(\mathbb{C})^{-1}(\overline{Q})=\overline{D_1}$ one has $D_1 \sim \overline{D_1}$. For any $E \in |D|$ one has $E \sim D_1$. This implies $\overline{E} \sim \overline{D_1}$ hence also $E \sim \overline{E}$. Now $\pi(\mathbb{C})$ corresponds to a line L in $|D|$ that is invariant under this complex conjugation. If $|D|$ would be real it would imply L is a real line, hence containing a real point. Such real point corresponds to a real divisor and this would be a fiber of $\pi(\mathbb{C})$ invariant under complex conjugation but such fibers do not exist.

This implies $\mathcal{O}_X(D)$ corresponds to a point of $\mathrm{Pic}^k(X)(\mathbb{R}) \setminus \mathrm{Pic}^k(X)(\mathbb{R})^+$, hence $k \equiv g+1 \pmod 2$ (see [14]). In this section we consider pencils corresponding to points in $\mathrm{Pic}^k(X)(\mathbb{R}) \setminus \mathrm{Pic}^k(X)(\mathbb{R})^+$ (in particular $k \equiv g+1 \pmod 2$).

Theorem 4. Let g and $k \equiv g+1 \pmod 2$ be nonnegative integers. In case $k \geq g+1$ and X is a real curve of genus g with $X(\mathbb{R})=\emptyset$ then there exists a morphism $\pi : X \to R_0$ of degree k. In case $k < g+1$ then there exists a real curve of genus g with $X(\mathbb{R})=\emptyset$ having a morphism $\pi : X \to R_0$ of degree k.

Proof. First assume $k \geq g+1$. Choose an effective divisor D such that $\mathcal{O}_X(D)$ corresponds to a point on $\mathrm{Pic}^k(X)(\mathbb{R}) \setminus \mathrm{Pic}^k(X)(\mathbb{R})^+$. Then $|D|$ is not defined over $\mathbb{R}$ but complex conjugation of divisors is defined on $|D|$. Choose $D_1 \in$

$|D|$ and let L be the line in $|D|$ connecting D_1 to $\overline{D_1}$. This corresponds to a morphism $\pi(\mathbb{C}) : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $L = \overline{L} \subset |D|$ it follows $\pi(\mathbb{C})$ is equivariant using the complex conjugation on $X(\mathbb{C})$ and some antiholomorphic involution on $\mathbb{P}^1(\mathbb{C})$ without fixed points. This implies $\pi(\mathbb{C})$ is induced by a morphism $\pi : X \to R_0$ defined over $\mathbb{R}$.

Next assume $k \leq g+1$ and $k \equiv g+1 \pmod 2$. Let X_1 be a real curve of genus $g-k+2$ having a morphism $f_1 : X_1 \to R_0$ of degree 2 (such curves exist since $g-k+2$ is odd, see [14, Section 6]). Then use $k-2$ times Construction V.

□

Remark 5. In his paper [6] the author obtains Theorem 4 by fixing a real divisor D on R_0 and using the Riemann Existence Theorem using coverings on $R_0(\mathbb{C}) \setminus D$. Although the author gives no argument for the existence of an antiholomorphic involution on a suited choice of such a covering this reasoning can be applied using Klein surfaces (see [1] for the foundations of that theory). Also in that paper he obtains Theorem 2 for the case $s = 0$ but in that argument details are omitted.

In his paper [7] the author also gives an argument for the existence of real curves X of given topological type having a morphism $\pi : X \to \mathbb{P}^1$ of degree k. For the case $a(X) = 0$ he uses deformations of hyperelliptic and trigonal curves. The case $a(X) = 1$ is contained in [5]. In that paper the author uses deformations of singular real curves on suited real rational surfaces using Brusotti-type arguments. We use arguments directly applied to coverings which is more natural and easier. In both cases it is not clear to us how to obtain the existence of all admissible topological degrees for π using those deformations. We also give some remarks on some of the arguments used in those papers (especially [7]) in Remark 11.

In [14, Section 8] one considers real trigonal curves and one asks about determining the topological types of such curves. In Theorem 2 we obtain that all admissable topological types can be realized as real trigonal curves. A (in principle similar) proof is described in [7, Proposition 2.1]. A different proof using Fuchsian and NEC (non-euclidean crystallographic) groups (and obtaining much more detailled information concerning the case of cyclic trigonal coverings) is given in [10].

3 Some Brill-Noether Theory for real pencils on real curves

On $\mathrm{Pic}^k(X)(\mathbb{C})$ we consider the closed subspace $W^1_k(X)$ representing invertible sheafs on $X_{\mathbb{C}}$ satisfying $h^0(L) \geq 2$. This subspace $W^1_k(X)$ is invariant under conjugation, hence it is defined over $\mathbb{R}$. We write $W^1_k(X)(\mathbb{R})$ to denote its set of real points and $W^1_k(X)(\mathbb{R})^+ = W^1_k(X)(\mathbb{R}) \cap \mathrm{Pic}^k(X)(\mathbb{R})^+$. In case $k \geq g+1$ one has $W^1_k(X) = \mathrm{Pic}^k(X)(\mathbb{C})$, so we assume from now on that $k \leq g$.

The study of those spaces $W^1_k(X)$ for general complex curves X of genus g is the Brill-Noether Theory for pencils. The main results are the following. First consider the Brill-Noether number $\rho^1_k(g) = 2k - g - 2$. For a general complex curve X one has $W^1_k(X) = \emptyset$ in case $\rho^1_k(g) < 0$ and $\dim(W^1_k(X)) = \rho^1_k(g)$ in case $\rho^1_k(g) \geq 0$. Also in case $\rho^1_k(g) \geq 0$ then $W^1_k(X) \neq \emptyset$ for all complex curves X. Here "general" means: the statement holds for all curves X on a dense open subset of the moduli space M_g of curves of genus g.

In [9] one considers Brill-Noether theory of $W^1_k(X)(\mathbb{R})$ for real curves X. There are important differences with the complex case. We consider the moduli space $M_{g/\mathbb{R}}$ of real curves of genus g representing the isomorphism classes over $\mathbb{R}$ of real curves of genus g. We recall some facts on it (for details see [19]). It is a semi-analytic real variety of dimension $3g-3$ (as a matter of fact it has a non-empty boundary, see [18] for its description). For each topological type (g, s, a) there is a unique connected component $M_{g/\mathbb{R}}(g, s, a)$ of $M_{g/\mathbb{R}}$. This connected component is the quotient of a Teichmuller space (which is a connected real analytic manifold) by means of a discontinuous action of a modular group. Inside such component $M_{g/\mathbb{R}}(g, s, a)$, curves having a certain type of real linear system give rise to semi-analytic subvarieties of $M_{g/\mathbb{R}}(g, s, a)$. Therefore there is in general no description of behavior on a dense open subset of $M_{g/\mathbb{R}}(g, s, a)$ and we need to modify the meaning of "general".

Definition 6. Let P be some property concerning real linear systems on real curves. We say there is *a general real curve* of topological type (g, s, a) satisfying property P if there exists a non-empty open subset U of $M_{g/\mathbb{R}}(g, s, a)$ such that P holds for all curves corresponding to a point of U.

Example 7. To illustrate the difference with the complex situation we consider W^1_3 for curves of genus 4 ($\rho^1_3(4) = 0 \geq 0$) (see [14, Section 8]). A real curve of type (4,0,1) has no real g^1_3 since all its real divisors have even degree. In case $s \neq 0$ there exists a general real curve of type (4,s,a) having a real g^1_3. In case $(s, a) \neq (1, 0)$ there also exists a general real curve of type $(4, s, a)$ having no real g^1_3. In Proposition 12, as an application of Theorem 2, we give a description of the situation in case $(s, a) = (1, 0)$.

In [9] it is proved that in case $\rho^1_k(g) \geq 0$ and $s \neq 0$ then there is a general real curve of type (g, s, a) such that $W^1_k(X)(\mathbb{R}) \neq \emptyset$. In order to prove this result the author constructs a real compactification of the real Hurwitz space of coverings of degree k. The restriction $s \neq 0$ has a technical cause. Working with families of real curves the author needs the existence of a section defined over $\mathbb{R}$. In particular such a section guaranties that the relative Picard scheme of such families represents the Picard functor (see [16]).

Now we are going to use the existence results from Section 2 to give another proof. In that way we obtain a finer statement involving the covering degree. In the argument we make intensive use of Brill-Noether theory for complex curves and we work with the space of morphisms to $\mathbb{P}^1$ instead of the Hurwitz space. This space of morphisms is defined using Hilbert schemes in [15, Section 4.c]. In particular we do not need relative Picard schemes to finish our arguments, hence we do not need the restriction $s \neq 0$. In case $s = 0$ we consider both $W^1_k(X)(\mathbb{R})^+$ and $W^1_k(X)(\mathbb{R}) \setminus W^1_k(X)(\mathbb{R})^+$.

In case $f : X \to R_0$ is a morphism of degree k then $f_{\mathbb{C}} : X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ corresponds to a linear system $g^1_k \subset |L|$ for some $L \in W^1_k(X)(\mathbb{R}) \setminus W^1_k(X)(\mathbb{R})^+$. In order to study $W^1_k(X)(\mathbb{R}) \setminus W^1_k(X)(\mathbb{R})^+$ using morphims, we also need the converse statement proved in [12, Example 1].

Lemma 8. Let $L \in W^1_k(X)(\mathbb{R}) \setminus W^1_k(X)(\mathbb{R})^+$ and assume $|L|$ is base point free of dimension 1. Then there is a morphism $f : X \to R_0$ of degree k such that $f_{\mathbb{C}}$ corresponds to $|L|$.

Let X be a real curve of topological type (g, s, a) and let g^1_k be a complete base point free linear system on X. We say g^1_k has topological degree $\underline{\delta}$ if the morphisms $f : X \to \mathbb{P}^1$ associated to g^1_k have topological degree $\underline{\delta}$.

Let ${}^\circ W^1_k(X)$ be the subspace of $W^1_k(X)$ parameterizing complete base point free linear systems g^1_k on $X_{\mathbb{C}}$. It is the complement of $W^1_{k-1}(X) + W^0_1(X)$ in $W^1_k(X)$, hence it is Zariski-open in $W^1_k(X)$. In particular ${}^\circ W^1_k(X)(\mathbb{R})$ is Zariski-open in $W^1_k(X)(\mathbb{R})$. Because the topological degree is a discrete invariant it is constant on connected components of ${}^\circ W^1_k(X)(\mathbb{R})$. Let ${}^\circ W^1_k(X)(\underline{\delta})$ be the sublocus of ${}^\circ W^1_k(X)(\mathbb{R})$ such that it parameterizes linear systems g^1_k of topological degree $\underline{\delta}$ and let $W^1_k(X)(\underline{\delta})$ be its closure in $W^1_k(X)(\mathbb{R})$ (it is a union of irreducible components of $W^1_k(X)(\mathbb{R})$).

Theorem 9. Let $\underline{\delta}$ be an admissible topological degree of base point free linear systems g^1_k on real curves of topological type (g, s, a) such that $\sum_{i=1}^{s} \delta_i \leq k - 2$ in case $a = 1$.

1. If $\rho^1_k(g) < 0$ then there is no general real curve X of type (g, s, a) such that $W^1_k(X)(\underline{\delta})$ is not empty.

2. If $\rho^1_k(g) \geq 0$ then there is a general real curve X of topological type (g, s, a) such that $W^1_k(X)(\underline{\delta})$ is non-empty and it is a real algebraic subset of $\mathrm{Pic}^k(X)(\mathbb{R})$ of dimension $\rho^1_k(g)$.

3. In case $s = 0$ and $k \equiv g + 1 \pmod 2$ then in case $\rho^1_k(g) < 0$ there is no general real curve X of topological type $(g, 0, 1)$ such that $W^1_k(X)(\mathbb{R}) \setminus W^1_k(X)(\mathbb{R})^+$ is not empty. In case $\rho^1_k(g) \geq 0$ then there is a general real curve X of topological type $(g, 0, 1)$ such that $W^1_k(X)(\mathbb{R}) \setminus W^1_k(X)(\mathbb{R})^+$ is not empty and has dimension $\rho^1_k(g)$.

In order to prove this theorem we are going to use spaces parameterizing morphisms to $\mathbb{P}^1$. First we introduce some definition that will be useful for the proof.

Definition 10. A *suited family of curves of genus g* is a projective mophism $\pi : \mathcal{C} \to S$ such that

- S is smooth, irreducible and quasi-projective,

- each fiber of π is a smooth connected curve of genus g,
- for each $s \in S$ the Kodaira-Spencer map $T_s(S) \to H^1(\pi^{-1}(s), T_{\pi^{-1}(s)})$ is bijective.

In case X is a given smooth curve of genus g then we say $\pi : \mathcal{C} \to S$ is a suited family for X if moreover some fiber of π is isomorphic to X. If moreover X is defined over $\mathbb{R}$ then we also assume $\pi : \mathcal{C} \to S$ is defined over $\mathbb{R}$ and X is isomorphic over $\mathbb{R}$ to some fiber of π over a real point of S.

Such suited families do exist (see [12, Section 3], in that paper one considers real curves with real points and one needs the family to have a real section, in our situation one does not need to use S' from loc. cit.). From a suited family we obtain a morphism $S \to M_g$ and in case $K = \mathbb{R}$ we also obtain a morphism $S(\mathbb{R}) \to M_{g/\mathbb{R}}$. From the bijectivity of the Kodaira-Spencer map it follows those morpisms to the moduli space have finite fibers. Therefore it is enough to prove the statements of Theorem 9 for real curves corresponding to fibers of π at general points of $S(\mathbb{R})$.

Let $\pi : \mathcal{C} \to S$ be a suited family of curves of genus g defined over K (being $\mathbb{C}$ or $\mathbb{R}$). For a noetherian S-scheme T we consider the set $\mathcal{H}_k(\pi)(T)$ of finite T-morphisms $\mathcal{C} \times_S T \to \mathbb{P}^1 \times T$ of degree k. This functor is representable by an S-scheme $\pi_k : H_k(\pi) \to S$ (see [15, Section 4.c]). Let $f : X \to \mathbb{P}^1_{\mathbb{C}}$ be a morphism corresponding to a point $[f]$ on $H_k(\pi)(\mathbb{C})$ and consider the associated exact sequence

$$0 \to T_X \to f^*(T_{\mathbb{P}^1}) \to N_f \to 0$$

From Horikawa's deformation theory of holomorphic maps (see [17], see also [23, 3.4.2]) it follows $T_{[f]}(H_k(\pi))$ is canonically identified with $H^0(X, N_f)$ and since $H^1(X, N_f) = 0$ it follows $H_k(\pi)$ is smooth of dimension $2k+2g-2$. In case $K = \mathbb{R}$ it follows $H_k(\pi)(\mathbb{R})$ is a smooth real manifold of dimension $2k + 2g - 2$ (of course $H_k(\pi)(\mathbb{R})$ need not be connected).

In case $K = \mathbb{R}$ then for a noetherian S-scheme T we consider the set $\mathcal{H}^R_k(\pi)(T)$ of finite T-morphisms $\mathcal{C} \times_S T \to R_0 \times T$ of degree k. This functor is represented by an S-scheme $\pi^R_k : H^R_k(\pi) \to S$ and using the same arguments we obtain $H^R_k(\pi)$ is smooth of dimension $2k + 2g - 2$ and $H^R_k(\pi)(\mathbb{R})$ is a smooth real manifold of dimension $2k + 2g - 2$ (again, it need not be connected).

Because $\dim(Aut(\mathbb{P}^1_{\mathbb{C}})) = 3$ the non-empty fibers of $\pi_k(\mathbb{C})$ (and $\pi^R_k(\mathbb{C})$) do have dimension at least 3. Hence the image of $\pi_k(\mathbb{C})$ (and of $\pi^R_k(\mathbb{C})$) has dimension at most $2k + 2g - 5 = 3g - 3 + \rho^1_k(g)$. Hence in case $\rho^1_k(g) < 0$ then a general point of $S(\mathbb{C})$ does not belong to the image of $\pi_k(\mathbb{C})$ (or $\pi^R_k(\mathbb{C})$). Now assume $\rho^1_k(g) \geq 0$. From the arguments in [24, Appendix] it follows that each component of $H_k(\pi)(\mathbb{C})$ (or $H^R_k(\pi)(\mathbb{C})$) dominates $S(\mathbb{C})$. Let $H'_k(\pi)(\mathbb{C})$ (resp. $H'^R_k(\pi)(\mathbb{C})$) be the closed subspace of $H_k(\pi)(\mathbb{C})$ (resp. $H^R_k(\pi)(\mathbb{C})$) consisting of the fibers of $\pi_k(\mathbb{C})$ (resp. $\pi^R_k(\mathbb{C})$) having dimension greater than $\rho^1_k(g) + 3$. Assume it contains an irreducible component of $H_k(\pi)(\mathbb{C})$. Since such component dominates $S(\mathbb{C})$ it would follow that it has dimension more than $3g-3+\rho_k^1(g)+3 = 2g+2k-2$, a contradiction. Hence it follows that $H'_k(\pi)(\mathbb{C})$ (resp. $H'^R_k(\pi)(\mathbb{C})$) does not contain irreducible components of $H_k(\pi)(\mathbb{C})$ (resp. $H^R_k(\pi)(\mathbb{C})$), hence its dimension is less than $2g+2k-2$. Moreover, those subspaces are invariant under complex conjugation, hence they are defined over $\mathbb{R}$. So we obtain closed subschemes $H'_k(\pi)$ and $H'^R_k(\pi)$ of $H_k(\pi)$ defined over $\mathbb{R}$ and we obtain $\dim\left(H'_k(\pi)(\mathbb{R})\right) < 2g+2k-2$ (resp. $\dim\left(H'^R_k(\pi)(\mathbb{R})\right) < 2g+2k-2$). For a complex curve X_s corresponding to a general point s on $S(\mathbb{C})$ it follows from Brill-Noether Theory that $W^r_k(X)$ has dimension $\rho^r_k(g) = g-(r+1)(g-k+r)$ (if $\rho^r_k(g)\geq 0$, otherwise it is empty). In case $r\geq 2$ this gives rise to a subspace of $\pi_k^{-1}(s)$ (resp. $\pi_k'^{-1}(s)$) of dimension $\rho^r_k(g)+2(r-1)+3 < \rho^1_k(g)+3$. Let $H^2_k(\pi)(\mathbb{C})$ (resp. $H^{2,R}_k(\pi)(\mathbb{C})$) be the closed subspace of $H_k(\pi)(\mathbb{C})$ (resp. $H^R_k(\pi)(\mathbb{C})$) corresponding to linear systems g^1_k that are not complete. Then we obtain that those are not components of $H_k(\pi)(\mathbb{C})$ (resp. $H^R_k(\pi)(\mathbb{C})$) because they have dimension less than $2g+2k-2$. Again $H^2_k(\pi)$ and $H^{2,R}_k(\pi)$ are defined over $\mathbb{R}$ and we find $\dim\left(H^2_k(\pi)(\mathbb{R})\right) < 2g+2k-2$ (resp. $\dim\left(H^{2,R}_k(\pi)(\mathbb{R})\right) < 2g+2k-2$).

Proof of Theorem 9. In case $\rho^1_k(g)<0$ then from $\dim\left(H_k(\pi)(\mathbb{R})\right) < 3g-3$ part 1 follows. Also in case $s=0$ one has $\dim\left(H^R_k(\pi)(\mathbb{R})\right) < 3g-3$, hence using Lemma 8 the first statement of part 3 follows. So assume $\rho^1_k(g)\geq 0$.

By Theorem 2 there is a real curve X of topological type (g,s,a) having a morphism $f: X\rightarrow \mathbb{P}^1$ of degree k having topological degree $\underline{\delta}$. Let $\pi:\mathcal{C}\rightarrow S$ be a suited family for X then f defines $[f]\in H_k(\pi)(\mathbb{R})$. From the previous consideration it follows we can assume $[f]\notin H'_k(\pi)(\mathbb{R})\cup H^2_k(\pi)(\mathbb{R})$. From $[f]\notin H'_k(\pi)(\mathbb{R})$ it follows $\dim\left(\pi_k(H_k(\pi)(\mathbb{R}))\right) = 3g-3$ and from $[f]\notin H^2_k(\pi)(\mathbb{R})$ it follows f corresponds to a complete g^1_k. Hence a general element of $\pi_k(H_k(\pi)(\mathbb{R}))$ corresponds to a real curve X' of type (g,s,a) with $\dim\left(W^1_k(X')(\underline{\delta})\right)\geq \rho^1_k(g)$. Also $\dim\left(H_k(\pi)(\mathbb{R})\right) = 2g+2k-2$ implies we obtain $\dim\left(W^1_k(X')(\underline{\delta})\right) = \rho^1_k(g)$, proving part 2.

In case $s=0$ and $k\equiv g+1 \pmod 2$, by Theorem 4 there exists a real curve X of topological type $(g,0,1)$ having a real morphism $f: X\rightarrow R_0$ of degree k. Now we find $[f]\in H^R_k(\pi)(\mathbb{R})$ and we can assume $[f]\notin H'^R_k(\pi)(\mathbb{R})\cup H^{2,R}_k(\pi)(\mathbb{R})$. As in the previous case we obtain the second statement of part 3 (again we also use Lemma 8).

□

Remark 11. The proof of Theorem 9 is intensively based on known dimension statements in the complex case. In order to be able to use those statements it is very important to know that $\dim\left(H_k(\pi)\right) = 2k+2g-2$ implies $\dim\left(H_k(\pi)(\mathbb{R})\right) = 2k+2g-2$. To make this conclusion it is very important to know that $H_k(\pi)(\mathbb{C})$ is smooth. In his paper [7] mentioned in Remark 5 the author uses similar arguments to obtain statements as those in Theorem 9 but without considering the topological degree. However the arguments are applied on the moduli space itself. Although the author claims the locus of k-gonal curves on the module space is smooth this is not clear (and in general it is not true). Moreover for his arguments he needs that the closure of that locus is smooth at the hyperelliptic or trigonal locus. To solve this technical problem one should use a suited family containing a hyperelliptic or trigonal curve and use the space parameterizing linear systems g^1_k on fibers of that family. Indeed, according to [2] that space is smooth. The construction of that space uses the relative Picard scheme and we did not use that space in our arguments in order to be able to use arguments also applicable in case $s = 0$. Also in his paper [5, End of proof of Theorem 0.1] the author uses similar arguments but again not all details are described completely. In particular the author uses an argument on Hurwitz schemes but it is not clear from it that one obtains real coverings with only simple ramification (in particular, that one obtains a point in the smooth locus of the compactified Hurwitz scheme). Those arguments in [5] can be replaced by using the Hilbert scheme parameterizing morphisms and the details described in this section.

In his paper [7] the author also shows that (with the same remarks on the proof as above), in case $\rho^1_k(g) < 0$, a general k-gonal real curve of topological type (g, s, a) has a unique linear system g^1_k. This fact can be obtained from our arguments (and fixing the topological degree of the linear system). As in [7] one uses that a general complex k-gonal curve has a unique g^1_k (see [3]). Since a general complex k-gonal curve has no multiple g^1_k (see [22]) one also obtains a similar result for general real k-gonal curves fixing the topological degrees.

We return to Example 7, finishing the case of curves of topological type (4,1,0). A non-hyperelliptic complex curve X of genus 4 is trigonal. By Riemann-Roch, in case g is a g^1_3 on X then $|K_X - g| = h$ is also a g^1_3 on X and it is well known that X has no other linear systems g^1_3. Hence X has either two (in case $g \neq h$) or one (in case $g = h$) linear systems g^1_3.

Proposition 12. Let X be a non-hyperelliptic real curve of topological type $(4, 1, 0)$. Then X has two real linear systems g^1_3. One of them has topological degree (3), the other one has topological degree (1).

Proof. Let T be a real Teichmüller space of real curves of topological type (4,1,0) and let $\pi_T : \mathcal{X} \to T$ be the associated universal family. This space T is a connected real manifold of dimension 9 and all real curves of topological type (4,1,0) do occur as fibers of that family. Inside T the closed subspace H of points having hyperelliptic fibers for π_H has codimension 2, hence $T \setminus H$ is still connected. Inside $T \setminus H$ we consider $T(1)$ (resp. $T(3)$) being the subspace of points such that the fiber X for π_H has a morphism $f : X \to \mathbb{P}^1$ of degree 3 of topological degree (1) (resp. (3)). From [13] we know $T(3) = T \setminus H$ and we need to prove $T(1) = T \setminus H$ too. Because of Theorem 2 we know $T(1) \neq \emptyset$, we assume that $T(1) \neq T \setminus H$.

The fiber of a point belonging to the closure of $T(1)$ in $T \setminus H$ is a real curve X that is the limit of real curves X_t having a morphism $f_t : X_t \to \mathbb{P}^1$ of topological degree (1). This corresponds to a linear system $g^1_3(t)$ on X_t and the limit is a linear system g^1_3 on X. Since X is not hyperelliptic this linear system has no base points and it is complete, hence it corresponds to a morphism $f : X \to \mathbb{P}^1$

being a limit of those morphisms f_t, hence f has topological degree (1) too. This implies $T(1)$ is closed in $T \setminus H$ hence it is not open in $T \setminus H$ since $T \setminus H$ is connected. Therefore $T(1)$ has a boundary point t_0 in $T \setminus H$, let X_0 be the fiber of π_T above t_0. Now let $\pi : \mathcal{C} \to S$ be a suited morphism of curves for X_0 (let $s_0 \in S(\mathbb{R})$ such that $\pi^{-1}(s_0) = X_0$).

Let $\pi_3 : H_3 \to S$ be the parameterspace for all coverings of degree 3 to $\mathbb{P}^1$ of fibers of π above points on S. Let $H_3(1)$ (resp. $H_3(3)$) be the closed open subsets of coverings of topological degree (1) (resp. (3)). For a covering $f : X \to \mathbb{P}^1$ we write L_f to denote the corresponding invertible sheaf. For $[f] \in H_3$ it follows from the deformation theory of Horikawa that $d_{[f]}(\pi_3)$ is surjective if and only if $\dim \left(H^0(L_f^{\otimes 2}) \right) = 3$ (indeed $H^0(X, N_f) \to H^1(X, T_X)$ is the tangent map of π_3 at $[f]$). In that case for $x = \pi_3([f])$ there is a neighborhood U of x in S such that $U \subset \mathrm{im}(\pi_3)$. In case $\dim \left(H^0(L_f^{\otimes 2}) \right) > 3$ then $L_f^{\otimes 2} \cong \omega_{\pi^{-1}(x)}$, hence L_f is half-canonical.

Let $f_0 : X_0 \to \mathbb{P}^1$ be the morphism of degree 3 on X_0 of topological degree (1). For each classical neighborhood U of s_0 in S there exists $s \in U \cap S(\mathbb{R})$ such that $\pi^{-1}(s) = \pi_T^{-1}(t)$ for some $t \notin T(1) \cup H$, hence $U \nsubseteq \pi_3(H_3(1))$. From the previous description of the tangent map of π_3 it follows L_{f_0} is half-canonical on X_0. In particular X_0 has no g^1_3 associated to an invertible sheaf different from L_{f_0} and this would imply there is no morphism $f : X_0 \to \mathbb{P}^1$ of topological degree (3), hence $t_0 \notin T(3)$ contradicting $T(3) = T \setminus H$.

□

Remark 13. In contrast with our Theorem 9 it is already mentioned that for each topological type $(4, s, a) \neq (4, 1, 0)$ there is a general real curve X having no real g^1_3. In [9] it is also proved that for each topological type $(8, s, a) \neq \{(8, 1, 0), (8, 0, 1)\}$ there is a general real curve X having no real g^1_5 (while Theorem 9 implies there is a general real curve X having a real g^1_5). On the other hand, the main result of [13] implies that there is no general real curve X of topological type $(8, 1, 0)$ having no base point free g^1_5 of topological degree (5). For real curves without real points it is proved in [20] that for each genus g there exist general real curves X of topological type $(g, 0, 1)$ such that $X^1_k(X)(\mathbb{R}) \setminus W^1_k(X)(\mathbb{R})^+$ is empty for each $k \leq g$. Our Theorem 9 implies that there exist general real curves of topological type $(g, g-1, 0)$ having a base point free g^1_{g-1} of topological degree $(1, \cdots, 1)$. From forthcoming work of the first author it follows that there also exist general real curves X of topological type $(g, g-1, 0)$ having no such linear system g^1_{g-1} and many similar statements.

4 Real 4-gonal curves having a g_4^1 with no totally non-real divisor and no component of non-zero degree

Definition 14. Let X be a real curve and let $f : X \to \mathbb{P}^1$ be a morphism defined over $\mathbb{R}$. We define the *covering number* $k(f)$ as follows. Consider the associated map $f(\mathbb{R}) : X(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$. If $f(\mathbb{R})$ is not surjective then $k(f) = 0$. Otherwise $k(f)$ is the minimal number k such that there exist connected components $C_1, \cdots, C_k$ of $X(\mathbb{R})$ such that $f(C_1 \cup \cdots \cup C_k) = \mathbb{P}^1(\mathbb{R})$.

Of course, in case $X(\mathbb{R}) = \emptyset$ then $k(f) = 0$ by definition. Also $k(f) = 1$ if and only if there is a connected component C of $X(\mathbb{R})$ such that $\delta_C(f) \geq 1$. In particular $k(f) = 1$ in case $\deg(f)$ is odd. Also in case $\deg(f) = 2$ then $k(f)$ is equal to 0 or to 1. So the case with $\deg(f) = 4$ is the first interesting case to study the possible values for $k(f)$. In the next theorem we prove there are no further restrictions on $k(f)$ for gonality 4.

Theorem 15. Let (g, s, a) be an admissable topological type for real curves with $s \geq 1$. Let $1 \leq k \leq s$. There exists a real curve X of topological type (g, s, a) such that there is a covering $f : X \to \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ such that $k(f) = k$ and for each connected component C of $X(\mathbb{R})$ one has $\delta_C(f) = 0$.

Proof. We first prove the case of M-curves ($s = g + 1$, in particular $a = 0$) with $k = g + 1$.

In case g is even take two hyperelliptic M-curves Y_1 and Y_2 of genus $g' = g/2$ having double coverings $f_i : Y_i \to \mathbb{P}^1$ defined over $\mathbb{R}$ with the following properties. Let $C_0^i, \cdots, C_g^i$ be the connected components of $Y_i(\mathbb{R})$ and let $I_j^i = f_i(C_j^i)$ then $I_{j_1}^1$ intersects $I_{j_2}^2$ if and only if one of the following holds:

$$\begin{cases} j_1 = j_2 & \text{with } 0 \leq j_1 \leq g' \\ j_1 = j_2 + 1 & \text{with } 1 \leq j_1 \leq g' \\ j_1 = 0 \text{ and } j_2 = g' & \end{cases}$$

and the non-empty intersections are connected (see Figure 9 with $g' = 2$).
Let $t \in I_0^1 \cap I_0^2$ and take $p_i \in f_i^{-1}(t)$. Let $X_0 = (Y_0 \cup Y_1)_{p_1 = p_2}$. It is a stable real curve of genus g having a node $p = (p_1 = p_2)$ and a morphism $f : X_0 \to \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$. Locally at p the curve X_0 is defined over $\mathbb{R}$ by the equation $x^2 - y^2 = 0$ and the morphism is given by $(x, y) \mapsto x$. Using a local deformation over $\mathbb{R}$ given by $x^2 - y^2 = t$ and gluing with other local coordinates one obtains a curve X_t defined over $\mathbb{R}$ having a covering $f_t : X_t \to \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ such that $g(X_t) = 2g' = g$. (The details are as in Construction 1.) The components of $X_t(\mathbb{R})$ are deformations $C_j^i(t)$ of C_j^i for $1 \leq j \leq g'$ and $i = 1; 2$ and of $\left(C_0^1 \cup C_0^2\right)_{p_1 = p_2}$ which is one component $C_0(t)$. Clearly $f_t\left(C_0(t)\right)$ is a deformation of $I_0^1 \cup I_0^2$ and $f_t\left(C_j^i(t)\right)$ is a deformation of I_j^i for $i = 1; 2$, $1 \leq j \leq g'$. It follows that their union is equal to $\mathbb{P}^1(\mathbb{R})$ but omitting one of

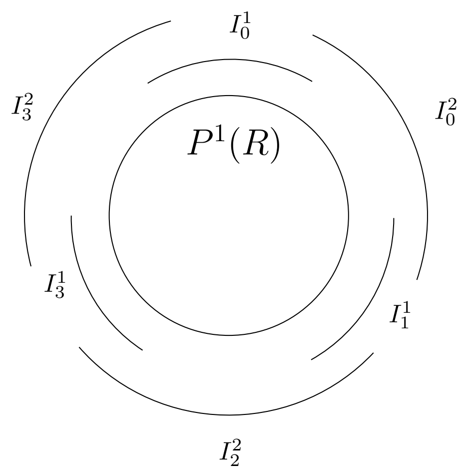

Figure 9: $s = k = g + 1$ and g even

them does not cover $\mathbb{P}^1(\mathbb{R})$. It follows $s(X_t) = 2g' + 1 = g + 1$ (hence X_t is an M-curve) and $k = g + 1$.

In case g is odd take two hyperelliptic M-curves Y_1 of genus $g' = (g-1)/2$ and Y_2 of genus $g' + 1 = (g+1)/2$. Use the notation C_j^i for the components of $Y_i(\mathbb{R})$ with $0 \leq j \leq g'$ in case $i = 1$ and $0 \leq j \leq g' + 1$ in case $i = 2$. Let $I_j^i = f_i\left(C_j^i\right)$ and assume $I_{j_1}^1$ intersects $I_{j_2}^2$ for $0 \leq j_1; j_2 \leq g'$ as in the previous case and $I_{g'+1}^2 \subset I_0^1$ (see Figure 10 with $g' = 2$).

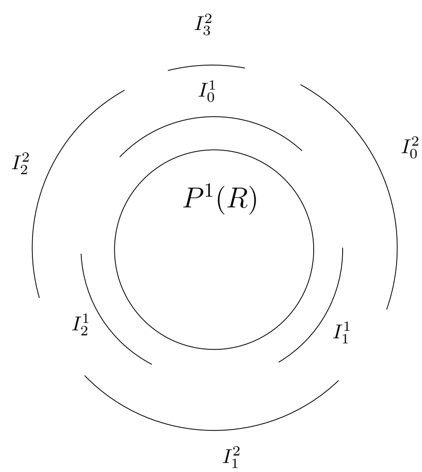

Figure 10: $s = k = g + 1$ and g odd

Let $t \in I_{g'+1}^2$ and take $p_i \in f_i^{-1}(t)$. Let X_0 be as before then arguing as before one obtains an M-curve X_t of genus g with a covering $f_t : X_t \rightarrow \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ such that the components of $X_t(\mathbb{R})$ are deformations $C^i_j(t)$ for $0 \leq j \leq g'$ and $i = 1; 2$ (with $j \neq 0$ for $i = 1$) and $C(t)$ of $\left(C^1_0 \cup C^2_{g'+1}\right)_{p_1=p_2}$. The images $f_t\left(C^i_j(t)\right)$ are deformations of I^i_j and the image $f_t(C(t))$ is a deformation of I^1_0. It follows $k = g + 1$.

Now we are going to prove the theorem for M-curves in case $k < g+1$. From the previous part of the proof we obtain the existence of an M-curve Y of genus $k-1$ having a covering $f_Y : Y \rightarrow \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ satisfying the following property. Let $C_1, \cdots, C_k$ be the components of $Y(\mathbb{R})$, let $I_i = f_Y(C_i)$ then I_i intersects I_j for $i \neq j$ if and only if

$$\begin{cases} j = i+1 & \text{for } 1 \leq i \leq k-1 \\ j = k \text{ and } i = 1 \end{cases}$$

and the non-empty intersections are connected. In case $k = 1$ then $Y(\mathbb{R})$ has a unique component C_1 dominating $\mathbb{P}^1(\mathbb{R})$ such that $\delta_{f_Y}(C_1) = 0$ and there is a connected closed subset $I \subset \mathbb{P}^1(\mathbb{R})$ such that $x \in I$ if and only if $f_y^{-1}(x) \subset Y(\mathbb{R})$.

Let $c_1, \cdots, c_{g-k+1}$ be different points on $I_1 \cap I_2$ in this order (with c_1 most close to $I_1 \setminus I_2$) and let $f_Y^{-1}(x_i) \cap C_1 = \{p_{i1}, p_{i2}\}$ (see Figure 11 with $k = 4$ and $g = 6$). In case $k = 1$ those are points in the inner part of I. Let $X_0 = Y_{p_{i1}=p_{i2} \text{ for } 1 \leq i \leq g-k+1}$. Then X_0 is defined over $\mathbb{R}$ and it has a covering $f_0 : X_0 \rightarrow \mathbb{P}^1$ defined over $\mathbb{R}$ of degree 4. Locally at the node $p_i = (p_{i1} + p_{i2})$ the curve X_0 is defined over $\mathbb{R}$ by $x^2 - y^2 = 0$ and the morphism by $(x, y) \mapsto x$. Using local deformations over $\mathbb{R}$ by the equation $x^2 - y^2 = t$ with $t \geq 0$ one obtains two new real ramification points close to c_i. Gluing one obtains a curve X_t defined over $\mathbb{R}$ having a morphism $f_t : X_t \rightarrow \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ with $g(X_t) = g$. (This is similar to applying $g-k+1$ times Construction I with real ramification.) For each $2 \leq i \leq k$ there is a component $C_i(t)$ of $X_t(\mathbb{R})$ that is a deformation of C_i. Because of the chosen local deformations, the deformation of C_1 is a union of $g-k+2$ components $C_1(t), C'_1(t), \cdots, C'_{g-k+1}(t)$. It follows $s(X_t) = g+1$ hence X_t is an M-curve. The images $f_t(C_i(t))$ are deformations of I_i for $2 \leq i \leq t$, the image $f_t(C_1(t))$ is a deformation of the connected component of $I_1 \setminus \{c_1\}$ containing $I_1 \setminus I_2$, the images of $f_t(C'_i(t))$ are deformations of the interval between c_i and c_{i+1} on I_1 for $1 \leq i \leq g-k$ and a deformation of the connected component of $I_1 \setminus \{c_{g-k+1}\}$ not containing $I_1 \setminus I_2$ for $i = g-k+1$. If follows $C_1(t), \cdots, C_k(t)$ is the only subset of k components C of $X_t(\mathbb{R})$ such that the union of the intervals $f_t(C)$ equals $\mathbb{P}^1(\mathbb{R})$. This implies $k(f_t) = k$.

Now we finish the proof for the orientable case $a = 0$ in case X is not an M-curve (in particular $s \leq g-1$). Let $b = (g+1-s)/2$, which is an integer. From the prevous part of the proof we obtain the existence of an M-curve Y of genus $g-b = s+b-1$ (hence $Y(\mathbb{R})$ has $s+b$ connected components) having a covering $f_Y : Y \rightarrow \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ satisfying the following property. Let $C_1, \cdots, C_{s+b}$ be the components of $Y(\mathbb{R})$ and let $I_i = f_Y(C_i)$. Then I_j and $I_{j'}$ with $1 \leq j; j' \leq k+b$ do have a point in common if and only if

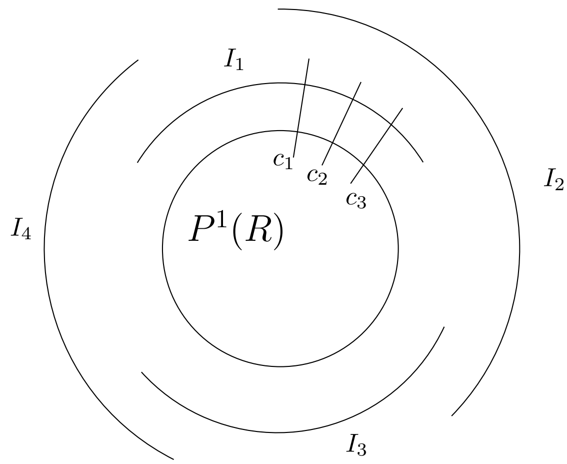

Figure 11: $k < s = g+1$

$$\begin{cases} j' = j+1 & \text{for } 1 \leq j \leq k+b-1 \\ j = k+b \text{ and } j' = 1 & \end{cases}$$

and the non-empty intersections are connected and $I_{k+b+1}, \cdots, I_{s+b}$ is contained in $I_1 \setminus (I_2 \cup I_{k+b})$ (see Figure 12 with $k+b=4$ and $s+b=6$).

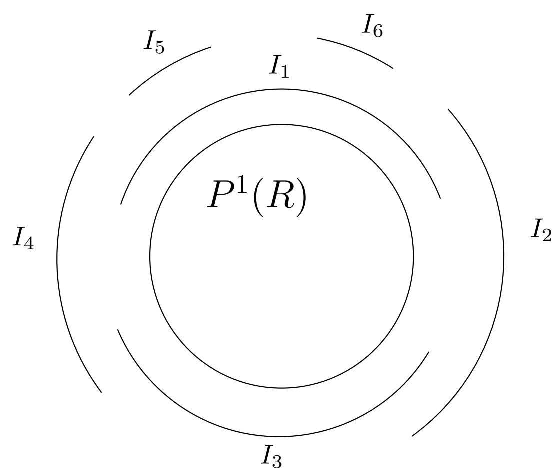

Figure 12: $a = 0$; $s < g+1$

For $1 \leq j \leq b$ let $p_j \in C_j$ and $q_j \in C_{j+1}$ with $f_Y(p_j) = f_Y(q_j) \in I_j \cap I_{j+1}$. Let $X_0 = Y_{p_j = q_j \text{ for } 1 \leq j \leq b}$. Using locally real deformations at the nodes $p_j = q_j$ one obtains a real curve X_t of genus $g(Y) + b = g$ and a covering $f_t = X_t \to \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$. The components of $X_t(\mathbb{R})$ are a deformation $C(t)$ of $(C_1 \cup \cdots \cup C_{b+1})_{p_j=q_j \text{ for } 1\le j\le b}$ and deformations $C_i(t)$ of C_i for $b+2 \le i \le s+b$. In particular $s(X_t) = s$. The image $f(C(t))$ is a deformation of $I_1 \cup \cdots \cup I_{b+1}$ and the image $f(C_i(t))$ for $b+2 \le i \le s+b$ are deformations of I_i. It follows that $k(f_t) = k$.

Finally we consider the non-orientable case (hence $a = 1$). First assume $s \equiv g \pmod 2$. Let Y be an orientable curve of genus $g-1$ with $s(Y) = s$ such that Y has a covering $f_Y : Y \to \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ with $k(f_Y) = k$. Let $c \in \mathbb{P}^1(\mathbb{R})$ such that $f_Y^{-1}(c)$ is not a totally real divisor and let $P + \overline{P} \subset f_Y^{-1}(c)$ for a non-real pair $P + \overline{P}$. Let $X_0 = Y_{P=\overline{P}}$, it is a real curve of genus $g+1$ having a covering of degree 4 defined over $\mathbb{R}$. Locally at the node $P = \overline{P}$ the curve is defined over $\mathbb{R}$ by $x^2 + y^2 = 0$ and the morphism by $(x, y) \mapsto x$. Take a local deformation of the the type $Z(x^2+y^2-t)$ with $t < 0$ and glue it to obtain a real curve X_t (this corresponds to Construction II without real ramification). Clearly X_t is non-orientable, $g(X_t) = g$, $s(X_t) = s$ and it has a covering $f_t : X_t \to \mathbb{P}^1$ defined over $\mathbb{R}$ of degree 4 with $k(f_Y) = k$. Finally assume $s \equiv g + 1 \pmod 2$. Since $s \le g - 1$ we can take a non-orientable real curve Y of genus $g-1$ such that $s(Y) = s$ and there is a covering $f_Y : Y \to \mathbb{P}^1$ of degree 4 defined over $\mathbb{R}$ with $k(f_Y) = k$. Repeating the previous construction one obtains the curve X_t one is looking for.

□

References

[1] N.L. Alling and N. Greenleaf, *Foundations of the Theory of Klein Surfaces*, Lecture Notes in Mathematics, vol. 219, Springer-Verlag, Berlin, 1971.

[2] E. Arbarello and M. Cornalba, *Su una congettura di Petri*, Comment. Math. Helv. **56** (1981), 1-38.

[3] ———, *Footnotes to a paper of Beniamino Segre*, Math. Ann. **256** (1981), 341-362.

[4] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves Volume I*, Grundlehren den Math. Wiss., vol. 267, Springer-Verlag, Berlin, 1985.

[5] E. Ballico, *Real algebraic curves and real spanned bundles*, Ricerche di Matematica **50** (2001), 223-241.

[6] ———, *Real curves with fixed gonality and empty real locus*, Le matematiche **60** (2005), 129-131.

[7] ———, *Real ramification points and real Weierstrass points of real projective curves*, Glasnik Matematički **41** (2006), 233-238.

[8] E. Biswas, *On line bundles over real algebraic curves*, Bull. Sci. Math. **134** (2010), 447-449.

[9] S. Chaudhary, *The Brill-Noether theorem for real algebraic curves*, Ph.-D. Thesis, Duke University, 1995.

[10] A.F. Costa and M. Izquierdo, *On real trigonal Riemann surfaces*, Math. Scand. **98** (2006), 53-68.

[11] M. Coppens, *Totally non-real divisors in linear systems on smooth real curves*, Adv. Geometry **8** (2008), 551-555.

[12] M. Coppens and G. Martens, *Linear pencils on real algebraic curves*, Journal Pure Applied Algebra **214** (2010), 841-849.

[13] A. Gabard, *Sur la representation conforme des surfaces de Riemann à bord et une caractérisation des courbes séparantes*, Comment. Math. Helv. **81** (2006), 945-964.

[14] B.H. Gross and J. Harris, *Real algebraic curves*, Ann. scient. Ec. Norm. Sup. **14** (1981), 157-182.

[15] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géometrie algébrique IV Les schémas de Hilbert*, Séminaire Bourbaki **221** (1960/61).

[16] ______, *Techniques de construction et théorèmes d'existence en géometrie algébrique V Les schémas de Picard*, Séminaire Bourbaki **232** (1961/62).

[17] Horikawa, *On deformations of holomorphic maps I*, J. Math. Soc. Japan **25** (1973), 372-396.

[18] J. Huisman, *Real quotient singularities and nonsingular real algebraic curves in the boundary of the moduli space*, Compos. Math. **118** (1999), 42-60.

[19] ______, *Real Teichmüller spaces and moduli of real algebraic curves*, Contemporary Mathematics **253** (2000), 145-179.

[20] G. Martens, *Minimale Blätterzahl bei Uberlagerungen Kleinschen Flächen der projektiven Ebene*, Achiv der Math. **30** (1978), 481-486.

[21] J.-Ph. Monnier, *Very special divisors on real algebraic curves*, Bull. London Math. Soc., to appear.

[22] B. Segre, *Sui moduli delle curve poligonali e sopra un complemento al teorema di esistenza di Riemann*, Math. Ann. **100** (1928), 537-551.

[23] E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Math. Wiss., vol. 333, Springer-Verlag, Berlin, 2006.

[24] M. Teixidor, *Half-canonical series on algebraic curves*, Transactions of the AMS **302** (1987), 99-115.

[25] G. Weichold, *Uber symmetrische Riemann'sche Flächen und die Periodicitätsmoduln zugehörigen Abel'schen Normalintegrale erster Gattung*, Zeit. f. Math. u. Phys. **28** (1883), 321-351.

Marc Coppens
Katholieke Hogeschool Kempen
Departement Industrieel Ingenieur en Biotechniek
KULeuven Dept. Wiskunde Groep Algebra
Kleinhoefstraat 4
B 2440 Geel Belgium
E-mail: marc.coppens@khk.be
Partially supported by the Fund of Scientific Research - Flanders (G.0318.06)

Johannes Huisman
Université Européenne de Bretagne
France
and
Université de Brest; CNRS, UMR 6205
Laboratoire de Mathématiques de Brest
ISSTB
6, avenue Victor Le Gorgeu

CS 93837
29238 Brest Cedex 3 France
E-mail: johannes.huisman@univ-brest.fr
Home page: http://pageperso.univ-brest.fr/∼huisman