Real hypersurfaces having many pseudo-hyperplanes

J. Huisman

Abstract

Let n and d be natural integers satisfying $n \geq 3$ and $d \geq 10$. Let X be an irreducible real hypersurface X in $\mathbb{P}^n$ of degree d having many pseudo-hyperplanes. Suppose that X is not a projective cone. We show that the arrangement $\mathcal{H}$ of all $d-2$ pseudo-hyperplanes of X is trivial, i.e., there is a real projective linear subspace L of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$ such that $L \subseteq H$ for all $H \in \mathcal{H}$. As a consequence, the normalization of X is fibered over $\mathbb{P}^1$ in quadrics. Both statements are in sharp contrast with the case $n = 2$; the first statement also shows that there is no Brusotti-type result for hypersurfaces in $\mathbb{P}^n$, for $n \geq 3$.

MSC 2000: 14P25, 52C35

Keywords: real hypersurface, quasi-hyperplane, pseudo-hyperplane, pseudo-line, arrangement of pseudo-lines, arrangement of pseudo-hyperplanes, hyperelliptic curve, fibration in quadrics

1 Introduction

Let X be an irreducible real algebraic curve in $\mathbb{P}^2$. Let d be its degree. A pseudo-line of X is a global real analytic branch of the set of real points of X that is homotopically nontrivial in $\mathbb{P}^2(\mathbb{R})$ and that is homeomorphic to $\mathbb{P}^1(\mathbb{R})$. It can be shown that X has at most $d-2$ pseudo-lines [1]. The curve X is said to have many pseudo-lines if it has exactly $d-2$ pseudo-lines.

One can easily construct real plane curves having many pseudo-lines. Indeed, let $L_1, \ldots, L_{d-2}$ be real projective lines in general position in $\mathbb{P}^2$. Let $C \subseteq \mathbb{P}^2$ be an irreducible real conic such that its set of real points $C(\mathbb{R})$ does not intersect $L_i(\mathbb{R})$, for $i = 1, \ldots d-2$. Let $\overline{X}$ be the union of C and the lines $L_1, \ldots, L_{d-2}$. Note that $\overline{X}$ has real singularities as well as nonreal singularities. A well known result of Brusotti [2] states that one can "deform away" the nonreal singularities of $\overline{X}$, keeping the real singularities of $\overline{X}$. What one obtains is an irreducible real plane curve X having many pseudo-lines.

An irreducible real plane curve having many pseudo-lines determines an arrangement of pseudo-lines in the real projective plane $\mathbb{P}^2(\mathbb{R})$. For example, the plane curve constructed above determines a generic arrangement of pseudo-lines, i.e., any 3 pseudo-lines of X have empty intersection. An interesting problem is to determine the arrangements of pseudo-lines that can be realized as arrangement of pseudo-lines of an irreducible real plane curve having many pseudo-lines. For example, one may wonder whether any arrangement of true lines in the real projective plane can be realized as an arrangement of pseudo-lines of an irreducible real plane curve having many pseudo-lines. When studying such questions, one is naturally led to study real hypersurfaces in real projective n-space having many pseudo-hyperplanes, since plane sections of such hypersurfaces are real plane curves having many pseudo-lines.

Let X be an irreducible real hypersurface in $\mathbb{P}^n$. Let d be its degree. A pseudo-hyperplane of X is a global real analytic branch of the set of real points of X that is homotopically nontrivial in $\mathbb{P}^n(\mathbb{R})$ and that is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$. It can be shown that X has at most $d-2$ pseudo-hyperplanes [3] (cf. also Section 2). The hypersurface X is said to have many pseudo-hyperplanes if it has exactly $d-2$ pseudo-hyperplanes.

We show the following. Let X be an irreducible real hypersurface in $\mathbb{P}^n$ of degree d having many pseudo-hyperplanes. Suppose that $n \geq 3$ and $d \geq 10$, i.e, X has at least 8 pseudo-hyperplanes. Assume that X is not a projective cone. Then the arrangement $\mathcal{H}$ of all $d-2$ pseudo-hyperplanes of X is trivial, i.e., there is a real projective linear subspace L of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$ such that $L \subseteq H$ for all $H \in \mathcal{H}$ (Theorem 3.3). To put it otherwise, the only irreducible real hypersurfaces in $\mathbb{P}^n$, $n \geq 3$, of degree ≥ 8 having many pseudo-hyperplanes such that the arrangement of pseudo-hyperplanes is nontrivial, are projective cones over real plane curves having many pseudo-lines!

An interesting consequence of this fact is that we do not have a Brusotti-type result in higher dimension. More explicitly, Let $H_1, \ldots, H_8$ be real projective hyperplanes in $\mathbb{P}^n$ in general position, with $n \geq 3$. Let $C \subseteq \mathbb{P}^n$ be a real quadric such that $C(\mathbb{R}) \cap H_i(\mathbb{R}) = \emptyset$ for $i = 1, \ldots, 8$. Then any irreducible deformation of the union $\overline{X}$ of C and $H_1, \ldots, H_8$ has to smooth out some of the real singularities of $\overline{X}$! If, however, the real projective hyperplanes $H_1, \ldots, H_8$ constitute a trivial arrangement, there is an irreducible deformation of $\overline{X}$ that "deforms away" only nonreal singularities.

Another consequence of Theorem 3.3 concerning irreducible real hypersurfaces having many pseudo-hyperplanes, is that the normalization of such a hypersurface X is fibered over $\mathbb{P}^1$ in quadrics. Such varieties can be considered to be higher-dimensional analogues of hyperelliptic curves. And indeed, we show that an irreducible real plane curve having many pseudo-lines determines a trivial arrangement of pseudo-lines if and only if its normalization is hyperelliptic (Theorem 3.2).

Concerning the problem of determining the arrangements of pseudo-lines that are realizable as the arrangement of an irreducible real plane curve having many pseudo-lines, Theorem 3.3 shows that, unfortunately, the class of irreducible real hypersurfaces having many pseudo-hyperplanes seems to be too restricted in order to produce interesting real plane curves having many pseudo-lines that have a nontrivial or nongeneric arrangement of pseudo-lines. Probably, one should study the larger class of irreducible real hypersurfaces having many quasi-hyperplanes (see Section 2).

Convention and notation. Projective n-space over $\mathbb{R}$ is denoted by $\mathbb{P}^n$ instead of $\mathbb{P}^n_{\mathbb{R}}$.

2 Real hypersurfaces having many quasi-hyperplanes

Let n be a nonzero natural integer. Let $X \subseteq \mathbb{P}^n$ be a real hypersurface, i.e., X is defined by a nonconstant homogeneous real polynomial in the variables $X_0, \ldots, X_n$. Note that we do not assume X to be reduced, irreducible or smooth. The set of real points $X(\mathbb{R})$ of X is a real analytic subvariety of $\mathbb{P}^n(\mathbb{R})$. Let C be an irreducible global real analytic branch of $X(\mathbb{R})$. Then C is a compact connected real analytic subvariety of $\mathbb{P}^n(\mathbb{R})$. Its dimension is at most $n-1$. By [4], C realizes a $\mathbb{Z}/2\mathbb{Z}$-homology class $[C]$ in $H_{n-1}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. This homology class vanishes if $\dim(C) < n-1$. The real branch C of X is a *quasi-hyperplane* of X if $[C] \neq 0$. Equivalently, C is a quasi-hyperplane if any real projective line L in $\mathbb{P}^n(\mathbb{R})$ that is not contained in C, intersects C in an odd number of points, when counted with multiplicities. A quasi-hyperplane C of X is a *pseudo-hyperplane* of X if C is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$. In case $n = 2$, a quasi-hyperplane (resp. pseudo-hyperplane) of X is called a *quasi-line* (resp. *pseudo-line*). In case $n = 3$, it is called a *quasi-plane* (resp. *pseudo-plane*).

Note that the dimension of a quasi-hyperplane of X is equal to $n-1$. Also, a pseudo-hyperplane is a quasi-hyperplane, but not conversely. A trivial example of a quasi-hyperplane that is not a pseudo-hyperplane is the following. Let $X \subseteq \mathbb{P}^2$ be the real plane curve given by the affine equation $y^2 = x^2(x+1)$. Then $X(\mathbb{R})$ is an irreducible global real analytic subvariety of $\mathbb{P}^2(\mathbb{R})$. Clearly, $X(\mathbb{R})$ is a quasi-line of X. Since $X(\mathbb{R})$ is not homeomorphic to $\mathbb{P}^1(\mathbb{R})$, $X(\mathbb{R})$

is not a pseudo-line. Refer to Example 2.4.2 for a less trivial example of a quasi-hyperplane that is not a pseudo-hyperplane.

The following three statements haven been shown in [3].

Proposition 2.1. *Let n and d be nonzero natural integers. Let X be a real hypersurface of $\mathbb{P}^n$ of degree d. Then, the number of quasi-hyperplanes of X, when counted with multiplicities, is congruent to d (mod 2).* □

Proposition 2.2. *Let n and d be nonzero natural integers. Let X be a real hypersurface of $\mathbb{P}^n$ of degree d. Then, X has at most d quasi-hyperplanes, when counted with multiplicities.* □

Proposition 2.3. *Let n and d be nonzero natural integers. Let X be a real hypersurface of $\mathbb{P}^n$ of degree d. Then, X has exactly d quasi-hyperplanes if and only if X is the scheme-theoretic union of d real hyperplanes.* □

By Propositions 2.1, 2.2 and 2.3, a real hypersurface of degree d having at least $d-1$ quasi-hyperplanes is the scheme-theoretic union of real hyperplanes. While these hypersurfaces are interesting from a combinatorial point of view, they do not seem to be very interesting from a geometric point of view. This motivates the following definition.

Let n and d be nonzero natural integers, with $d \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a real hypersurface of degree d. We say that X *has many quasi-hyperplanes* if X has exactly $d-2$ quasi-hyperplanes.

Examples 2.4.

1. A real hypersurface X in $\mathbb{P}^1$ of degree $d \geq 2$ has many quasi-hyperplanes if and only if the number of real points of X is equal to $d-2$, when counted with multiplicities.

2. Let $n \geq 2$. Any irreducible real cubic hypersurface in $\mathbb{P}^n$ has exactly 1 quasi-hyperplane by Propositions 2.1, 2.2 and 2.3. Therefore, an irreducible real cubic hypersurface X in $\mathbb{P}^n$ has many quasi-hyperplanes. If $n = 3$ and X is smooth, the quasi-plane C of X is homeomorphic to the i-fold connected sum of $\mathbb{P}^2(\mathbb{R})$, for some $i \in \{1, 3, 5, 7\}$. Moreover, each of these 4 homeomorphism types can be realized. These results are due to Segre (cf. [5, §6.1]). In particular, a smooth quasi-hyperplane of a real hypersurface having many quasi-hyperplanes is not necessarily a pseudo-hyperplane.

3. Let $\tilde{X}$ be an abstract, i.e., non embedded, proper smooth connected real algebraic curve. Let g be the genus of $\tilde{X}$, and suppose that $\tilde{X}(\mathbb{R})$ has at least g connected components. Such curves abound. Choose any divisor D on $\tilde{X}$ of degree $g+2$, and having odd degree on g connected

components of $\tilde{X}(\mathbb{R})$. Then the linear system $|D|$ defines a birational embedding of $\tilde{X}$ in $\mathbb{P}^2$ whose image X is a real curve in $\mathbb{P}^2$ having many quasi-lines [7].

4. Let n and d be natural integers, with $n \geq 2$ and $d \geq 3$. Choose two relatively prime separable polynomials $P, Q \in \mathbb{R}[X_1]$ such that $\deg(P) = d$, $\deg(Q) = d - 3$, all roots of Q are real, and having the following property. Between any two consecutive real roots of Q there is exactly one real root of P, and any real root of Q is between two consecutive real roots of P. Let $X \subseteq \mathbb{P}^n$ be the real hypersurface defined by the affine equation
$$(X_2^2 + \cdots + X_n^2)Q(X_1) = P(X_1).$$
Then, X is a real hypersurface in $\mathbb{P}^n$ of degree d having many pseudo-hyperplanes. Indeed, X is the real hypersurface of revolution obtained from the real plane curve defined by the affine equation $X_2^2 Q(X_1) = P(X_1)$. The latter curve is easily seen to have many pseudo-lines [6]. Therefore, X has many pseudo-lines.

The following statement allows to construct, out of the above examples, yet other examples of real hypersurfaces having many quasi-hyperplanes.

Proposition 2.5. *Let n be a nonzero natural integer. Let $X \subseteq \mathbb{P}^n$ be a real hypersurface. Let $\hat{X} \subseteq \mathbb{P}^N$ be a projective cone over X, for some $N \geq n$. Then, $\hat{X}$ has many quasi-hyperplanes if and only if X has many quasi-hyperplanes.*

Proof. The degree of $\hat{X}$ is equal to the degree of X. Moreover, the number of quasi-hyperplanes of $\hat{X}$ is equal to the number of quasi-hyperplanes of X. $\square$

Let n be a nonzero natural integer. Let $X \subseteq \mathbb{P}^n$ be a real hypersurface having many quasi-hyperplanes. We say that X is *primitive* if X is not a projective cone over a real hypersurface of dimension strictly less than the dimension of X.

Proposition 2.6. *Let n and d be natural integers, $d \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a real hypersurface of degree d having many quasi-hyperplanes. Then, there is a unique irreducible real hypersurface $X^\heartsuit$ contained in X that has many quasi-hyperplanes. Moreover, $X^\heartsuit$ is reduced, and X is the scheme-theoretic union of $X^\heartsuit$ and a finite number of projective hyperplanes of $\mathbb{P}^n$.*

Proof. Let $X_1, \ldots, X_k$ be the irreducible components of X with the induced scheme structure. Then each X_i is a real hypersurface in $\mathbb{P}^n$. Let d_i be the degree of X_i. Then $d = d_1 + \cdots + d_k$. Let p_i be the number of quasi-hypersurfaces of X_i. Then $d-2 = p_1 + \cdots + p_k$. By Proposition 2.2, $p_i \leq d_i$, and by Proposition 2.1, $p_i \equiv d_i \pmod 2$. Hence, there is exactly one i such that $p_i = d_i - 2$, and $p_j = d_j$ for $j \neq i$. It follows that $X^\heartsuit = X_i$ is the unique irreducible real hypersurface $X^\heartsuit$ contained in X that has many quasi-hyperplanes. Moreover, by Proposition 2.3, each X_j is a real hyperplane with multiplicity d_j. □

In the sequel we restrict our attention to primitive irreducible real hypersurfaces having many quasi-hyperplanes.

Proposition 2.7. *Let n and d be nonzero natural integers, with $n, d \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a primitive irreducible real hypersurface of degree d having many quasi-hyperplanes. Suppose that X has two distinct quasi-hyperplanes H and H'. Let $I = H \cap H'$. Then, all points of I are smooth points of H and of H'. Moreover, the intersection of H and H' is transverse. Furthermore, I is a real projective linear subspace of $\mathbb{P}^n(\mathbb{R})$ of codimension 2.*

Proof. Let $P \in I$. We show that P is a smooth point of H. Suppose to the contrary, that P is not a smooth point of H. Let $K \subseteq \mathbb{P}^n$ be any projective linear hyperplane of $\mathbb{P}^n$ not containing P. Let Q be a point of $H' \cap K(\mathbb{R})$, and let L be the real projective line passing through P and Q. We show that $L \subseteq X(\mathbb{R})$.

Suppose that $L \not\subseteq X(\mathbb{R})$. Then L is not contained in any global real analytic branch of X. Since P is a singular point of H, the degree of the intersection product $L \cdot H$ is at least 2. Since H is a quasi-hyperplane, this degree is at least 3. Also, since L intersects H' in 2 distinct points P and Q, and since H' is a quasi-hyperplane, the degree of the intersection product $L \cdot H'$ is at least 3. For each of the other $d-4$ quasi-hyperplanes H'' of X one has that the degree of the intersection product $L \cdot H''$ is at least 1. Let $\overline{L}$ be the Zariski closure of L in $\mathbb{P}^n$. Then the intersection product $L \cdot X$ has degree at least $3+3+(d-4) = d+2$. Contradiction by Bezout's Theorem. Therefore $L \subseteq X(\mathbb{R})$.

It follows that the the real projective cone over $H' \cap K(\mathbb{R})$ with vertex P in $\mathbb{P}^n(\mathbb{R})$ is entirely contained in $X(\mathbb{R})$. Hence, its Zariski closure Y in $\mathbb{P}^n$ is contained in X. Since $[H'] \cdot [K(\mathbb{R})] \neq 0$ in $H_{n-2}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$, the intersection $H' \cap K(\mathbb{R})$ has dimension $n-2$. It follows that Y is a real hypersurface in $\mathbb{P}^n$. Since X is irreducible, $Y = X$. This contradicts the fact that X is primitive. Therefore, P is a smooth point of H. By symmetry, P is a smooth point of H'.

Let us show that the intersection of H and H' is transverse. Let $P \in I$. Suppose that the intersection of H and H' is not transverse at P. Let L be a real projective line in $\mathbb{P}^n(\mathbb{R})$ that is tangent to H at P. Then L is also tangent to H' at P. Using Bezout's Theorem as above, L is contained in $X(\mathbb{R})$. It follows that the real projective tangent space to H at P is contained in $X(\mathbb{R})$. Contradiction. Therefore, the intersection of H and H' is transverse.

Since $[H]\cdot[H'] \neq 0$ in $H_{n-2}(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$, the intersection $I = H \cap H'$ is nonempty and of dimension $n-2$. We show that I is a real projective linear subspace of $\mathbb{P}^n(\mathbb{R})$. We may assume that $n \geq 3$. Using Bezout's Theorem as above, any real projective line in $\mathbb{P}^n(\mathbb{R})$ that intersects I in at least 2 points is contained in $X(\mathbb{R})$. Suppose that I is not a projective linear subspace of $\mathbb{P}^n(\mathbb{R})$. Then there is a point $P \in I$ such that the real projective cone in $\mathbb{P}^n(\mathbb{R})$ on $I \setminus \{P\}$ with vertex P has dimension at least $n-1$. By the preceding remark, this cone is entirely contained in $X(\mathbb{R})$. This contradicts the fact that X is primitive. Hence, I is a real projective linear subspace of dimension $n-2$. □

3 Real hypersurfaces having many pseudo-hyperplanes

Theorem 3.1. *Let n and d be natural integers, with $n \geq 3$ and $d \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a primitive irreducible real hypersurface of degree d having many pseudo-hyperplanes. If $d \geq 4$ then each pseudo-hyperplane of X contains at most 3 real projective linear subspaces of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$.*

Proof. Since $d \geq 4$, X has at least 2 pseudo-hyperplanes. Let H and H' be distinct pseudo-hyperplanes of X. By Proposition 2.7, the intersection $I = H \cap H'$ is a real projective linear subspace of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$. Since the points of I are smooth points of H, the restriction to $H \setminus I$ of the linear projection from $\mathbb{P}^n(\mathbb{R}) \setminus I$ onto $\mathbb{P}^1(\mathbb{R})$ extends to a real analytic map

$$\pi\colon H \to \mathbb{P}^1(\mathbb{R}).$$

Let L be a real projective linear subspace of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$ that is contained in H and that is distinct from I. Since H is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$, the intersection $L \cap I$ is a real projective linear subspace of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-3$. Let K be the hyperplane of $\mathbb{P}^n$ such that $K(\mathbb{R})$ contains I and L. By Proposition 2.1, 2.2 and 2.3, $K(\mathbb{R})$ intersects H in exactly 3 real projective linear subspaces of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$, say I, L and L'. It follows that the union $L \cup L'$ is a fiber of π.

Now, let L'' be any real projective linear subspace of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$ contained in H. We show that $L'' \in \{I, L, L'\}$. We may assume that $L'' \neq I$. Then, as above, L'' is contained in a fiber of π. Since H is homeomorphic to $\mathbb{P}^{n-1}(\mathbb{R})$, the intersection $L'' \cap L$ is nonempty. It follows that $L'' \subseteq L \cup L'$. Hence, $L'' \in \{I, L, L'\}$. □

It is interesting to compare Theorem 3.1 with the main result of [1] that states that each pseudo-line of a real plane curve having many pseudolines, contains exactly 3 inflection points.

Let n and d be natural integers, with $n, d \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a primitive irreducible real hypersurface of degree d having many pseudo-hyperplanes. By Proposition 2.7, the set of pseudo-hyperplanes of X constitutes an arrangement of pseudo-hyperplanes in $\mathbb{P}^n(\mathbb{R})$. Let $\mathcal{H}$ be that arrangement. We say that $\mathcal{H}$ is *trivial* if there is a real projective linear subspace L of $\mathbb{P}^n(\mathbb{R})$ of dimension $n-2$ such that L is contained in each member H of $\mathcal{H}$.

Theorem 3.2. *Let d be a natural integer satisfying $d \geq 2$. Let $X \subseteq \mathbb{P}^2$ be an irreducible real algebraic curve of degree d having many pseudo-lines. Let $\mathcal{H}$ be the arrangement of pseudo-lines of X. Suppose that $d \geq 4$, and assume that the normalization $\tilde{X}$ of X is of genus $d-2$. Then, $\mathcal{H}$ is trivial if and only if $\tilde{X}$ is hyperelliptic.*

Proof. Suppose that $\mathcal{H}$ is trivial. Let P be the real point of $\mathbb{P}^2$ such that all pseudo-lines of X contain P. The linear projection from $\mathbb{P}^2 \setminus \{P\}$ onto $\mathbb{P}^1$ induces a morphism from $\tilde{X}$ onto $\mathbb{P}^1$ of degree 2. It follows that $\tilde{X}$ is hyperelliptic.

Conversely, suppose that $\tilde{X}$ is hyperelliptic. Let $g = d-2$ be the genus of $\tilde{X}$. By [7], the morphism f from $\tilde{X}$ into $\mathbb{P}^2$ is determined by a complete linear system $|D|$, where D is an effective divisor on $\tilde{X}$ of degree d having odd degree at g connected components $C_1, \ldots, C_g$ of $\tilde{X}(\mathbb{R})$. Let E be the hyperelliptic divisor on $\tilde{X}$, i.e., E is an effective divisor on $\tilde{X}$ of degree 2 such that $|E|$ is 1-dimensional. By Riemann-Roch, the summation map

$$C_1 \times \cdots \times C_g \longrightarrow \operatorname{Pic}^g(\tilde{X})$$

is surjective on a connected component of $\operatorname{Pic}^g(\tilde{X})$. It follows that there is an effective divisor F on $\tilde{X}$ of the form $P_1 + \cdots + P_g$, where $P_i \in C_i$, such that $E + F \in |D|$. This means that $f(P_i) = f(P_j)$ for all $i, j \in \{1, \ldots, g\}$. Hence, $\mathcal{H}$ is trivial. □

The condition in Theorem 3.2 on X to have a normalization of genus $d-2$ is not a severe restriction (cf. Example 2.4.3). Theorem 3.2 implies that "most" real algebraic curves in $\mathbb{P}^2$ of degree ≥ 5 having many pseudo-lines define a nontrivial arrangement of pseudo-lines. The next statement shows that in higher dimension the opposite is true.

Theorem 3.3. *Let n and d be natural integers, with $n \geq 3$ and $d \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a primitive irreducible real hypersurface of degree d having many pseudo-hyperplanes. Let $\mathcal{H}$ be the arrangement of pseudo-hyperplanes of X. If $d \geq 10$ then $\mathcal{H}$ is trivial.*

Proof. We define a line geometry on the finite set $\mathcal{H}$ as follows. A *line* in $\mathcal{H}$ is a subset of $\mathcal{H}$ of the form

$$\{H \in \mathcal{H} \,|\, I \subseteq H\},$$

for some real projective linear subspace I of $\mathbb{P}^n(\mathbb{R})$ that is the intersection of two distinct pseudo-hyperplanes of X. Let $\mathcal{L}$ be the set of lines in $\mathcal{H}$. By definition, a line in $\mathcal{H}$ contains at least 2 elements. By Proposition 2.7, two distinct lines in $\mathcal{H}$ intersect in at most 1 element of $\mathcal{H}$. Moreover, through two distinct elements of $\mathcal{H}$ passes exactly one line.

Now, suppose that $\mathcal{H}$ is not a trivial arrangement of pseudo-hyperplanes in $\mathbb{P}^n(\mathbb{R})$. Let $h = \#\mathcal{H}$ and $\ell = \#\mathcal{L}$. Since $\mathcal{H}$ is not trivial, there are at least two lines in $\mathcal{H}$. By Theorem 3.1, any $H \in \mathcal{H}$ is contained in at most 3 distinct lines. It follows that any line of $\mathcal{H}$ contains at most 3 elements. Let us show that any $H \in \mathcal{H}$ is contained in exactly 3 distinct lines. Suppose that $H \in \mathcal{H}$ is contained in strictly less than 3 lines. Since $\mathcal{H}$ is nontrivial, H is contained in exactly 2 lines L and L'. Then, $L \cup L' = \mathcal{H}$. It follows that $h \leq 3 + 3 - 1 = 5$. Contradiction, since $h = d - 2 \geq 8$. Therefore, each $H \in \mathcal{H}$ is contained in exactly 3 lines.

We show, similarly, that each line of $\mathcal{H}$ contains exactly 3 elements. Suppose that $L \in \mathcal{L}$ contains strictly less than 3 elements. Then L contains exactly 2 elements. Let $H \in L$. Since H is contained in exactly 3 lines, let L' and L'' be the other lines containing H. Then $\mathcal{H} = L \cup L' \cup L''$. Since any line of $\mathcal{H}$ contains at most 3 elements, $h \leq 2 + 3 + 3 - 1 - 1 = 6$. Contradiction. Therefore, each $L \in \mathcal{L}$ contains exactly 3 elements.

Counting the elements of $\mathcal{H}$ line by line, one gets $3\ell = 3h$, i.e., $\ell = h$. Moreover, any unordered pair of distinct points of $\mathcal{H}$ determines a line. Counting the lines of $\mathcal{H}$ in this way, one gets $\frac{1}{2}(h^2 - h) = 3\ell$. Hence, $\frac{1}{2}(h^2 - h) = 3h$. It follows that $h = 0$ or 7. Contradiction. □

Theorem 3.4. *Let n and d be natural integers, with $n \geq 3$ and $d \geq 2$. Let $X \subseteq \mathbb{P}^n$ be a primitive irreducible real hypersurface of degree d having many pseudo-hyperplanes. Let $\tilde{X}$ be the normalization of X. If $d \geq 10$ then $\tilde{X}$ is fibered over $\mathbb{P}^1$ in quadrics.*

Proof. According to Theorem 3.3, there is a real projective linear subspace I of $\mathbb{P}^n(\mathbb{R})$ of dimension $n - 2$ that is contained in every pseudo-hyperplane of X. Let $\overline{I}$ be the Zariski closure of I in $\mathbb{P}^n$. Then, $\overline{I}$ is a projective linear subspace of $\mathbb{P}^n$ of dimension $n - 2$ which is contained in X. The restriction to $X \setminus \overline{I}$ of the linear projection from $\mathbb{P}^n \setminus \overline{I}$ onto $\mathbb{P}^1$ induces a morphism

$$\pi \colon \tilde{X} \longrightarrow \mathbb{P}^1.$$

Since all pseudo-hyperplanes of X contain I, the morphism π is a fibration in quadrics. $\square$

References

[1] Huisman, J. Inflection points on real plane curves having many pseudo-lines. Beiträge Algebra Geom. **2001** *42*, 509–516.

[2] Brusotti, L. Sulla generazzione di curve piane algebriche reali mediante "piccola variazione" di una curva spezzata. Annali di Mat. (3) **1913** *22*, 117–169.

[3] Huisman, J. Real cubic hypersurfaces and group laws. (submitted)

[4] Borel, A.; Haefliger, A. La classe d'homologie fondamentale d'un espace analytique. Bull. Soc. Math. Fr. **1961** *89*, 461–513.

[5] Benedetti, R.; Silhol, R. Spin and Pin^- structures, immersed and embedded surfaces and a result of Segre on real cubic surfaces. Topology **1995** *34*, 651–678.

[6] Fichou, G. Loi de groupe sur la composante neutre de la jacobienne d'une courbe hyperelliptique réelle ayant beaucoup de composantes réelles. (submitted)

[7] Huisman, J. Nonspecial divisors on real algebraic curves and embeddings into real projective spaces. Ann. Mat. Pura Appl. (4) (to appear)

Institut de Recherche Mathématique de Rennes
Université de Rennes 1
Campus de Beaulieu
35042 Rennes Cedex
France
E-mail: huisman@univ-rennes1.fr
Home page: http://www.maths.univ-rennes1.fr/~huisman/

Typeset by $\mathcal{AMS}$-LaTeX