EVERY ORIENTABLE SEIFERT 3-MANIFOLD IS A REAL COMPONENT OF A UNIRULED ALGEBRAIC VARIETY

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ABSTRACT. We show that any orientable Seifert 3-manifold is diffeomorphic to a connected component of the set of real points of a uniruled real algebraic variety, and prove a conjecture of János Kollár.

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1. INTRODUCTION

A smooth compact connected 3-manifold M is a *Seifert manifold* if it admits a smooth fibration $f\colon M \to S$ over a smooth surface S, whose fibers are circles, such that f is locally trivial, with respect to the ramified Grothendieck topology on S. More precisely, for every point P of S, there is a ramified covering $\tilde{U} \to U$ of an open neighborhood U of P such that the fiber product $M \times_S \tilde{U} \to \tilde{U}$ is a locally trivial smooth circle bundle over $\tilde{U}$ (see Section 2 for another—but equivalent—definition).

A smooth, projective and geometrically irreducible real algebraic variety X is called *ruled* if there is a real algebraic variety Y such that $Y \times \mathbb{P}^1$ and X are birational. The variety X is *uniruled* if there is a real algebraic variety Y, with $\dim(Y) = \dim(X) - 1$, and a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$. Of course, a ruled real algebraic variety is uniruled, but not conversely.

Let X be a uniruled real algebraic variety of dimension 3 such that $X(\mathbb{R})$ is orientable. János Kollár has proved that each connected component of $X(\mathbb{R})$ belongs to a given list of manifolds, containing the Seifert manifolds [Ko01, Th. 6.6]. He conjectured, conversely, that each orientable Seifert manifold is diffeomorphic to a connected component of the set of real points of a uniruled real algebraic variety [Ko01, Conj. 6.7.2]. In this paper we prove that conjecture.

Theorem 1.1. *Every orientable Seifert manifold is diffeomorphic to a real component of a uniruled real algebraic variety.*

The strategy of our proof is the following. Let M be an orientable Seifert manifold, and let $f\colon M \to S$ be a Seifert fibration as above. In case M admits a spherical geometry, Kollár has constructed a uniruled real algebraic variety having a real component diffeomorphic to M. Therefore, we may assume that M does not admit a spherical geometry. Then, we show that there is a ramified Galois covering $p\colon \tilde{S} \to S$ of smooth surfaces such that the fiber product

$$\tilde{f}\colon \tilde{M} = M \times_S \tilde{S} \longrightarrow \tilde{S}$$

is a locally trivial circle bundle (Theorem 2.3). In particular, there is a finite group G acting on the fiber bundle $\tilde{f}$ such that $\tilde{f}/G \cong f$. We show that there is a

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structure of a real algebraic surface on $\tilde{S}$, and that there is a real algebraic vector bundle $\tilde{L}$ of rank 2 on $\tilde{S}$ admitting

(1) a real algebraic action of G on the total space of $\tilde{L}$, and
(2) a G-equivariant real algebraic metric λ on $\tilde{L}$,

such that the unit circle bundle in $\tilde{L}$ is G-equivariantly diffeomorphic to $\tilde{M}$. The statement of Theorem 1.1 will then follow.

Conventions. A manifold is smooth, compact and connected, and without boundary, unless stated otherwise. A Riemann surface is compact and connected. A real algebraic variety is smooth projective and geometrically irreducible, unless stated otherwise.

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2. Seifert fibrations

Let $S^1 \times D^2$ be the *solid torus* where S^1 is the unit circle $\{u \in \mathbb{C}, |u| = 1\}$ and D^2 is the closed unit disc $\{z \in \mathbb{C}, |z| \leq 1\}$. A *Seifert fibration* of the solid torus is a smooth map

$$f_{p,q} : S^1 \times D^2 \to D^2 \ , (u, z) \mapsto u^q z^p \ ,$$

where p, q are relatively prime integers satisfying $0 \leq q < p$.

Definition 2.1. *Let M be a 3-manifold. A* Seifert fibration *of M is a smooth map $f : M \to S$ to a surface S having the following property. Every $P \in S$ has a closed neighborhood U such that the restriction of f to $f^{-1}(U)$ is diffeomorphic to a Seifert fibration of the solid torus. More precisely, there are relatively prime integers p and q, with $0 \leq q < p$, and there are diffeomorphisms $g\colon U \to D^2$ and $h\colon f^{-1}(U) \to S^1 \times D^2$ such that the diagram*

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow[h]{} & S^1 \times D^2 \\ \Big\downarrow {\scriptstyle f_{|f^{-1}(U)}} & & \Big\downarrow {\scriptstyle f_{p,q}} \\ U & \xrightarrow[g]{} & D^2 \end{array}$$

commutes. We will say that M is a Seifert manifold *if M admits a Seifert fibration.*

In the literature, e.g. [Sc83], nonorientable local models are also allowed. Following Kollár, we kept Seifert's original definition of a Seifert manifold.

Let us show that Seifert fibrations, as defined above, satisfy the property mentioned in the Introduction. The converse is easy to prove, and is left to the reader.

Proposition 2.2. *Let $f\colon M \to S$ be a Seifert fibration. Then, for every $P \in S$, there is a ramified covering $\tilde{U} \to U$ of an open neighborhood U of P such that the the fiber product $M \times_S \tilde{U} \to \tilde{U}$ is a locally trivial smooth circle bundle over $\tilde{U}$.*

Note that, the fiber product $M \times_S \tilde{U}$ is to be taken in the category of smooth manifolds. We will see that, in general, it does not coincide with the set-theoretic fiber product!

Proof. According to the definition of a Seifert fibration, it suffices to show that there is a ramified covering $g\colon D^2 \to D^2$ such that the fiber product

$$\tilde{f}\colon \tilde{F} = (S^1 \times D^2) \times_{f_{p,q}, D^2, g} D^2 \longrightarrow D^2$$

is a trivial smooth circle bundle over D^2.

Let $g\colon D^2 \to D^2$ be the ramified covering $g(w) = w^p$. Then, the set theoretic fiber product of $S^1 \times D^2$ and D^2 over D^2 is the set

$$F = \{(u, z, w) \in S^1 \times D^2 \times D^2 | u^q z^p = w^p\}.$$

It is easy to see that F is not necessarily smooth along the subset $S^1 \times \{0\} \times \{0\}$. Let $f\colon F \to D^2$ be the map defined by $f(u, z, w) = w$. Let $g'\colon F \to S^1 \times D^2$ be the map defined by $g(u, z, w) = (u, z)$, so that the diagram

$$\begin{array}{ccc} F & \xrightarrow[g']{} & S^1 \times D^2 \\ \big\downarrow f & & \big\downarrow f_{p,q} \\ D^2 & \xrightarrow[g]{} & D^2 \end{array}$$

commutes.

The fiber product $\tilde{F}$ is the manifold defined by

$$\tilde{F} = \{(u, z, x) \in S^1 \times D^2 \times S^1 | u^q = x^p\}.$$

The desingularization map is the map $\delta\colon \tilde{F} \to F$ defined by $\delta(u, z, x) = (u, z, xz)$. Let $\tilde{f}\colon \tilde{F} \to D^2$ be the map defined by $\tilde{f}(u, z, x) = xz$. Then, the diagram

$$\begin{array}{ccc} \tilde{F} & \xrightarrow[\delta]{} & F \\ \big\downarrow \tilde{f} & & \big\downarrow f \\ D^2 & \xrightarrow[\mathrm{id}]{} & D^2 \end{array}$$

commutes, i.e., the fibration $\tilde{f}$ is the desingularization of f.

Now, the map $\tilde{f}$ is a trivial fibration. Indeed, $\tilde{F}$ is a homogeneous space over D^2 under the action of S^1 defined by $v \cdot (u, z, x) = (v^p u, v^{-q} z, v^q x)$. Moreover, $\tilde{f}$ admits a smooth section s defined by $s(z) = (1, z, 1)$. Therefore, $\tilde{f}$ is a trivial fibration of $\tilde{F}$ over D^2. □

A point P on a 2-dimensional orbifold S is a *cone point* with *cone angle* $2\pi/p$ if a neighbourhood of P is orbifold diffeomorphic to the orbifold quotient $\mathbb{C}//\mu_p$, where μ_p is the group of p-th roots of unity. A cone point is a *trivial* cone point, or a *smooth* cone point, if its cone angle is equal to 2π.

Let $f\colon M \to S$ be a Seifert fibration. It follows from the local description of f that the surface S has a natural structure of an orbifold with only finitely many nontrivial cone points. Indeed, with the notation above, the fibration $f_{p,q}$ is the quotient of the trivial fibration $\tilde{f}$ by the action of μ_p defined on $\tilde{F}$ by $\xi \cdot (u, z, x) = (u, z, \xi x)$. The target of $f_{p,q}$ acquires the orbifold structure of $D^2//\mu_p$.

Recall that a manifold M *admits a geometric structure* if M admits a complete, locally homogeneous metric. In that case, the universal covering space M' of M admits a complete homogeneous metric. The manifold M has then a geometric structure *modeled* on the $(Isom(M'), M')$-geometry.

More generally, a *geometry* is a pair (I, V) where V is a simply connected manifold and I a real Lie group acting smoothly and transitively on V with compact point stabilisers. We will only consider geometries (I, V) that *admit a compact quotient*, i.e., there is a subgroup $H \subset I$ such that the projection $V \to V/H$ is a covering map onto a compact quotient. Two geometries (I, V) and (I', V') are equivalent if I is isomorphic to I' and there is a diffeomorphism $\varphi\colon V \to V'$ which transform the action $\rho\colon I \to \mathrm{Diff}(V)$ onto the action $\rho'\colon I' \to \mathrm{Diff}(V')$. We restrict ourselves to geometries where I is maximal. Namely, if I' is a strict subgroup of I

and $\rho'\colon I' \to \mathrm{Diff}(V)$ is the restriction of $\rho\colon I \to \mathrm{Diff}(V)$, we will only consider the (I, V) geometry.

Thurston has classified the 3-dimensional geometries: there are eight of them. In general, a 3-manifold does not possess a geometric structure. However, it turns out that every Seifert manifold admits a geometric structure and that the geometry involved is unique [Sc83, Sec. 4]. Let M be a Seifert manifold, the geometry of M is modeled on one of the six following models (see [Sc83] for a detailed description of each geometry):

$$S^3, S^2 \times \mathbb{R}, E^3, \mathrm{Nil}, H^2 \times \mathbb{R}, \widetilde{\mathrm{SL}_2\,\mathbb{R}}$$

where E^3 is the 3-dimensional euclidean space and H^2 is the hyperbolic plane.

The appropiate geometry for a Seifert bundle can be determined from the two invariants χ and e, where χ is the Euler number of the base orbifold and e is the Euler number of the Seifert bundle [Sc83, Table 4.1].

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	E^3	$H^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}_2\,\mathbb{R}}$

Table 1: Geometries for Seifert manifolds.

Theorem 2.3. *Let M be an orientable Seifert manifold that does not admit a spherical geometry, and let $f\colon M \to S$ be a Seifert fibration. Then there is an orientable surface $\tilde{S}$ and a finite ramified Galois covering $\tilde{S} \to S$ such that the fiber product*

$$\tilde{f}\colon \tilde{M} = M \times_S \tilde{S} \longrightarrow \tilde{S}$$

is a locally trivial smooth circle bundle.

A group of isometries of a Riemannian manifold B is *discrete* if for any $x \in B$, the orbit of x intersects a small neighborhood of x only finitely many times. The quotient B/Γ of B by a discrete group Γ of isometries, is a surface. The projection map $B \to B/\Gamma$ is a local homeomorphism except at points x where the isotropy subgroup Γ_x is nontrivial. In that case, Γ_x is a cylic group $\mathbb{Z}/p\mathbb{Z}$ for some $p > 1$, and the projection is similar to the projection of a meridian disk cutting across a singular fiber of a Seifert fibration.

For convenience, let us denote by $S^2(p, q)$, $1 \le q < p$, the orbifold whose underlying surface is S^2 with two cone points with angles $2\pi/q$ and $2\pi/p$. If $q = 1$, $S^2(p, 1) = S^2(p)$ is the teardrop orbifold. From [Sc83, Th. 2.3, Th. 2.4 and Th. 2.5], we can state the following:

Theorem 2.4 (Scott)**.** *Every closed 2-dimensional orbifold S with only cone points, and which is different from $S^2(p, q)$, $p \neq q$, is finitely covered by a smooth surface $\tilde{S}$.*

For the convenience of the reader we recall the main ideas of the proof.

Proof. Every 2-dimensional orbifold with only cone points, and which is different from $S^2(p, q)$, $p \neq q$, is isomorphic, as an orbifold, to the quotient of S^2, E^2 or H^2 by some discrete group of isometries Γ. We need to show that any finitely generated, discrete group Γ of isometries of S^2, E^2 or H^2 with compact quotient space contains a torsion free subgroup of finite index. This is trivial for S^2 and easy for E^2. For H^2 it is a corollary of Selberg's Lemma below.

Let us denote by $\Gamma' \subset \Gamma$ a torsion-free normal subgroup of finite index, the orbifold quotient $\tilde{S} = S^2/\Gamma'$, E^2/Γ' or H^2/Γ', respectively, is then a smooth surface. □

Selberg's Lemma. [Ra94, Chap. 7] *Every finitely generated subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *contains a torsion-free normal subgroup of finite index.* □

Remark 2.5. The fact that any finitely generated, discrete group of isometries of H^2 admits a torsion free subgroup of finite index was conjectured by Fenchel and the first proof was completed by Fox [Fo52]. From the modern point of view, however, this result is a corollary of the Selberg's Lemma

Proof of Theorem 2.3. Let M be an orientable Seifert manifold that does not admit a spherical geometry. Let $M \to S$ be a Seifert fibration. Let us show that the base orbifold S is not isomorphic to one of the orbifolds $S^2(p,q)$, with $1 \le q < p$. Indeed, the Euler number χ of $S^2(p,q)$ is strictly positive, and the Euler number e of any Seifert fibration over $S^2(p,q)$ is nonzero (see [Sc83], in particular, the discussion before Lemma 3.7). Therefore, by Table 1, if $S \cong S^2(p,q)$, the manifold M would admit a spherical geometry. This shows that S is not isomorphic to $S^2(p,q)$. It follows from Theorem 2.4 that there is a finite ramified Galois covering $\tilde{S} \to S$ of the orbifold S by a smooth surface $\tilde{S}$. Moreover, we may assume $\tilde{S}$ to be orientable, taking the Galois closure of the orientation double covering if necessary. It is clear that the fiber product $\tilde{f}$ is locally trivial. □

3. Klein surfaces

In this section we recall the definition of a Klein surface, and we prove some statements that we need for the proof of Theorem 1.1. Classically, a Klein surface is defined as a topological surface endowed with an atlas whose transition functions are either holomorphic or antiholomorphic [AG71]. This seems to be a less suitable point of view for what we need since, with that definition, a Klein surface is not a locally ringed space. In particular, the definition of a line bundle over such a Klein surface is cumbersome, and, we would not have at our disposal a first cohomology group of the type $H^1(S, \mathcal{O}^\star)$ classifying all line bundles on a given Klein surface S. Therefore, we will use another definition of a Klein surface, giving rise to a category equivalent to the category of Klein surfaces of [AG71].

Let $\mathbb{D}$ be the double open half plane $\mathbb{C}\setminus\mathbb{R}$. On $\mathbb{D}$ one has the sheaf of holomorphic functions $\mathcal{H}$. The Galois group $\Sigma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts naturally on $\mathbb{D}$. Let σ denote complex conjugation in Σ. We consider the following *algebraic* action of Σ on the sheaf $\mathcal{H}$ over the action of Σ on $\mathbb{D}$. If $U \subseteq \mathbb{D}$ is open and f is a section of $\mathcal{H}$ over U, then we define $\sigma \cdot f \in \mathcal{H}(\sigma \cdot U)$ by

$$(\sigma \cdot f)(z) = \overline{f(\overline{z})}$$

for all $z \in \sigma \cdot U$. Let $(\mathbb{H}, \mathcal{O})$ be the quotient of $(\mathbb{D}, \mathcal{H})$ by the action of Σ in the category of locally ringed spaces. In particular, $\mathbb{H} = \mathbb{D}/\Sigma$ is homeomorphic to the open upper half plane—or lower half plane for that matter. Let $p\colon \mathbb{D} \to \mathbb{H}$ be the quotient map. Then $\mathcal{O}$ is the sheaf $(p_\star\mathcal{H})^\Sigma$ of Σ-invariant sections over $\mathbb{H}$. The sheaf $\mathcal{O}$ on $\mathbb{H}$ is a sheaf of local $\mathbb{R}$-algebras. Each stalk of $\mathcal{O}$ is noncanonically isomorphic to the $\mathbb{R}$-algebra $\mathbb{C}\{z\}$ of complex convergent power series in z.

A *Klein surface* is a locally ringed space $(S, \mathcal{O})$, where $\mathcal{O}$ is a sheaf of local $\mathbb{R}$-algebras, such that S is compact connected and separated, and $(S, \mathcal{O})$ is locally isomorphic to $(\mathbb{H}, \mathcal{O})$. With the obvious definition of morphisms of Klein surfaces, we have the category of Klein surfaces. Note that, here, we have only defined the notion of a compact connected Klein surface without boundary. For a more general definition of Klein surfaces, the reader may refer to [Hu02].

Basic examples of Klein surfaces are the following. A Riemann surface is a Klein surface. More generally, let S be a Riemann surface. A *Klein action* of a finite group G on S is an action of G on S as a Klein surface. Let be given a Klein action of G on S. Suppose that S contains only finitely many fixed points for the action of G. Then the quotient S/G has a natural structure of a Klein surface.

The following statement is well known.

Theorem 3.1. *Let S be a compact connected smooth surface. Then S admits the structure of a Klein surface.*

Proof. If S is orientable, then S admits the structure of a Riemann surface. In particular, S admits the structure of a Klein surface. Therefore, we may assume that S is not orientable. This means that S is a $(g+1)$-fold connected sum of real projective planes, for some natural integer g. Let C be any real algebraic curve of genus g without real points. Then, the set of complex points $C(\mathbb{C})$ of C is a Riemann surface of genus g. Since C is real, the Galois group $\Sigma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $C(\mathbb{C})$ by holomorphic or antiholomorphic automorphisms. Since C has no real points, the action of Σ on $C(\mathbb{C})$ is fixed point-free, i.e., we are in the presence of a Klein action of Σ on S. Therefore, the quotient $C(\mathbb{C})/\Sigma$ has a natural structure of a Klein surface. It is clear that $C(\mathbb{C})/\Sigma$ is diffeomorphic to S as a smooth surface. Hence, S admits the structure of a Klein surface. □

There is also a Klein version of the Riemann Existence Theorem. It can either be proven as the Riemann Existence Theorem, or it can be proven using the Riemann Existence Theorem.

Theorem 3.2. *Let S be a Klein surface and let $\tilde{S}$ be a compact connected smooth surface. If $f\colon \tilde{S} \to S$ is a ramified covering of smooth surfaces, then there is a unique structure of a Klein surface on $\tilde{S}$ such that f is a morphism of Klein surfaces.* □

Let $(S,\mathcal{O})$ be a Klein surface. A *line bundle* over S is an invertible sheaf of $\mathcal{O}$-modules. The group of isomorphism classes of line bundles is isomorphic to the group $H^1(S,\mathcal{O}^\star)$.

We will also need the notion of a smooth line bundle over S. Let $\mathcal{C}^\infty$ be the sheaf of smooth complex valued functions on the open double half plane $\mathbb{D}$. The Galois group $\Sigma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathcal{C}^\infty$, in a similar way as its action on $\mathcal{H}$. This action extends the action of Σ on $\mathcal{H}$. Denote by $\mathcal{C}$ the induced sheaf of Σ-invariant sections on $\mathbb{H}$. We call it the sheaf of *smooth functions* on $\mathbb{H}$. The sheaf $\mathcal{C}$ on $\mathbb{H}$ contains $\mathcal{O}$ as a subsheaf. It is now clear that a Klein surface $(S,\mathcal{O})$ carries an induced sheaf $\mathcal{C}$ of smooth functions which contains the sheaf $\mathcal{O}$ as a subsheaf.

A *smooth line bundle* on a Klein surface $(S,\mathcal{O})$ is an invertible sheaf of $\mathcal{C}$-modules. Again, the group of isomorphism classes of smooth line bundles on S is isomorphic to $H^1(S,\mathcal{C}^\star)$. Of course, if L is a line bundle on S, then $L\otimes_{\mathcal{O}}\mathcal{C}$ is a smooth line bundle on S. Let L' be a smooth line bundle on S. We say that L' admits the structure of a *Klein bundle* if there is a line bundle L over S such that $L\otimes_{\mathcal{O}}\mathcal{C}\cong L'$. The following statement shows that every smooth line bundle on a Klein surface does admit the structure of a Klein bundle.

Theorem 3.3. *Let $(S,\mathcal{O})$ be a Klein surface and let $\mathcal{C}$ be the induced sheaf of smooth functions on S. If L' is a smooth line bundle on S then there is a line bundle L on S such that*

$$L\otimes_{\mathcal{O}}\mathcal{C}\cong L'.$$

Proof. We show that the natural map

$$H^1(S,\mathcal{O}^\star)\longrightarrow H^1(S,\mathcal{C}^\star)$$

is surjective. As in [Hu02], we have exponential morphisms

$$\exp\colon \mathcal{O}\to\mathcal{O}^\star \quad\text{and}\quad \exp\colon \mathcal{C}\to\mathcal{C}^\star.$$

They are both surjective, and their kernels are isomorphic. Let $\mathcal{K}$ denote their kernels. Then we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \to & \mathcal{O} & \to & \mathcal{O}^\star & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{K} & \to & \mathcal{C} & \to & \mathcal{C}^\star & \to & 0 \end{array}$$

It induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} H^1(S,\mathcal{O}) & \to & H^1(S,\mathcal{O}^\star) & \to & H^2(S,\mathcal{K}) & \to & H^2(S,\mathcal{O}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(S,\mathcal{C}) & \to & H^1(S,\mathcal{C}^\star) & \to & H^2(S,\mathcal{K}) & \to & H^2(S,\mathcal{C}) \end{array}$$

Now, $\mathcal{C}$ is a fine sheaf. Hence, $H^1(S,\mathcal{C}) = 0$, and the map

$$H^1(S,\mathcal{C}^\star) \longrightarrow H^2(S,\mathcal{K})$$

is injective. Moreover, $H^2(S,\mathcal{O}) = 0$ [Hu02]. Hence, the map

$$H^1(S,\mathcal{O}^\star) \longrightarrow H^2(S,\mathcal{K})$$

is surjective. It follows that the natural map

$$H^1(S,\mathcal{O}^\star) \longrightarrow H^1(S,\mathcal{C}^\star)$$

is surjective. □

4. Equivariant line bundles on Riemann surfaces

Let S be a Riemann surface and let L be a smooth complex line bundle on S. Let be given a Klein action of a finite group G on S. A *smooth Klein action* of G on L is an action of G on L over the action of G on S such that $g \in G$ acts antilinearly on L if and only if g acts antiholomorphically on S, for all $g \in G$. If, moreover, L is a holomorphic line bundle on S and G acts by holomorphic or antiholomorphic automorphisms on the total space L, then the smooth Klein action is a *Klein action* of G on L.

Theorem 4.1. *Let S be a Riemann surface and let L be a smooth complex line bundle over S. Let be given a faithful Klein action of a finite group G on S and a smooth Klein action of G on L. Then, there is a structure of a holomorphic line bundle on L such that the smooth Klein action of G is a Klein action of G on L.*

Proof. Since the action of G on S is a Klein action, S contains finitely many fixed points $P_1, \ldots, P_n$. Let $G_1, \ldots, G_n$ be the isotropy groups of $P_1, \ldots, P_n$, respectively. Since the action of G on S is a Klein action, each isotropy group G_i is a finite cyclic group of order p_i, acting holomorphically on S. Let ρ_i be the induced 1-dimensional representation of G_i on the complex tangent space T_iS of S at P_i. Since ρ_i is a faithful representation and since the induced action of G_i on the fiber L_i over P_i is complex linear, there is a unique integer $q_i \in \{0, \ldots, p_i - 1\}$ such that the 1-dimensional representation L_i is isomorphic to $\rho_i^{q_i}$.

Let K be the holomorphic line bundle $\mathcal{O}(\sum q_i P_i)$ on S. It is clear that K comes along with a Klein action of G. Then, $K \otimes L$ is a smooth complex line bundle on S with a smooth Klein action of G such that the group G_i acts trivially on the fiber $K_i \otimes L_i$ over P_i. Now, it suffices to show that $K \otimes L$ admits a structure of a holomorphic complex line bundle such that the smooth action of G on $K \otimes L$ is a Klein action. Therefore, replacing L by $K \otimes L$, we may assume that $q_i = 0$, for $i = 1, \ldots, n$. More precisely, we may assume that, for each i, the action of G_i on the fiber L_i of L over P_i is trivial.

Let S' be the quotient Klein surface S/G, and let $p\colon S \to S'$ be the quotient map. Since the action of G_i on L_i is trivial, there is a smooth line bundle L' on S' such that $p^\star L'$ is G-equivariantly isomorphic to L. By Theorem 3.3, L' has a structure of a Klein bundle over S'. It follows that $p^\star L'$ has the structure of a holomorphic line bundle such that the action of G on $p^\star L'$ is a Klein action. □

5. Uniruled algebraic models

Proof of Theorem 1.1. Let M be an orientable Seifert manifold. We show that there is a uniruled real algebraic variety X such that M is diffeomorphic to a connected component of $X(\mathbb{R})$. That statement is known to be true if M admits a spherical geometry [Ko99, Ex. 10.4]. Therefore, we may assume that M does not admit a spherical geometry.

Choose a Seifert fibration $f\colon M \to S$ of M. By Theorem 2.3, there is a ramified Galois covering

$$p\colon \tilde{S} \longrightarrow S$$

such that the Seifert fibration

$$\tilde{f}\colon \tilde{M} \longrightarrow \tilde{S},$$

obtained from f by base change, is a locally trivial circle fibration. Moreover, we may assume $\tilde{S}$ and $\tilde{M}$ to be oriented. Let G be the Galois group of $\tilde{S}$ over S. Then G acts naturally on $\tilde{M}$ and $\tilde{f}$ is G-equivariant. The quotient of $f\colon \tilde{M} \to \tilde{S}$ by G is isomorphic to $f\colon M \to S$ as Seifert fibrations. This means that there is a quotient map

$$\tilde{p}\colon \tilde{M} \longrightarrow M$$

for the action of G on $\tilde{M}$ such that the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{p}} & M \\ \big\downarrow{\scriptstyle \tilde{f}} & & \big\downarrow{\scriptstyle f} \\ \tilde{S} & \xrightarrow{p} & S \end{array}$$

commutes.

Since S is a compact connected surface without boundary, S admits a structure of a Klein surface by Theorem 3.1. By the Riemann Existence Theorem for Klein surfaces, there is a unique structure of a Klein surface on $\tilde{S}$ such that the map $p\colon \tilde{S} \to S$ is a morphism of Klein surfaces. In particular, the group G acts on $\tilde{S}$ by automorphisms of $\tilde{S}$. In fact, since $\tilde{S}$ is an oriented Klein surface without boundary, $\tilde{S}$ is a Riemann surface. The action of G on the Riemann surface $\tilde{S}$ is by holomorphic or antiholomorphic automorphisms, i.e., we are in the presence of a so-called Klein action of G on the Riemann surface $\tilde{S}$.

Choose a smooth relative Riemannian metric μ on $\tilde{M}/\tilde{S}$. Since G is finite, one may assume that μ is G-equivariant. Since $\tilde{f}$ is locally trivial, and since $\tilde{M}$ and $\tilde{S}$ are oriented, there is a relative orientation of $\tilde{M}/\tilde{S}$. Hence, the structure group of the locally trivial circle bundle $\tilde{M}/\tilde{S}$ is $\mathrm{SO}(2)$. Since $\mathrm{SO}(2) = \mathrm{SU}(1)$, there is a smooth complex line bundle $\tilde{L}$ on $\tilde{S}$, that comes along with a hermitian metric, whose unit circle bundle is $\tilde{M}$. We also have an action of G on $\tilde{L}$ over the action of G on $\tilde{S}$ that extends the action of G on $\tilde{M}$. The action of G on $\tilde{L}$ is a smooth Klein action since G acts by orientation preserving automorphisms on $\tilde{M}$. By Theorem 4.1, there is a structure of a homolomorphic line bundle on $\tilde{L}$ such that the action of G is a Klein action. By the GAGA-principle, $\tilde{S}$ is a complex algebraic curve and $\tilde{L}$ is a complex algebraic line bundle on $\tilde{S}$. Moreover, the action of G on $\tilde{L}$ is by algebraic or antialgebraic automorphisms. The restriction of scalars $R(\tilde{S})$ is a real algebraic surface whose set of real points is diffeomorphic to $\tilde{S}$ [Hu92, Hu00]. The restriction of scalars $R(\tilde{L})$ is a real algebraic vector bundle over $R(\tilde{S})$ of rank 2, whose set of real points is diffeomorphic to $\tilde{L}$. The action of G on $\tilde{L}$ induces an algebraic action of G on $R(\tilde{L})$.

Let U be an affine Zariski open subset of $R(\tilde{S})$ containing the real points of $R(\tilde{S})$ and which is G-equivariant. Since $R(\tilde{L})$ is a real algebraic vector bundle over an affine real algebraic variety, there is a vector bundle V over U such that the direct sum

$$V \oplus (R(\tilde{L})_{|U})$$

is trivial. Since $V \oplus (R(\tilde{L})_{|U})$ is trivial, there is a real algebraic metric λ on the restriction of $R(\tilde{L})$ to U. Since G is finite, we may assume that λ is G-equivariant. Let $\tilde{T}$ be the unit circle bundle $\lambda = 1$ in $R(\tilde{L})_{|U}$. Then $\tilde{T}(\mathbb{R})$ is G-equivariantly diffeomorphic to $\tilde{M}$. In particular, the quotient $\tilde{T}(\mathbb{R})/G$ is diffeomorphic to M. Let X be a smooth projective model of $\tilde{T}/G$. Since G acts fixed point-freely on $\tilde{T}(\mathbb{R})$, the quotient $\tilde{T}(\mathbb{R})/G$ is smooth and is, therefore, a connected component of $X(\mathbb{R})$. The real algebraic variety X is uniruled since a smooth projective model of $\tilde{T}$ is ruled. $\square$

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