

Inflection points on real plane curves having many pseudo-lines

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Abstract

A pseudo-line of a real plane curve C is a global real branch of $C(\mathbb{R})$ that is not homologically trivial in $\mathbb{P}^2(\mathbb{R})$. A geometrically integral real plane curve C of degree d has at most $d - 2$ pseudo-lines, provided that C is not a real projective line. Let C be a real plane curve of degree d having exactly $d - 2$ pseudo-lines. Suppose that the genus of the normalization of C is equal to $d - 2$. We show that each pseudo-line of C contains exactly 3 inflection points. This generalizes the fact that a nonsingular real cubic has exactly 3 real inflection points.

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1 INTRODUCTION

Let $C \subseteq \mathbb{P}^2$ be a real algebraic plane curve. The set $C(\mathbb{R})$ of real points is a real analytic subset of $\mathbb{P}^2(\mathbb{R})$ and has a finite number of global branches [1]. Let B be such a branch. Since B is a real analytic subset of $\mathbb{P}^2(\mathbb{R})$, it has a fundamental class $[B]$ in the homology group $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ of $\mathbb{P}^2(\mathbb{R})$ [2]. We say that B is a *pseudo-line* of the curve C if $[B] \neq 0$ (see Figure 1). A convenient and equivalent way to express that B is a pseudo-line is that there is a projective real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ intersecting B in an odd number of points (counted with multiplicity). In fact, if B is a pseudo-line then *any* projective real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ intersects B in an odd number of points (provided, of course, that $L \neq B$). These statements all follow from the fact that $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, that the intersection product on $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is nondegenerate, and that the fundamental class $[L]$ of a projective real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ is nonzero in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ [8].

Real plane curves with many pseudo-lines have rarely been studied. The reason may be that such curves are necessarily singular. Indeed, if $C \subseteq \mathbb{P}^2$ is a

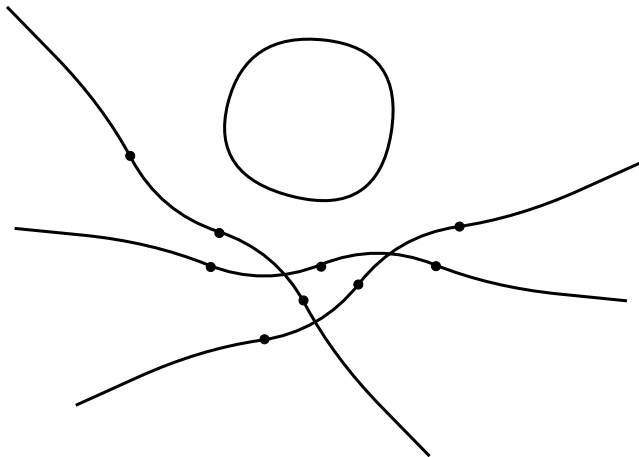


Figure 1: A real plane curve having 4 real branches, exactly 3 of them are pseudo-lines. The marked points are the real inflection points of the curve.

real algebraic curve having many pseudo-lines then two distinct pseudo-lines B and B' of C intersect each other in a singular point of C (since $[B] \cdot [B'] \neq 0$). Curves with many pseudo-lines seem interesting to study because they turn out to have a more uniform behavior than, for example, nonsingular plane curves. In fact, one may think of the class of curves having many pseudo-lines and the class of nonsingular plane curves as lying at opposite ends on the spectrum of all plane curves of given degree. A study of the passage from curves having many pseudo-lines to curves having less pseudo-lines might shed a different light on the geometry of nonsingular real plane curves. In this paper we study the geometry of curves with many pseudo-lines.

Let us make precise what we mean by real plane curves having many pseudo-lines. Let C be a geometrically integral real plane curve, i.e., its complexification $C \times_{\mathbb{R}} \mathbb{C}$ is reduced and irreducible [5]. Let d be the degree of C . We say that C has *many pseudo-lines* if C has exactly $d - 2$ pseudo-lines and if the genus of the normalization of C is equal to $d - 2$ (Figure 1 is a picture of a curve of degree 5 having many pseudo-lines). In Section 2 we give a motivation for the present definition of a real plane curve having many pseudo-lines.

There are many examples of real plane curves having many pseudo-lines: all nonsingular real conics and all nonsingular real cubics have many pseudo-lines. In fact, there are curves having many pseudo-lines of arbitrary degree $d \geq 2$. Indeed, choose a nonsingular real conic X in \mathbb{P}^2 and choose $d - 2$ real projective lines L_1, \dots, L_{d-2} in \mathbb{P}^2 in general position such that $L_i(\mathbb{R})$

does not intersect $X(\mathbb{R})$ for $i = 1, \dots, d - 2$. Let C' be the union of X and L_1, \dots, L_{d-2} . Then, by a well known result of Brusotti one can “deform away” all the nonreal singularities of the real plane curve C' . What one gets is a real plane curve C of degree d having exactly $d - 2$ pseudo-lines such that the genus of the normalization of C is equal to $d - 2$, i.e., C is a real plane curve of degree d having many pseudo-lines (see [6] for another proof of the existence of such curves).

The paper is devoted to the study of real inflection points on real plane curves having many pseudo-lines. The main result is the following:

Theorem 1. *Let C be a real plane curve having many pseudo-lines. Then, each pseudo-line of C contains exactly 3 inflection points.*

Theorem 1 generalizes that what is known for nonsingular real cubics [9]: A nonsingular real cubic has a unique pseudo-line which admits the structure of a real Lie group isomorphic to the circle group. The set of inflection points on the pseudo-line coincide with the 3-torsion subgroup. The latter subgroup is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Therefore, the pseudo-line contains exactly 3 inflection points.

Using Bezout’s Theorem and the fact that a real projective line necessarily intersects a pseudo-line, it can easily be seen that a real branch of C that is not a pseudo-line cannot contain inflection points. Hence, Theorem 1 implies that C has exactly $3(d - 2)$ real inflection points. This total of $3(d - 2)$ real inflection points can also be obtained from the generalized Klein Equation [10, 11]. However, that equation does not imply anything concerning the distribution of the real inflection points over the different real branches.

In order to prove Theorem 1, one may show that a smooth pseudo-line of any real plane curve contains at least 3 inflection points. Then, applying the generalized Klein Equation, one deduces that each pseudo-line of C contains exactly 3 inflection points. However, we present here a proof which does not use the generalized Klein Equation. The proof is inspired on the case of a nonsingular real cubic, in spite of the absence of a natural structure of a Lie group on a pseudo-line of C .

Theorem 1 has already been proved in [6] in the case where the curve C has, besides the $d - 2$ pseudo-lines, yet another real branch. The proof in [6] does not seem to generalize to the case of curves having many pseudo-lines.

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2 THE NUMBER OF PSEUDO-LINES OF A REAL PLANE CURVE

In this section we briefly justify the present definition of a real curve having many pseudo-lines.

Proposition 2. *Let $C \subseteq \mathbb{P}^2$ be a geometrically integral real plane curve of degree d . If C has at least $d - 1$ pseudo-lines then C is a real projective line in \mathbb{P}^2 .*

Proof. Let \tilde{C} be the normalization of C and let \tilde{g} be its genus. By the genus formula,

$$\tilde{g} = \frac{1}{2}(d - 1)(d - 2) - \mu,$$

where μ is the multiplicity of the singular locus of C [3]. By hypothesis, C has at least $d - 1$ pseudo-lines. Since any two distinct pseudo-lines of C intersect each other, μ is greater than or equal to $\frac{1}{2}(d - 1)(d - 2)$. Hence, $\tilde{g} = 0$ and C is a rational curve. Then, C can have at most 1 pseudo-line. It follows that $d - 1 \leq 1$, i.e., $d = 1$ or $d = 2$. Then, C is a real projective line or a real conic. But a geometrical integral real conic has no pseudo-lines. Since C is supposed to have at least $d - 1$ pseudo-lines, C is not a conic. Therefore, C is a real projective line in \mathbb{P}^2 . \square

By the above proposition, real plane curves of degree d having at least $d - 1$ pseudo-lines do not constitute an interesting class of real curves as far as geometry is concerned. Therefore, we concentrate on real plane curves having exactly $d - 2$ pseudo-lines:

Proposition 3. *Let $C \subseteq \mathbb{P}^2$ be a geometrically integral real plane curve of degree d having exactly $d - 2$ pseudo-lines. Then, the genus \tilde{g} of the normalization of C is equal to $d - 3$ or $d - 2$. Moreover, one has $\tilde{g} = d - 2$ if and only if any two distinct pseudo-lines of C intersect in one point only—the intersection being transverse—and C has no other singularities.*

Proof. Let \tilde{C} be the normalization of C . By the genus formula [3], the genus \tilde{g} of \tilde{C} satisfies:

$$\tilde{g} \leq \frac{1}{2}(d - 1)(d - 2) - \frac{1}{2}(d - 2)(d - 3) = d - 2.$$

Equality holds if and only if any two distinct pseudo-lines of C intersect in one point only—the intersection being transverse—and C has no other singularities. Note that, by definition, the intersection of two distinct real branches of C is transverse in a point P if and only if both branches are smooth at P and both tangent lines are distinct.

Harnack's Inequality [4] states that the number s of real branches of C satisfies the inequality

$$s \leq \tilde{g} + 1.$$

Since C has at least $d - 2$ real branches, one has $d - 2 \leq s \leq \tilde{g} + 1$, i.e., $d - 3 \leq \tilde{g}$. Therefore, \tilde{g} is equal to $d - 3$ or $d - 2$. \square

In view of the two preceding propositions, the definition of a real plane curve having many pseudo-lines now seems reasonable.

3 INFLECTION POINTS ON PSEUDO-LINES

We fix, throughout this section, an integer $d \geq 2$ and a geometrically integral curve $C \subseteq \mathbb{P}^2$ of degree d having many pseudo-lines. In particular, C has exactly $d - 2$ pseudo-lines. Let $\nu: \tilde{C} \rightarrow C$ be the normalization of C . By hypothesis, the genus \tilde{g} of \tilde{C} is equal to $d - 2$.

Let us collect in the following statement some immediate properties satisfied by C , some of which have already been mentioned above:

Proposition 4. 1. *The curve C has either $d - 2$ or $d - 1$ real branches.*

2. *Two distinct pseudo-lines of C intersect in one point only. These singularities are the only singularities of C . They are all real ordinary multiple points. In particular, each real branch of C is a smooth real analytic curve in $\mathbb{P}^2(\mathbb{R})$.*

3. *The curve C has only ordinary real inflection points and no real multi-tangent lines.*

Proof. 1. By Harnack's Inequality [4] the number s of real branches of C satisfies $s \leq \tilde{g} + 1 = d - 1$. Since C has many pseudo-lines, $s \geq d - 2$. Therefore, $s = d - 2$ or $d - 1$.

2. These properties have already been shown above (cf. Proposition 3).

3. Suppose that there is a real branch B of C containing a nonordinary real inflection point P . Let $L \subseteq \mathbb{P}^2$ be a real projective line such that $L(\mathbb{R})$ is tangent to B at P . By hypothesis, the order of contact of L at P is at least 4. Since $L(\mathbb{R})$ intersects at least $d - 3$ pseudo-lines of C different from B , the degree of the intersection product $L \cdot C$ is at least $4 + (d - 3) = d + 1$. Contradiction by Bezout's Theorem since C is of degree d . Therefore, C does not have nonordinary real inflection points.

Let us make precise what we mean by real multitangent lines. Let $L \subseteq \mathbb{P}^2$ be a real line. We say that L is a multitangent line of C if either L is tangent to C at a nonreal closed point, or L is tangent to C at two distinct real

points. This is in accordance with the fact that a nonreal closed point is a point of degree 2.

Suppose that L is a multitangent line of C that is tangent to C at a nonreal point P . Since $L(\mathbb{R})$ intersects each of the $d - 2$ pseudo-lines of C , the degree of $L \cdot C$ is at least $(d - 2) + 2 \deg(P) = d + 2$. This contradicts the fact that $L \cdot C$ is of degree d . Hence, there are no multitangent lines of C that are tangent at a nonreal point.

Suppose that L is a multitangent line of C that is tangent to C at two distinct real points P and P' . Let B (resp. B') be the real branch of C to which $L(\mathbb{R})$ is tangent at P (resp. P'). There are 3 cases to consider:

1. B and B' are distinct pseudo-lines of C ,
2. B and B' are one and the same pseudo-line of C , and
3. B or B' is not a pseudo-line of C .

We show that each of these cases leads to a contradiction. In each of the first two cases, the main observation is that $L(\mathbb{R})$ intersects each pseudo-line of C in an odd number of points.

In case 1, the degrees of $L(\mathbb{R}) \cdot B$ and $L(\mathbb{R}) \cdot B'$ are at least 3. Since $L(\mathbb{R})$ intersects each of the remaining $d - 4$ pseudo-lines, the degree of $L \cdot C$ is at least $3 + 3 + (d - 4) = d + 2$. Contradiction.

In case 2, the degree of $L(\mathbb{R}) \cdot B$ is at least 5. Since $L(\mathbb{R})$ intersects each of the remaining $d - 3$ pseudo-lines, the degree of $L \cdot C$ is at least $5 + (d - 3) = d + 2$. Contradiction.

In case 3, $L(\mathbb{R})$ is tangent to a real branch O of C that is not a pseudo-line, i.e., the degree of $L(\mathbb{R}) \cdot O$ is at least 2. Since $L(\mathbb{R})$ intersects each of the $d - 2$ pseudo-lines of C and since the degree of $L \cdot C$ is equal to $d = 2 + (d - 2)$, the real line $L(\mathbb{R})$ is not tangent to a pseudo-line of C and is tangent to O at only one point. Contradiction, since L was supposed to be tangent to C at two distinct real points. \square

Let us, for completeness, include a proof of the following statement:

Lemma 5. *Any pseudo-line of C contains at least 1 inflection point.*

Proof. We show first the more general statement that, for any geometrically integral real plane curve D , the dual curve B^\vee of a smooth real branch B of D is homologically trivial in $\mathbb{P}^2(\mathbb{R})^\vee$. Indeed, choose a general point $P \in \mathbb{P}^2(\mathbb{R})$ such that $P \notin B$. Let $p: B \rightarrow \mathbb{P}^1(\mathbb{R})$ be the restriction of the linear projection from $\mathbb{P}^2(\mathbb{R}) \setminus \{P\}$ onto $\mathbb{P}^1(\mathbb{R})$ with center P . Then, p is a nonconstant real analytic map from the real analytic curve B into the real analytic curve $\mathbb{P}^1(\mathbb{R})$.

Such a map is necessarily ramified at an even number of points of B . Therefore, the number of lines passing through P and tangent to B is even. Dually, this means that the line dual to P intersects B^\vee in an even number of points. Therefore, B^\vee is homologically trivial in $\mathbb{P}^2(\mathbb{R})^\vee$.

Now, let B be a pseudo-line of C . It is clear that B^\vee is a real branch of C^\vee . By the preceding paragraph, B^\vee is not a smooth real branch of C^\vee since otherwise $B = (B^\vee)^\vee$ would be homologically trivial. Therefore, B^\vee contains singularities. By Proposition 4.3, B^\vee can only contain ordinary cusps as singularities. Hence, B^\vee contains at least 1 ordinary cusp. It follows that B contains at least 1 inflection point. \square

Proof of Theorem 1. Let B be a pseudo-line of C . By Proposition 4.2, B is a smooth real analytic curve in $\mathbb{P}^2(\mathbb{R})$. Let Q be a point on B and let $L \subseteq \mathbb{P}^2(\mathbb{R})$ be the tangent line at Q to B . Since B is a pseudo-line, L has to intersect B in yet another point P . Using Bezout's Theorem, one sees that the point $P \in B$ is uniquely determined by Q . Let $\alpha: B \rightarrow B$ be the map defined by $\alpha(Q) = P$. It is clear that α is continuous.

An inflection point of B is a fixed point of α and conversely. Therefore, we have to show that α has exactly 3 fixed points. The idea is to show that α is a topological covering of B of degree -2 . It will then follow from Lemma 7 below that α has exactly 3 fixed points.

For $P \in B$, let $\pi_P: \mathbb{P}^2 \setminus \{P\} \rightarrow \mathbb{P}_P^1$ be the linear projection with center P . Here, \mathbb{P}_P^1 denotes the real algebraic curve of projective lines in \mathbb{P}^2 passing through P . Of course, \mathbb{P}_P^1 is isomorphic to \mathbb{P}^1 , however, not canonically. That will be crucial below. By definition, the π_P -image of a real point Q of $\mathbb{P}^2 \setminus \{P\}$ is the real projective line passing through P and Q .

Let $f_P: \tilde{C} \rightarrow \mathbb{P}_P^1$ be the unique morphism such that $f_P = \pi_P \circ \nu$ on $\tilde{C} \setminus \nu^{-1}(P)$. Then, f_P is a morphism of degree $d - \mu$, where μ is the multiplicity of C at P or, equivalently, μ is the number of real branches of C passing through P . By Proposition 4.2, one may identify a real branch of C with the corresponding real branch of \tilde{C} . The morphism f_P is ramified at a point $Q \in B$ if and only if the real line $L \subseteq \mathbb{P}^2(\mathbb{R})$ through P and Q is tangent to B at Q . It follows that the fiber $\alpha^{-1}(P)$ is equal to the set of ramification points of the restriction of f_P to B .

Let B' be a pseudo-line of C not passing through P . Since B' is contained in $\mathbb{P}^2(\mathbb{R}) \setminus \{P\}$, the restriction of f_P to B' is a continuous map from B' into $\mathbb{P}_P^1(\mathbb{R})$ of odd topological degree (see [7] for the notion of topological degree mod 2). In particular, the restriction of f_P to B' is surjective. Since there are $(d - 2) - \mu$ pseudo-lines of C not passing through P and since the degree of f_P is equal to $d - \mu$, each fiber of the restriction of f_P to B has cardinality at most 2. In fact, more precisely, for all points $R \in \mathbb{P}_P^1(\mathbb{R})$, the degree of

the divisor $(f_P|_B)^*(R)$ on B is at most 2. Since the topological degree of the restriction of f_P to B is even, there are either 0 or 2 points of B at which f_P is ramified. In the former case $f_P|_B$ is not null-homotopic, in the latter case $f_P|_B$ is null-homotopic.

Let $\mathcal{T} = \mathcal{T}_{B/\mathbb{P}^2(\mathbb{R})}$ be the restriction to B of the tangent bundle of $\mathbb{P}^2(\mathbb{R})$. Since B is a pseudo-line, the real analytic vector bundle \mathcal{T} is isomorphic to the direct sum of a trivial line bundle and a nontrivial, i.e. a Möbius line bundle on B . Denote by $\mathbb{P}(\mathcal{T})$ the projectivization of \mathcal{T} . The total space of $\mathbb{P}(\mathcal{T})$ is a Klein bottle. The fiber $\mathbb{P}(\mathcal{T})_P$ of $\mathbb{P}(\mathcal{T})$ over a point P of B is canonically isomorphic to $\mathbb{P}_P^1(\mathbb{R})$, i.e., we have made the collection of all real projective lines $\{\mathbb{P}_P^1(\mathbb{R})\}_{P \in B}$ into a locally trivial real analytic fiber bundle over B . Define

$$F: B \times B \longrightarrow \mathbb{P}(\mathcal{T})$$

by $F(P, Q) = f_P(Q)$ for all $(P, Q) \in B \times B$. The map F is real analytic, and, when we consider $B \times B$ to be fibered over B through the projection on the first factor, F is a map of locally trivial real analytic fiber bundles over B . The fiber F_P of F over $P \in B$ is the map $f_P|_B$, i.e., we have made the collection of all maps $f_P|_B$ into a real analytic family of maps over B .

Above, we have seen that $f_P|_B$ is ramified at exactly 0 or 2 points of B . Moreover, $f_P|_B$ is not null homotopic in the former case and is null homotopic in the latter case. Since the maps $f_P|_B$ vary continuously in a connected family, either all maps $f_P|_B$ are null homotopic, or all maps $f_P|_B$ are not null homotopic. Hence, either all maps $f_P|_B$ are unramified, or all maps $f_P|_B$ are ramified at exactly 2 points. Now, there is, of course, a point $P_0 \in B$ such that $\alpha^{-1}(P_0)$ is nonempty. But then, as we have seen above, $f_{P_0}|_B$ is ramified. Hence, all maps $f_P|_B$ are ramified at exactly 2 points of B . Moreover, since, for all $R \in \mathbb{P}_P^1(\mathbb{R})$, the degree of the divisor $(f_P|_B)^*(R)$ is at most 2, the image $(f_P|_B)(B)$ is an interval I_P in $\mathbb{P}_P^1(\mathbb{R})$. The union I of all I_P is an interval subbundle of $\mathbb{P}(\mathcal{T})$. Since the latter fiber bundle is not globally trivial, the interval bundle I is a Möbius bundle over B . This implies that the ramification locus of F , i.e., the union of the ramification loci of the maps $f_P|_B$, is a nontrivial topological covering of the base B of degree ± 2 . Now, this ramification locus is the transpose of the graph of α . Hence, α is of degree ± 2 .

In order to show that α is of degree -2 , recall that B has at least 1 inflection point by Lemma 5 and that such an inflection point is necessarily ordinary by Proposition 4.3. A local study of α at an ordinary inflection point of B reveals that α is orientation-reversing. Hence, the topological degree of α is equal to -2 . It follows from Lemma 7 below that the number of fixed points of α is equal to 3. Therefore, B contains exactly 3 inflection

points. □

Before proving Lemma 7 one needs the following preliminary statement:

Lemma 6. *Let $\alpha, \beta: S^1 \rightarrow S^1$ be topological coverings either both orientation-preserving or both orientation-reversing. Then, the product $\alpha\beta: S^1 \rightarrow S^1$, defined by $(\alpha\beta)(z) = \alpha(z) \cdot \beta(z)$ for any $z \in S^1$, is also a topological covering.*

Proof. Let $p: \mathbb{R} \rightarrow S^1$ be the universal covering defined by $p(t) = \exp(2\pi it)$ for $t \in \mathbb{R}$. Let $\tilde{\alpha}, \tilde{\beta}: \mathbb{R} \rightarrow \mathbb{R}$ be liftings of α and β , respectively, i.e., $p \circ \tilde{\alpha} = \alpha \circ p$ and $p \circ \tilde{\beta} = \beta \circ p$. Then, $\tilde{\alpha} + \tilde{\beta}$ is a lifting of $\alpha\beta$. Since α and β are topological coverings, $\tilde{\alpha}$ and $\tilde{\beta}$ are homeomorphisms of \mathbb{R} onto itself. Since α and β are either both orientation-preserving or both orientation-reversing, $\tilde{\alpha}$ and $\tilde{\beta}$ are either both strictly ascending or both strictly descending real functions. It follows that $\tilde{\alpha} + \tilde{\beta}$ is strictly ascending or strictly descending. In particular, $\tilde{\alpha} + \tilde{\beta}$ is a homeomorphism and, therefore, $\alpha\beta$ is a topological covering. □

Lemma 7. *Let $\alpha: S^1 \rightarrow S^1$ be a topological covering of degree $-e$, for some $e > 0$. Then, α has exactly $e + 1$ fixed points.*

Proof. Let $\beta: S^1 \rightarrow S^1$ be the standard topological covering of degree $-e$, i.e., $\beta(z) = \bar{z}^e$ for $z \in S^1$. Then, α and β are isotopic coverings, i.e., there is a homotopy $F: S^1 \times [0, 1] \rightarrow S^1$ such that $F_0 = \alpha$, $F_1 = \beta$ and F_t is a topological covering for all $t \in [0, 1]$. Define $F': S^1 \times [0, 1] \rightarrow S^1$ by $F'(z, t) = F(z, t) \cdot \bar{z}$ for $(z, t) \in S^1 \times [0, 1]$. By Lemma 6, F' is an isotopy of topological coverings of S^1 . One has $F'_1(z) = \bar{z}^{e+1}$. Hence, the fiber $F'^{-1}_1(1)$ consists of $e + 1$ points. Then, the fiber $F'^{-1}_0(1)$ consists of $e + 1$ points too, i.e., there are exactly $e + 1$ points $z \in S^1$ such that $\alpha(z)\bar{z} = 1$. Therefore, α has exactly $e + 1$ fixed points. □

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