ON COMPONENTS OF SPACES OF $\mathrm{PSL}_2(\mathbb{C})$-REPRESENTATIONS OF EXTENDED QUASIFUCHSIAN GROUPS

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ABSTRACT. We study components of the space $\mathcal{R}(G)$ of all $\mathrm{PSL}_2(\mathbb{C})$-representations of a group G of finite presentation. We apply to the case of G being a co-compact extended quasifuchsian group of the first kind acting freely on its domain of discontinuity. It turns out that $\mathcal{R}(G)$ can have many components. We show that only one of these components contain faithful discrete representations of G.

1. Components of $\mathrm{PSL}_2(\mathbb{C})$-representation spaces

Let G be a group of finite presentation. We study the set $\mathcal{R}(G)$ of all $\mathrm{PSL}_2(\mathbb{C})$-representations of G, i.e., the set of all homomorphisms from G into $\mathrm{PSL}_2(\mathbb{C})$.

The set $\mathcal{R}(G)$ has a natural structure of an algebraic variety. Indeed, choose a finite presentation

$$G = <x_1, \dots, x_n \,|\, R_1, \dots, R_m>$$

of G. Each R_i is a word in the free group generated by the symbols $x_1, \dots, x_n$. Hence, each R_i induces, in a natural way, a map from $\mathrm{PSL}_2(\mathbb{C})^n$ into $\mathrm{PSL}_2(\mathbb{C})$. This map will again be denoted by R_i. Then, the map

$$f\colon \mathcal{R}(G) \longrightarrow \mathrm{PSL}_2(\mathbb{C})^n$$

defined by $f(\rho) = (\rho(x_1), \dots, \rho(x_n))$ is a bijection of $\mathcal{R}(G)$ onto the subset

$$\bigcap_{i=1}^{m} R_i^{-1}(\{1\})$$

of $\mathrm{PSL}_2(\mathbb{C})^n$. Since this subset is obviously algebraic, the set $\mathcal{R}(G)$ acquires the structure of an affine algebraic variety. Note that this structure on $\mathcal{R}(G)$ does not depend on the presentation of G. All future references to a topology on $\mathcal{R}(G)$ concern the Zariski topology on $\mathcal{R}(G)$. We are going to study certain components, i.e., certain open and closed subsets of the space $\mathcal{R}(G)$ of $\mathrm{PSL}_2(\mathbb{C})$-representations of G. In what follows we identify $\mathcal{R}(G)$ with its f-image in $\mathrm{PSL}_2(\mathbb{C})^n$.

Let μ_2 be the subgroup $\{\pm 1\}$ of $\mathrm{SL}_2(\mathbb{C})$. Let π be the natural morphism from the group $\mathrm{SL}_2(\mathbb{C})^n$ onto $\mathrm{PSL}_2(\mathbb{C})^n$. The kernel of π is equal to μ_2^n. Put

$$\widetilde{\mathcal{R}}(G) = \pi^{-1}(\mathcal{R}(G)).$$

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Then, $\widetilde{\mathcal{R}}(G)$ is also an affine algebraic variety. Again, all references to a topology on $\widetilde{\mathcal{R}}(G)$ concern the Zariski topology on $\widetilde{\mathcal{R}}(G)$.

One has an induced action of μ_2^n on $\widetilde{\mathcal{R}}(G)$ and the restriction of π to $\widetilde{\mathcal{R}}(G)$, considered as a map into $\mathcal{R}(G)$, is a quotient map for this action. For $\varepsilon \in \mu_2^m$, let

$$\widetilde{\mathcal{R}}(G)_\varepsilon = \bigcap_{i=1}^{m} R_i^{-1}(\{\varepsilon(i)\}),$$

where R_i is now considered as a map from $\mathrm{SL}_2(\mathbb{C})^n$ into $\mathrm{SL}_2(\mathbb{C})$. Clearly, the subset $\widetilde{\mathcal{R}}(G)_\varepsilon$ is open and closed in $\widetilde{\mathcal{R}}(G)$. In particular, $\widetilde{\mathcal{R}}(G)_\varepsilon$ is itself an affine algebraic variety. Note that $\widetilde{\mathcal{R}}(G)_\varepsilon$ is not necessarily nonempty or connected.

Let $(\mu_2^m)'$ be the subset of $\varepsilon \in \mu_2^m$ such that $\widetilde{\mathcal{R}}(G)_\varepsilon$ is nonempty. Then, $\widetilde{\mathcal{R}}(G)$ is the disjoint union of the subsets $\widetilde{\mathcal{R}}(G)_\varepsilon$, $\varepsilon \in (\mu_2^m)'$, i.e.,

$$\widetilde{\mathcal{R}}(G) = \coprod_{\varepsilon \in (\mu_2^m)'} \widetilde{\mathcal{R}}(G)_\varepsilon.$$

A subset C of $\widetilde{\mathcal{R}}(G)$ is called a *component* of $\widetilde{\mathcal{R}}(G)$ if there is $\varepsilon \in (\mu_2^m)'$ such that $C = \widetilde{\mathcal{R}}(G)_\varepsilon$. A subset C of $\mathcal{R}(G)$ is called a *component* of $\mathcal{R}(G)$ if C is the π-image of a component of $\widetilde{\mathcal{R}}(G)$. We will show that the components of $\mathcal{R}(G)$ do not depend on the presentation of G (Proposition 1.5). But first we need some preparation.

Lemma 1.1. *There is a unique action of μ_2^n on $(\mu_2^m)'$ such that*

$$\delta \cdot \widetilde{\mathcal{R}}(G)_\varepsilon \subseteq \widetilde{\mathcal{R}}(G)_{\delta\cdot\varepsilon},$$

for all $\delta \in \mu_2^n$ and $\varepsilon \in (\mu_2^m)'$.

Proof. Uniqueness of the action is clear. In order to show existence, define, for $i = 1, \ldots, m$, *signature morphisms*

$$\sigma_i \colon \mu_2^n \longrightarrow \mu_2$$

by requiring that

$$\sigma_i(\delta) R(\alpha_1, \ldots, \alpha_n) = R_i(\delta(1)\alpha_1, \ldots, \delta(n)\alpha_n)$$

for all $\alpha_1, \ldots, \alpha_n \in \mathrm{SL}_2(\mathbb{C})$. Then, define a left action of μ_2^n on μ_2^m by

$$(\delta \cdot \varepsilon)(i) = \sigma_i(\delta)\varepsilon(i)$$

for all $i = 1, \ldots, m$.

Let $\widetilde{\rho} \in \widetilde{\mathcal{R}}(G)$ and $\delta \in \mu_2^n$. It is clear that $\delta \cdot \widetilde{\rho}$ is in $\widetilde{\mathcal{R}}(G)_{\delta\cdot\varepsilon}$ if $\widetilde{\rho}$ is in $\widetilde{\mathcal{R}}(G)_\varepsilon$. Hence, the action of μ_2^n on μ_2^m induces an action of μ_2^n on $(\mu_2^m)'$ and this induced action has the required property. □

Let $\sigma \colon \mu_2^n \longrightarrow \mu_2^m$ be the *total signature morphism* $(\sigma_1, \ldots, \sigma_m)$. Let K be the kernel of σ and let I be its image. It follows from the definition of the action of μ_2^n on μ_2^m that K is equal to the isotropy group of any element of μ_2^m. In particular, K acts trivially on μ_2^m. Therefore, we get an induced free action of I on μ_2^m and also on $(\mu_2^m)'$.

For $\varepsilon \in (\mu_2^m)'/I$ define $\mathcal{R}(G)_\varepsilon$ to be the π-image of $\widetilde{\mathcal{R}}(G)_\varepsilon$. The following statement is then clear.

Proposition 1.2. *For all $\varepsilon \in (\mu_2^m)'/I$, the subset $\mathcal{R}(G)_\varepsilon$ is a component of the space $\mathcal{R}(G)$ of $\mathrm{PSL}_2(\mathbb{C})$-representations of G. And, for any component C of $\mathcal{R}(G)$, there is a unique $\varepsilon \in (\mu_2^m)'/I$ such that $C = \mathcal{R}(G)_\varepsilon$. In particular,*

$$\mathcal{R}(G) = \coprod_{\varepsilon \in (\mu_2^m)'/I} \mathcal{R}(G)_\varepsilon.$$

Moreover, any component of $\mathcal{R}(G)$ is isomorphic—as an algebraic variety—to the quotient of a component of $\widetilde{\mathcal{R}}(G)$ by the free action of K on such a component. □

Remark 1.3. In view of the last statement of the proposition, it would be interesting to study connectedness of the components of $\widetilde{\mathcal{R}}(G)$; if $\widetilde{\mathcal{R}}(G)_\varepsilon$ is connected, for some $\varepsilon \in (\mu_2^m)'$, then $\mathcal{R}(G)_\varepsilon$ is connected too and there is a surjective morphism from the fundamental group of $\mathcal{R}(G)_\varepsilon$ onto the group K. In particular, if $\widetilde{\mathcal{R}}(G)_\varepsilon$ is connected and K is not trivial, then $\mathcal{R}(G)_\varepsilon$ is not simply connected.

Before giving some examples of components of spaces of $\mathrm{PSL}_2(\mathbb{C})$-representations, we show that these components are intrinsically defined, i.e., do not depend on the presentation of the group.

Recall that a *central extension* of μ_2 by G is a short exact sequence $\mathcal{E}$

$$1 \longrightarrow \mu_2 \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

such that the image of μ_2 is contained in the center of $\widetilde{G}$. Two central extensions $\mathcal{E}$ and $\mathcal{E}'$ of μ_2 by G are *isomorphic* if there is a morphism $f\colon \widetilde{G} \to \widetilde{G}'$ such that its restriction to μ_2 is the identity and such that the induced morphism on G is equal to the identity too. A central extension $\mathcal{E}$ is *trivial* if it is isomorphic to the trivial extension

$$1 \longrightarrow \mu_2 \longrightarrow \mu_2 \times G \longrightarrow G \longrightarrow 1.$$

Example 1.4. Let $\rho \in \mathcal{R}(G)$ be any $\mathrm{PSL}_2(\mathbb{C})$-representation of G. Letting G_ρ be the fiber product of G and $\mathrm{SL}_2(\mathbb{C})$ over $\mathrm{PSL}_2(\mathbb{C})$, one gets a central extension $\mathcal{E}_\rho$

$$1 \longrightarrow \mu_2 \longrightarrow G_\rho \longrightarrow G \longrightarrow 1$$

of μ_2 by G.

Proposition 1.5. *Two elements ρ and ρ' of $\mathcal{R}(G)$ belong to the same component if and only if the extensions $\mathcal{E}_\rho$ and $\mathcal{E}_{\rho'}$ are isomorphic. In particular, the components of $\mathcal{R}(G)$ do not depend on the presentation of G.*

Proof. Let $\rho \in \mathcal{R}(G)$ be a representation of G. Choose $\widetilde{\rho} \in \widetilde{\mathcal{R}}(G)$ such that $\pi(\widetilde{\rho}) = \rho$. Let $\varepsilon \in \mu_2^m$ be such that $\widetilde{\rho} \in \widetilde{\mathcal{R}}(G)_\varepsilon$. Define the group G_ε by the presentation

$$G_\varepsilon = \langle x_1, \dots, x_n, -1 \mid (-1)^2, [-1, x_i] \text{ and } \varepsilon(i)R_i \text{ for } i = 1, \dots, m \rangle.$$

Let $q\colon G_\varepsilon \to G$ be the morphism defined by $q(x_i) = x_i$, for $i = 1, \dots, n$, and $q(-1) = 1$. The morphism q is obviously surjective. The element $\widetilde{\rho}$ of $\widetilde{\mathcal{R}}(G)_\varepsilon$ can be considered, in a natural way, as a morphism from G_ε into $\mathrm{SL}_2(\mathbb{C})$. Then, the diagram

$$\begin{array}{ccc} G_\varepsilon & \xrightarrow{\;q\;} & G \\ {\scriptstyle\widetilde{\rho}}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathrm{SL}_2(\mathbb{C}) & \longrightarrow & \mathrm{PSL}_2(\mathbb{C}) \end{array}$$

is commutative. It follows that the short sequence $\mathcal{E}_\varepsilon$

$$1 \longrightarrow \mu_2 \longrightarrow G_\varepsilon \longrightarrow G \longrightarrow 1$$

is exact. Hence, $\mathcal{E}_\varepsilon$ is a central extension of μ_2 by G. It also follows that the morphisms $\widetilde{\rho}$ and q induce a morphism f from G_ε into the fiber product G_ρ of G and $\mathrm{SL}_2(\mathbb{C})$ over $\mathrm{PSL}_2(\mathbb{C})$. It then follows immediately that the central extensions $\mathcal{E}_\varepsilon$ and $\mathcal{E}_\rho$ are isomorphic.

Now, suppose that ρ and ρ' belong to the same component, say $\mathcal{R}(G)_\varepsilon$, of $\mathcal{R}(G)$. It follows from the statement above that the extensions $\mathcal{E}_\rho$ and $\mathcal{E}_{\rho'}$ are both isomorphic to $\mathcal{E}_\varepsilon$. In particular, the extensions $\mathcal{E}_\rho$ and $\mathcal{E}_{\rho'}$ are isomorphic.

Let ε and ε' be two elements of μ_2^m. Observe that the two central extensions $\mathcal{E}_\varepsilon$ and $\mathcal{E}_{\varepsilon'}$ are isomorphic if and only if there is $\delta \in \mu_2^n$ such that $\delta \cdot \varepsilon = \varepsilon'$.

Now, suppose that ρ and ρ' are elements of $\mathcal{R}(G)$ such that the extensions $\mathcal{E}_\rho$ and $\mathcal{E}_{\rho'}$ are isomorphic. Let ε and ε' be elements of μ_2^m such that ρ and ρ' belong to $\mathcal{R}(G)_\varepsilon$ and $\mathcal{R}(G)_{\varepsilon'}$ respectively. It follows from the statement above that the extensions $\mathcal{E}_\varepsilon$ and $\mathcal{E}_{\varepsilon'}$ are isomorphic. By the observation above, there is $\delta \in \mu_2^n$ such that $\delta \cdot \varepsilon = \varepsilon'$. Therefore, $\mathcal{R}(G)_\varepsilon = \mathcal{R}(G)_{\varepsilon'}$. □

The next examples do not only illustrate the results of this section but will also be useful in the next section.

Example 1.6. Let G be the fundamental group of a compact Riemann surface of genus $s \geq 1$. Taking a standard presentation of G, $n = 2s$, $m = 1$ and the group μ_2^n acts trivially on μ_2^m. Since the commutator map

$$[\cdot,\cdot]\colon \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

is surjective, both subsets $\widetilde{\mathcal{R}}(G)_1$ and $\widetilde{\mathcal{R}}(G)_{-1}$ are nonempty. Hence, $\mathcal{R}(G)$ has 2 components $\mathcal{R}(G)_1$ and $\mathcal{R}(G)_{-1}$.

It is interesting to observe that the component $\mathcal{R}(G)_1$ is the only one containing discrete representations of G. Indeed, let $\rho\colon G \to \mathrm{PSL}_2(\mathbb{C})$ be a discrete representation of G. Since the group G has no 2-torsion, ρ lifts to $\mathrm{SL}_2(\mathbb{C})$ [1], or, to put it differently, the central extension $\mathcal{E}_\rho$ is trivial. Therefore, $\rho \in \mathcal{R}(G)_1$. It follows that $\mathcal{R}(G)_{-1}$ does not contain any discrete representation of G.

Example 1.7. Let $\widetilde{r} \geq 1$. Let G be the group given by the presentation

$$G = \langle x_1, \ldots, x_{\widetilde{r}} \,|\, x_1^2 \cdots x_{\widetilde{r}}^2 \rangle.$$

This group is of particular interest since it has a faithful co-compact representation as an extended Fuchsian group acting freely on its domain of discontinuity if $\widetilde{r} \geq 3$ (see the next section). For the given presentation of G, one has $n = \widetilde{r}$ and $m = 1$ and the action of μ_2^n on μ_2^m is trivial. Since the square map

$$\mathrm{sq}\colon \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

is surjective, both subsets $\widetilde{\mathcal{R}}(G)_1$ and $\widetilde{\mathcal{R}}(G)_{-1}$ are nonempty. Hence, the space $\widetilde{\mathcal{R}}(G)$ has 2 components $\mathcal{R}(G)_1$ and $\mathcal{R}(G)_{-1}$.

Using [1] again, one can prove that the component $\mathcal{R}(G)_{-1}$ of $\mathcal{R}(G)$ does not contain any discrete $\mathrm{PSL}_2(\mathbb{C})$-representation of G if $\widetilde{r} \geq 2$.

2. Extended quasifuchsian groups and their $\mathrm{PSL}_2(\mathbb{C})$-representations

Recall that a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is called an *extended Fuchsian group* if it is not Fuchsian but does preserve a Euclidean circle in $\mathbb{P}^1(\mathbb{C})$ [4]. It is clear that such a group contains a unique Fuchsian subgroup of index 2. An extended Fuchsian group is said to be of the *first kind* if its Fuchsian subgroup of index 2 is so; otherwise, it is said to be of the *second kind.* A subgroup of $\mathrm{PSL}_2(\mathbb{C})$ that is a quasiconformal deformation of an extended Fuchsian group is called an *extended quasifuchsian group.* Such a group is of the *first kind* if it is a quasiconformal deformation of an extended Fuchsian group of the first kind; otherwise, it is of the *second kind.* It is the class of extended quasifuchsian groups of the first kind that will have our attention.

Let us fix the following notation. Let r, $\widetilde{r}$ and s be natural integers. Denote by $G_{r,\widetilde{r},s}$ the group given by the presentation

$$<x_1,\dots,x_{2r+\widetilde{r}+2s}\,|\,R_1,\dots,R_{2r+1}>, \tag{1}$$

where

$$R_{2i-1}=[x_{2i-1},x_{2i}] \quad\text{and}\quad R_{2i}=x_{2i}^2,$$

for $i=1,\dots,r$, and

$$R_{2r+1}=x_1x_3\cdots x_{2r-1}x_{2r+1}^2x_{2r+2}^2\cdots x_{2r+\widetilde{r}}^2\cdot \\ \cdot[x_{2r+\widetilde{r}+1},x_{2r+\widetilde{r}+2}]\cdots[x_{2r+\widetilde{r}+2s-1},x_{2r+\widetilde{r}+2s}].$$

Example 2.1. Observe that $G_{0,0,s}$, with $s\geq 1$, is the group studied in Example 1.6, and that $G_{0,\widetilde{r},0}$, with $\widetilde{r}\geq 3$, is the group studied in Example 1.7.

Theorem 2.2. *Let G be a co-compact extended quasifuchsian group of the first kind acting freely on its domain of discontinuity. Then, there are natural integers r, $\widetilde{r}$ and s such that G is isomorphic to the group $G_{r,\widetilde{r},s}$. Moreover, one necessarily has $r+\widetilde{r}+2s\geq 3$ and $r+\widetilde{r}\neq 0$.*

Conversely, let r, $\widetilde{r}$ and s be natural integers such that $r+\widetilde{r}+2s\geq 3$ and $r+\widetilde{r}\neq 0$. Then, there is a faithful co-compact representation $\rho\colon G_{r,\widetilde{r},s}\to\mathrm{PSL}_2(\mathbb{C})$ such that $\rho(G_{r,\widetilde{r},s})$ is an extended quasifuchsian group of the first kind acting freely on its domain of discontinuity.

Although it does not explicitly contain the above statement, one is referred to the paper [2] for a proof.

Remark 2.3. Let G be a co-compact extended quasifuchsian group of the first kind acting freely on its domain of discontinuity Ω. Let r, $\widetilde{r}$ and s be natural integers such that $G_{r,\widetilde{r},s}\cong G$. The integers r, $\widetilde{r}$ and s have the following geometric interpretation.

The compact Riemann surface $X=\Omega/G$ comes equipped with an antiquasiconformal involution ζ. The quotient $S=X/<\zeta>$ is homeomorphic to the connected sum of $\widetilde{r}$ projective planes and a compact oriented surface of genus s with r connected boundary components. Note that, according to the classification of such surfaces, the integers r, $\widetilde{r}$ and s are not uniquely determined by the group G (see also Remark 2.4). The condition $r+\widetilde{r}\neq 0$ is equivalent to S not being a closed orientable surface. The genus g of X is easily seen to be equal to $r+\widetilde{r}+2s-1$. Therefore, the condition $r+\widetilde{r}+2s\geq 3$ is equivalent to $g\geq 2$.

The group $G_{r,\widetilde{r},s}$ has the following geometric interpretation. Any topological surface with boundary can be considered in a natural way as a $\mathbb{Z}/2\mathbb{Z}$-orbifold by giving the points of the interior trivial isotropy groups and the points in the boundary non-trivial isotropy groups. Then, the group $G_{r,\widetilde{r},s}$ is isomorphic to the fundamental group of S, considered as a $\mathbb{Z}/2\mathbb{Z}$-orbifold. Or, to put it in the terminology of [2], the group $G_{r,\widetilde{r},s}$ is isomorphic to the $<\zeta>$-equivariant fundamental group of X.

Remark 2.4. We should note that the groups $G_{r,\widetilde{r},s}$ are not mutually nonisomorphic. More precisely, let $r, r', \widetilde{r}, \widetilde{r}', s, s'$ be natural integers. Let $g = r + \widetilde{r} + 2s - 1$ and $g' = r' + \widetilde{r}' + 2s' - 1$. We state under what conditions the groups $G_{r,\widetilde{r},s}$ and $G_{r',\widetilde{r}',s'}$ are isomorphic.

Firstly, note that the group $G_{r,\widetilde{r},s}$ is finite if and only if $g = -1$ or $g = 0$, and in that case $G_{r,\widetilde{r},s}$ is isomorphic to the group 0 or $\mathbb{Z}/2\mathbb{Z}$, respectively. Therefore, we may assume, as far as classification is concerned, that g and g' are both strictly positive. Then, the groups $G_{r,\widetilde{r},s}$ and $G_{r',\widetilde{r}',s'}$ are isomorphic if and only if the following three conditions hold.

1. $g = g'$,
2. $r = r'$,
3. $\widetilde{r} = 0 = \widetilde{r}'$ or $\widetilde{r} \neq 0 \neq \widetilde{r}'$.

As a consequence, the group $G_{r,\widetilde{r},s}$ is isomorphic to the group $G_{r,\widetilde{r}+2s,0}$ if $\widetilde{r} \neq 0$. In particular, when studying the groups $G_{r,\widetilde{r},s}$, one may assume that either $\widetilde{r} = 0$ or $s = 0$.

We refer to [3] for further details.

We will now study the space $\mathcal{R}(G)$ of $\mathrm{PSL}_2(\mathbb{C})$-representations of $G = G_{r,\widetilde{r},s}$, for all natural integers r, $\widetilde{r}$ and s. We may assume that G is not the trivial group, i.e., that not all integers r, $\widetilde{r}$ and s are zero.

Using the presentation (1) of G and with the same notation as in Section 1, we have

$$n = 2r + \widetilde{r} + 2s \text{ and } m = 2r + 1.$$

For $\delta \in \mu_2^n$ one has

$$\sigma_{2i-1}(\delta) = 1 \text{ and } \sigma_{2i}(\delta) = 1 \text{ for } i = 1, \dots, r, \text{ and}$$

$$\sigma_{2r+1}(\delta) = \prod_{i=1}^{r} \delta(2i-1).$$

Therefore, the total signature map $\sigma\colon \mu_2^n \to \mu_2^m$ is trivial if $r = 0$. If $r \neq 0$, the total signature map has its kernel K of index 2 in μ_2^n and its image I equal to the subgroup $\{1\}^{m-1} \times \mu_2^1$ of μ_2^m.

In order to determine the components of $\mathcal{R}(G)$ we need some preparation.

The proof of the following statement is easy and is left to the reader.

Lemma 2.5. *Let $\alpha \in \mathrm{SL}_2(\mathbb{C})$.*

1. *There is a $\beta \in \mathrm{SL}_2(\mathbb{C})$ such that $[\alpha, \beta] = 1$ and $\beta^2 = 1$.*
2. *There is no $\beta \in \mathrm{SL}_2(\mathbb{C})$ such that $[\alpha, \beta] = -1$ and $\beta^2 = 1$.*
3. *There is a $\beta \in \mathrm{SL}_2(\mathbb{C})$ such that $[\alpha, \beta] = 1$ and $\beta^2 = -1$ if and only if α is not parabolic.*
4. *There is a $\beta \in \mathrm{SL}_2(\mathbb{C})$ such that $[\alpha, \beta] = -1$ and $\beta^2 = -1$ if and only if $\alpha^2 = -1$.* □

Let $T \subseteq \mathrm{SL}_2(\mathbb{C})$ be the subset defined by

$$T = \{\alpha \in \mathrm{SL}_2(\mathbb{C}) \mid \alpha^2 = -1\} = \{\alpha \in \mathrm{SL}_2(\mathbb{C}) \mid \mathrm{tr}(\alpha) = 0\}.$$

It is easy to check that $T \cdot T = \mathrm{SL}_2(\mathbb{C})$. The following statement then easily follows.

Lemma 2.6. *Let $r \geq 2$. Then, $T^r = \mathrm{SL}_2(\mathbb{C})$.* □

Theorem 2.7. *Let G be the group $G_{r,\widetilde{r},s}$, where r, $\widetilde{r}$ and s are natural integers, not all equal to zero. Let $\varepsilon \in \mu_2^m$. Then the following statements hold.*

1. *If $(\varepsilon(2i-1), \varepsilon(2i)) = (-1, 1)$ for some i, with $1 \leq i \leq r$, then the subset $\widetilde{\mathcal{R}}(G)_\varepsilon$ of $\widetilde{\mathcal{R}}(G)$ is empty.*
2. *Suppose that $r \neq 0$ and $(r, \widetilde{r}, s) \neq (1, 0, 0)$. The converse of 1 holds, i.e., if $(\varepsilon(2i-1), \varepsilon(2i)) \neq (-1, 1)$ for all $i = 1, \dots, r$, then the subset $\widetilde{\mathcal{R}}(G)_\varepsilon$ of $\widetilde{\mathcal{R}}(G)$ is not empty.*
3. *Suppose that $r = 0$. Then, the subset $\widetilde{\mathcal{R}}(G)_\varepsilon$ of $\widetilde{\mathcal{R}}(G)$ is not empty.*
4. *Suppose that $(r, \widetilde{r}, s) = (1, 0, 0)$. Then, the subset $\widetilde{\mathcal{R}}(G)_\varepsilon$ is not empty if and only if $\varepsilon(1) = 1$.*

Proof. Statement 1 follows immediately from Lemma 2.5.2. Statement 3 can be shown as in Examples 1.6 and 1.7. Or, if one admits Remark 2.4, one may assume that either $\widetilde{r} = 0$ or $s = 0$, and then Statement 3 actually follows from Examples 1.6 and 1.7.

In order to show Statement 4, observe that the group $G_{1,0,0}$ is the group given by the presentation $<x_1, x_2 \,|\, [x_1, x_2], x_2^2, x_1>$. It is then clear that $\widetilde{\mathcal{R}}(G)_\varepsilon$ is empty if $\varepsilon(1) = -1$, and that $\widetilde{\mathcal{R}}(G)_\varepsilon$ is not empty if $\varepsilon(1) = 1$.

Now, we show Statement 2. Suppose that $(\varepsilon(2i-1), \varepsilon(2i)) \neq (-1, 1)$ for all $i = 1, \dots, r$. Denote an element of $\mathrm{SL}_2(\mathbb{C})^{2r+\widetilde{r}+2s}$ by $X = (\alpha_1, \dots, \alpha_{2r+\widetilde{r}+2s})$. We have to show that there is an $X \in \mathrm{SL}_2(\mathbb{C})$ that belongs to $\widetilde{\mathcal{R}}(G)_\varepsilon$.

Suppose first that $\widetilde{r} > 0$ or $s > 0$. By Lemma 2.5, one can choose $\alpha_1, \dots, \alpha_{2r} \in \mathrm{SL}_2(\mathbb{C})$ such that

$$[\alpha_{2i-1}, \alpha_{2i}] = \varepsilon(2i-1) \quad \text{and} \quad \alpha_{2i}^2 = \varepsilon(2i),$$

for all $i = 1, \dots, r$. Since the square map

$$\mathrm{sq}\colon \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

and the commutator map

$$[\cdot,\cdot]\colon \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

are surjective and since $\widetilde{r} > 0$ or $s > 0$, there are $\alpha_{2r+1} \dots, \alpha_{2r+\widetilde{r}+2s} \in \mathrm{SL}_2(\mathbb{C})$ such that $R_{2r+1}(X) = \varepsilon(2r+1)$. But then $X \in \widetilde{\mathcal{R}}(G)_\varepsilon$.

Finally, suppose that $\widetilde{r} = s = 0$. By hypothesis, $r \geq 2$. By Lemma 2.6, there are $\alpha_1, \alpha_3, \dots .\alpha_{2r-1} \in \mathrm{SL}_2(\mathbb{C})$ such that $\alpha_{2i-1}^2 = -1$, for $i = 1, \dots, r$, and $\alpha_1 \cdots \alpha_{2r-1} = \varepsilon(2r+1)$. Note that the elements α_{2i-1} are, in particular, non-parabolic. Then, by Lemma 2.5, there are $\alpha_2, \alpha_4, \dots, \alpha_{2r} \in \mathrm{SL}_2(\mathbb{C})$ such that X belongs to $\widetilde{\mathcal{R}}(G)_\varepsilon$. □

Theorem 2.8. *Let G be the group $G_{r,\widetilde{r},s}$, where r, $\widetilde{r}$ and s are natural integers.*

1. *If $(r, \widetilde{r}, s) = (0, 0, 0)$ the the space $\mathcal{R}(G)$ of $\mathrm{PSL}_2(\mathbb{C})$-representations of G has 1 component.*
2. *If $r = 0$ or $(r, \widetilde{r}, s) = (1, 0, 0)$ then the space $\mathcal{R}(G)$ of $\mathrm{PSL}_2(\mathbb{C})$-representations of G has 2 components.*
3. *Otherwise, i.e., if $r \neq 0$ and $(r, \widetilde{r}, s) \neq (1, 0, 0)$, the space $\mathcal{R}(G)$ of $\mathrm{PSL}_2(\mathbb{C})$-representations of G has 3^r components.*

Proof. The first statement is trivial since the group G is trivial if $(r,\widetilde{r},s)=(0,0,0)$. Therefore, suppose that $(r,\widetilde{r},s)\neq(0,0,0)$. By Theorem 2.7, the space $\widetilde{\mathcal{R}}(G)$ has 2 components if $r=0$, 4 components if $(r,\widetilde{r},s)=(1,0,0)$ and $2\cdot 3^r$ components otherwise. By the discussion above, the group $I\cong\mu_2$ acts trivially on $(\mu_2^m)'$ if $r=0$ and nontrivially if $r\neq 0$. The second and third statement now follow from Proposition 1.2. □

According to Theorem 2.8, the representation space $\mathcal{R}(G_{r,\widetilde{r},s})$ can have many components. The following statement asserts that one of these components contain all faithful discrete representations.

Theorem 2.9. *Let G be the group $G_{r,\widetilde{r},s}$, where r, $\widetilde{r}$ and s are natural integers. There is a unique component $\mathcal{R}_{\mathrm{fd}}(G)$ of $\mathcal{R}(G)$ such that all faithful discrete representations $\rho\in\mathcal{R}(G)$ belong to $\mathcal{R}_{\mathrm{fd}}(G)$.*

Proof. If G is trivial, then the statement is trivial. If $(r,\widetilde{r},s)=(0,1,0)$ then the statement is easy to verify. Therefore, suppose $(r,\widetilde{r},s)\neq(0,0,0)$ and $(0,1,0)$.

If $r=0$, the statement can be proven as in Examples 1.6 and 1.7. Or again, by Remark 2.4, one may assume that either $\widetilde{r}=0$ or $s=0$, and then the statement of the proposition actually follows from Examples 1.6 and 1.7.

If $(r,\widetilde{r},s)=(1,0,0)$, then G is isomorphic to the group $G_{0,1,0}$ which has already been treated. Hence, we may assume, moreover, that $r\neq 0$ and $(r,\widetilde{r},s)\neq(1,0,0)$.

Let $\rho\in\mathcal{R}(G)$ be faithful (and discrete). Choose $\varepsilon\in(\mu_2^m)'$ such that $\rho\in\mathcal{R}(G)_\varepsilon$. We show that $\varepsilon(2i-1)=1$ and $\varepsilon(2i)=-1$, for $i=1,\ldots,r$. Since $r\neq 0$, all such ε are in the same orbit for the action of μ_2^n on $(\mu_2^m)'$. Therefore, the statement will be proven.

Choose $\widetilde{\rho}\in\widetilde{\mathcal{R}}(G)$ such that $\pi(\widetilde{\rho})=\rho$. Write $\widetilde{\rho}=(\alpha_1,\ldots,\alpha_{2r+\widetilde{r}+2s})$. Since ρ is faithful and $\rho(x_{2i})$ is of order 2, one has $\alpha_{2i}^2=-1$, i.e., $\varepsilon(2i)=-1$, for $i=1,\ldots,r$. Now, since α_{2i-1} cannot have finite order, $[\alpha_{2i-1},\alpha_{2i}]=1$ by Lemma 2.5. It follows that $\varepsilon(2i-1)=1$, for $i=1,\ldots,r$. □

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