Clifford's Inequality for real algebraic curves

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Abstract

We improve Clifford's Inequality for real algebraic curves. As an application we improve Harnack's Inequality for real space curves having a certain number of pseudo-lines. Another application involves the number of ovals that a real space curve can have.

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1 Introduction

Clifford's Inequality states that the dimension of a nonempty special complete linear system on an algebraic curve is bounded by half of its degree [2, p. 252]. Using the real topology of the linear system, we give a sharper upper bound on the dimension of such a linear system, for a real algebraic curve (Theorem 3.1).

As an application, we improve, for a real space curve having a certain number of pseudo-lines, the upper bound on the number of real branches that one obtains by a direct application of Castelnuovo's and Harnack's Inequality (Theorem 4.1). For example, let C be a nondegenerate real algebraic curve in $\mathbb{P}^3$ of degree 7 that has at least two pseudo-lines. Then, we show that C has at most 5 real branches, whereas a direct application of Castelnuovo's and Harnack's Inequality would only have given that C has at most 7 real branches.

Another application involves real space curves of genus g having at least g real branches. We show that such a curve in $\mathbb{P}^n$ of degree d has at least $n+1$ ovals if $g > d-n$ (Corollary 5.2). We show that if $g \leq d-n$ then almost any number of ovals, between 0 and $g+1$, can be obtained (Theorem 5.4).

Convention and notation. A real algebraic curve is a smooth geometrically irreducible proper scheme over $\mathbb{R}$ of dimension 1. Projective n-space over $\mathbb{R}$ is denoted by $\mathbb{P}^n$ instead of $\mathbb{P}^n_{\mathbb{R}}$.

Let C be a real algebraic curve. A *real branch* of C is a connected component of the set of real points $C(\mathbb{R})$ of C. Since C is proper and smooth, a real branch of C is necessarily homeomorphic to the unit circle.

Let D be a divisor on C and let X be a real branch of C. We define the *degree of D on X* to be the natural integer

$$\deg_X(D) = \sum_{P \in X} n_P P,$$

if $D = \sum_{P \in C} n_P P$.

The following lemma turns out to be crucial in the study of linear systems on real algebraic curves:

Lemma 2.1. *Let C be a real algebraic curve. Let D be a divisor on C and let d be its degree. Let δ be the number of real branches X of C such that the degree $\deg_X(D)$ of D on X is odd. If $d < \delta$ then $h^0(D) = 0$.*

Proof. Suppose that $h^0(D) \neq 0$. Then there is an effective divisor E on C that is linearly equivalent to D. For a real branch X of C, the degree $\deg_X(E)$ of E on X satisfies

$$\deg_X(E) \equiv \deg_X(D) \pmod 2.$$

Then, it follows from the hypothesis that $\deg_X(E) \not\equiv 0 \pmod 2$ for δ real branches X of C. In particular, the support of E contains at least δ real points. Since E is effective, $\deg(E) \geq \delta$. This contradicts the fact that $\deg(E) = \deg(D) = d$. □

The preceding lemma allows yet another proof of Harnack's Inequality for real algebraic curves [3]:

Corollary 2.2 (Harnack's Inequality). *Let C be a real algebraic curve and let g be its genus. Let s be the number of real branches of C. Then,*

$$s \leq g + 1.$$

Proof. We may assume that $s \neq 0$. Choose real points $P_1, \ldots, P_s$ on C, each belonging to a different real branch. Let D be the divisor $P_1 + \cdots + P_{s-1} - P_s$. Let d be the degree of D and let δ be the number of real branches of C on which D has odd degree. Then, $d = s - 2$ and $\delta = s$. By Lemma 2.1, $h^0(D) = 0$. By Riemann-Roch, $-h^0(K - D) = d - g + 1$, where K is canonical divisor on C. In particular, $s - g - 1 = d - g + 1 \leq 0$. Hence, $s \leq g + 1$. □

Lemma 2.1 is crucial since it implies the somewhat complementary key results of [6] and [5]:

Theorem 2.3 of [6]. *Let C be a real algebraic curve and let g be its genus. Let D be a divisor on C and let d be its degree. Let δ be the number of connected components X of $C(\mathbb{R})$ such that $\deg_X(D)$ is odd. If $d+\delta \geq 2g-1$ then D is nonspecial.*

Proof. Let K be a canonical divisor on C. We want to apply Lemma 2.1 to the divisor $K-D$. Note that $\deg_X(K)$ is even for all real branches X of C (see [1, Corollary 4.2.2] or [6, Lemma 2.2]). Therefore, $\deg_X(K-D) \equiv \deg_X(D)$ for all real branches X of C. In particular, $\deg_X(K-D)$ is odd for exactly δ real branches X of C. By hypothesis, $\deg(K-D) = 2g-2-d < \delta$. One can, therefore, apply Lemma 2.1 to $K-D$ in order to conclude that $h^0(K-D) = 0$, i.e. that D is nonspecial. □

Theorem 2.2 of [5]. *Let C be a real algebraic curve and let s be the number of real branches of C. Let $k < s$ be a natural integer and let $P_1, \ldots, P_k$ be real points of C such that no two of them belong to the same real branch of C. Let D be the divisor $\sum_{i=1}^k P_i$. Then, $h^0(D) = 1$.*

Proof. Since D is effective, $h^0(D) \geq 1$. In order to show that $h^0(D) \leq 1$, choose a real point P of C at a real branch different from the real branches that contain one of the points $P_1, \ldots, P_k$. Let E be the divisor $D - P$. Then E satisfies the conditions of Lemma 2.1. Hence, $h^0(E) = 0$. But then $h^0(D) = h^0(E+P) \leq 1$. □

3 Clifford's Inequality for real curves

Let us recall Clifford's Inequality for real algebraic curves. Let C be a real algebraic curve. Let D be a divisor on C and let d be its degree. If D is effective and special then the dimension of the linear system $|D|$ satisfies

$$\dim |D| \leq \tfrac{1}{2}d.$$

This is well known for complex algebraic curves [2, p. 251], and then immediately follows for real algebraic curves.

The following statement improves Clifford's Inequality for real algebraic curves.

Theorem 3.1. *Let C be a real algebraic curve and let s be the number of its real branches. Let D be a divisor on C and let d be its degree. Let δ be the number of real branches X of C such that $\deg_X(D)$ is odd. If D is effective then the following statements hold.*

1. If $d+\delta < 2s$ *then*

$$\dim|D| \leq \tfrac{1}{2}(d-\delta).$$

2. If $d+\delta \geq 2s$ *then*

$$\dim|D| \leq d-s+1.$$

Proof. 1. Since D is effective, $d \geq \delta$. Put $\ell = \frac{1}{2}(d-\delta)+1$ so that $\ell \geq 1$. Note that ℓ is an integer. From the hypotheses it follows that $\delta+\ell \leq s$. Hence, one can choose real points $P_1, \dots, P_\ell$ on C such that no two of them belong to the same real branch of C and such that none of them belong to a real branch X of C such that $\deg_X(D)$ is odd. Let $E = D - \sum_{i=1}^{\ell} P_i$. Then, the degree of E is odd on $\delta+\ell$ real branches of C. Moreover,

$$\deg(E) = d-\ell < \delta+\ell.$$

By Lemma 2.1, $h^0(E)=0$. Since $D = E+\sum_{i=1}^{\ell} P_i$, one has $h^0(D) \leq \ell$, i.e., $\dim|D| \leq \frac{1}{2}(d-\delta)$.

2. Again, $d \geq \delta$. Put $\ell = s-\delta$ and $e = \frac{1}{2}(d+\delta)-s+1$. Note that ℓ and e are natural integers. As before, choose real points $P_1, \dots, P_\ell$ on C such that no two of them belong to the same real branch of C and such that none of them belong to a real branch X of C such that $\deg_X(D)$ is odd. Moreover, choose a nonreal point P of C. Let $E = D - \sum_{i=1}^{\ell} P_i - eP$. Then, E has odd degree on all real branches of C. Moreover,

$$\deg(E) = d-\ell-2e = s-2 < s.$$

By Lemma 2.1, $h^0(E)=0$. Since $D = E+\sum_{i=1}^{\ell} P_i + eP$, one has $h^0(D) \leq \ell+2e = d-s+2$, i.e., $\dim|D| \leq d-s+1$. □

Theorem 3.1 is the most interesting for real algebraic curves having many real branches. Let C be a real algebraic curve C. Let g be its genus and let s be the number of its real branches. By Harnack's Inequality, $s \leq g+1$. We say that C has *many real branches* if $s \geq g$. Real algebraic curves of genus g that have many real branches abound. They come along in a $(3g-3)$-dimensional family if $g \geq 2$ [7, 4].

Theorem 3.2. *Let* C *be a real algebraic curve having many real branches. Let* D *be a divisor on* C *and let* d *be its degree. Let* δ *be the number of real branches* X *of* C *such that* $\deg_X(D)$ *is odd. If* D *is effective and special then*

$$\dim|D| \leq \tfrac{1}{2}(d-\delta).$$

Proof. Let g be the genus of C. By hypothesis, $s \geq g$. If $d+\delta \geq 2s$, then $d+\delta \geq 2g$ and D would be nonspecial by Theorem 2.3 of [6]. Since D is not nonspecial, $d+\delta < 2s$. Hence, $\dim|D| \leq \frac{1}{2}(d-\delta)$, by Theorem 3.1. □

Let $n \geq 2$ and let $C \subseteq \mathbb{P}^n$ be a real algebraic curve. We say that C is *nondegenerate* if C is not contained in a real hyperplane of $\mathbb{P}^n$. We assume, in what follows, that C is nondegenerate.

Let X be a real branch of C. Let $[X]$ be the homology class of X in the first homology group $H_1(\mathbb{P}^n(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. One says that X is a *pseudo-line* of C if $[X] \neq 0$. Otherwise, X is an *oval* of C. Equivalently, X is a pseudo-line of C if and only if each hyperplane H in $\mathbb{P}^n(\mathbb{R})$ intersects X in an odd number of points, when counted with multiplicities. Note that it makes sense to talk about multiplicities since X and H are real analytic submanifolds of $\mathbb{P}^n(\mathbb{R})$ that intersect each other in a finite number of points.

Let d be the degree of C and let g be its genus. Since C is nondegenerate, $d \geq n$ [2, p. 253]. Recall the Castelnuovo Inequality [2, p. 253]

$$g \leq \tfrac{1}{2}q(q-1)(n-1) + qr,$$

where q and r are the quotient and the remainder, respectively, in the Euclidean division of $d-1$ by $n-1$. The Castelnuovo Inequality is known to be sharp. For $d \leq 2n-1$, Castelnuovo's Inequality reads

$$g \leq d - n.$$

For $d \geq 2n$, the upper bound $\frac{1}{2}q(q-1)(n-1) + qr$ for g is strictly greater than $d-n$.

Let s be the number of real branches of C. Castelnuovo's and Harnack's Inequality together imply that

$$s \leq \tfrac{1}{2}q(q-1)(n-1) + qr + 1.$$

For $d \leq 2n-1$, this inequality reads $s \leq d-n+1$. For $d \geq 2n$, the upper bound $\frac{1}{2}q(q-1)(n-1) + qr + 1$ for s is strictly greater than $d-n+1$.

Since Castelnuovo's and Harnack's Inequality are both sharp, it is reasonable to think that the above inequality on s is sharp as well. The following statement implies that the above inequality is far from sharp when C has sufficiently many pseudo-lines.

Theorem 4.1. *Let $n \geq 2$ be an integer and let $C \subseteq \mathbb{P}^n$ be a nondegenerate real algebraic curve. Let d be the degree of C and let s be the number of real branches of C. Let δ be the number of pseudo-lines of C. If $d < 2n + \delta$ then $s \leq d - n + 1$.*

Proof. By hypothesis, $\frac{1}{2}(d-\delta) < n$. In particular, $\dim |D| > \frac{1}{2}(d-\delta)$, where D denotes a hyperplane section of C. By Theorem 3.1, $d + \delta \geq 2s$ and $\dim |D| \leq d - s + 1$. In particular, $n \leq d - s + 1$, i.e., $s \leq d - n + 1$. □

As observed above, if $d \leq 2n-1$ then Castelnuovo's and Harnack's Inequality already imply that $s \leq d-n+1$. Hence, Theorem 4.1 does not give anything new in that case. However, if $d \geq 2n$ and $d < 2n+\delta$ then the upper bound $d-n+1$ is much sharper than the upper bound $\frac{1}{2}q(q-1)(n-1)+qr+1$ for s.

Note that $d \equiv \delta \pmod 2$ in Theorem 4.1. Therefore, in view of the preceding observation, Theorem 4.1 gives nothing new for $\delta = 0$ or 1. For topological reasons, a real algebraic curve in $\mathbb{P}^2$ can have at most one pseudo-line. Therefore, Theorem 4.1 gives nothing new for real curves in $\mathbb{P}^2$. Let us reformulate Theorem 4.1 for real curves in $\mathbb{P}^3$.

Corollary 4.2. *Let $C \subseteq \mathbb{P}^3$ be a nondegenerate real algebraic curve. Let d be the degree of C and let s be the number of real branches of C. Let δ be the number of pseudo-lines of C. If $d < 6+\delta$ then $s \leq d-2$.* □

Note that a direct application of Castelnuovo's and Harnack's Inequality would have given the inequality

$$s \leq \begin{cases} \frac{1}{4}(d-2)^2+1 & \text{if } d \text{ is even, and} \\ \frac{1}{4}(d-1)(d-3)+1 & \text{if } d \text{ is odd,} \end{cases}$$

which is a weaker upper bound on s than the upper bound $d-2$ of Corollary 4.2, if $d \geq 6$.

Example 4.3. Let C be a nondegenerate real sextic in $\mathbb{P}^3$. Let s be the number of real branches of C. If C has a pseudo-line then $s \leq 4$, by Corollary 4.2.

This can also be seen directly. Indeed, the genus g of C is at most 4. Moreover, $g = 4$ if and only if C is a canonical curve [2, p. 253]. Now, a canonical curve has no pseudo-lines (see [1, Corollary 4.2.2] or [6, Lemma 2.2]). Hence, C is not a canonical curve. It follows that $g \leq 3$ and $s \leq 4$ by Harnack's Inequality.

In fact, the direct argument of the preceding example applies in general. Therefore, Theorem 4.1 does not give anything new in case $d = 2n$.

Here is the first example where Theorem 4.1 gives a sharper bound on the number of real branches of a real space curve.

Example 4.4. Let C be a nondegenerate real algebraic curve in $\mathbb{P}^3$ of degree 7. Let s be the number of real branches of C. If C has at least two pseudo-lines then $s \leq 5$, by Corollary 4.2. Note that, a priori, the genus of C is at most 6, so that a direct application of Harnack's Inequality would have given the bound $s \leq 7$ for the number of real branches of C.

5 Real space curves having many real branches

One may wonder whether the conditions of Theorem 4.1 imply that the genus g of C satisfies $g \leq d-n$. According to the next statement, this is the case, if C has many real branches.

Theorem 5.1. *Let $n \geq 2$ be an integer and let $C \subseteq \mathbb{P}^n$ be a nondegenerate real algebraic curve having many real branches. Let d be the degree of C and let g be the genus of C. Let δ be the number of pseudo-lines of C. If $d < 2n + \delta$ then $g \leq d - n$.*

Proof. By hypothesis, $\frac{1}{2}(d-\delta) < n$. In particular, $\dim |D| > \frac{1}{2}(d-\delta)$, where D denotes a hyperplane section of C. By Theorem 3.2, D is nonspecial, i.e., $\dim |D| = d - g$. Hence, $n \leq d - g$, i.e., $g \leq d - n$. □

When one turns around the formulation of Theorem 5.1, one gets a lower bound on the number of ovals for real space curves having many real branches of sufficiently high genus.

Corollary 5.2. *Let $n \geq 2$ be an integer and let $C \subseteq \mathbb{P}^n$ be a nondegenerate real algebraic curve having many real branches. Let d be the degree of C and let g be the genus of C. Let ε be the number of ovals of C. If $g > d - n$ then*

$$\varepsilon \geq \begin{cases} g-d+2n & \text{if } C \text{ has } g \text{ real branches, and} \\ g-d+2n+1 & \text{if } C \text{ has } g+1 \text{ real branches.} \end{cases}$$

In particular,

$$\varepsilon \geq \begin{cases} n+1 & \text{if } C \text{ has } g \text{ real branches, and} \\ n+2 & \text{if } C \text{ has } g+1 \text{ real branches.} \end{cases}$$

Proof. Let δ be the number of pseudo-lines of C. By Theorem 5.1, $d \geq 2n+\delta$. If C has exactly g real branches, $\delta + \varepsilon = g$. Hence, $d \geq 2n + g - \varepsilon$. Hence, $\varepsilon \geq (g-d+n)+n \geq n+1$. One proves similarly that $\varepsilon \geq (g-d+n)+n+1 \geq n+2$ if C has exactly $g+1$ real branches. □

Of course, if C is a real curve in $\mathbb{P}^2$, then $\delta = 0$ or 1, so that all real branches of C, except possibly one, are ovals. Hence, Corollary 5.2 is not interesting in that case. For real curves in $\mathbb{P}^3$, Corollary 5.2 has the following nice consequence.

Corollary 5.3. *Let $C \subseteq \mathbb{P}^3$ be a nondegenerate real algebraic curve having many real branches. Let d be the degree of C and let g be the genus of C. If $g > d - 3$ then C has at least 4 ovals.* □

One may wonder what one can say on the number of ovals of C when $C \subseteq \mathbb{P}^n$ is a real curve having many real branches and satisfying $g \leq d - n$. The following statement says that any number between 0 and $g + 1$ can be obtained, except $g + 1$ when d is odd.

Theorem 5.4. *Let n, d, g and ε be natural integers such that $n \geq 3$, $d \geq n$, $g \leq d-n$ and $\varepsilon \leq g+1$. If $\varepsilon \neq g+1$ or d is even then there is a nondegenerate real algebraic curve $C \subseteq \mathbb{P}^n$ having many real branches, of degree d and of genus g, such that the number of ovals of C is equal to ε. If $\varepsilon = g+1$ and d is odd then there is no such a real curve.*

Proof. Let us start by showing the last assertion. Let $C \subseteq \mathbb{P}^n$ be a real algebraic curve having many real branches, of degree d and of genus g, such that the number of ovals of C is equal to $g + 1$. By Harnack's Inequality, all real branches of C are ovals, i.e., $\delta = 0$. Since $d \equiv \delta \pmod 2$, d is even. This shows the last assertion.

We show the first assertion first in case $\varepsilon = 0$ and $g = d-n$, i.e., $n = d-g$. There are two cases to consider: the case $d - g$ is even, and the case $d - g$ is odd.

If $d-g$ is even, let C be a real algebraic curve of genus g having exactly g real branches. Since $d \equiv g \pmod 2$, there is a divisor D on C of degree d having odd degree on all real branches of C. Since $d + g \geq 2g + 3$, the divisor D is nonspecial by Theorem 2.3 of [6]. Hence, $\dim |D| = d - g = n$. In fact, since $d + g$ is "much" greater than $2g - 1$, the linear system $|D|$ is base point-free and induces an embedding of C into $\mathbb{P}^n$ (see Corollary 2.4 of [6]). It is clear that the image curve is nondegenerate, has degree d and genus g. Moreover, each real branch is a pseudo-line.

If $d - g$ is odd, one takes C a real algebraic curve of genus g having $g+1$ real branches. Then, the same argument as above shows that C can be embedded into $\mathbb{P}^n$ such that the image curve is nondegenerate, has degree d and genus g, and, moreover, each real branch is a pseudo-line.

This shows the first assertion of Theorem 5.4 in case $\varepsilon = 0$ and $g = d-n$. Now, let us treat the general case.

Suppose that $\varepsilon \neq g+1$ or that d is even. Let $d' = d+\varepsilon$ and let $n' = d'-g$. Then, by what we have seen above, there is a nondegenerate real algebraic curve $C \subseteq \mathbb{P}^{n'}$ of degree d' and of genus g such that all its real branches are pseudo-lines. Moreover, C' has g real branches if $d'-g$ is even, and has $g+1$ real branches if $d'-g$ is odd. We want to choose general real points $P_1, \ldots, P_\varepsilon$ on C' such that each one of them belongs to a different real branch of C'. This is certainly possible if $\varepsilon \neq g + 1$. If $\varepsilon = g + 1$ then, by hypothesis, d is even, hence, $d' - g = d+\varepsilon-g = d+1$ is odd and C' has $g+1$ real branches. Therefore, in that case, we can choose such real points as well.

The linear hull in $\mathbb{P}^{n'}$ of the real points $P_1, \ldots, P_\varepsilon$ has dimension at most $\varepsilon - 1$. Hence, its codimension in $\mathbb{P}^{n'}$ is at least $n' - \varepsilon + 1 \geq n + 1$. Therefore, we can choose a general real linear subspace $L \subseteq \mathbb{P}^{n'}$ of codimension $n+1$ containing the points $P_1 \ldots, P_\varepsilon$. Let $\pi \colon \mathbb{P}^{n'} \to \mathbb{P}^n$ be the linear projection with center L. Let C be the closure of $\pi(C' \setminus L)$. Then, C is a nondegenerate real algebraic curve in $\mathbb{P}^n$ of degree $d' - \varepsilon = d$ and of genus g. Moreover, the number of ovals of C is exactly equal to ε. $\square$

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