

HYPERBOLICITY OF GENERAL DEFORMATIONS

MIKHAIL ZAIDENBERG

ABSTRACT. This is the content of the talk given at the conference “Effective Aspects of Complex Hyperbolic Varieties”, Aver Wrac’h, France, September 10-14, '07. We present two methods of constructing low degree Kobayashi hyperbolic hypersurfaces in \mathbb{P}^n :

- **The projection method**
- **The deformation method**

The talk is based on joint works of the speaker with B. Shiffman and C. Ciliberto.

1. DIGEST on KOBAYASHI THEORY

1.1. Kobayashi hyperbolicity.

DEFINITION The *Kobayashi pseudometric* k_X on a complex space X satisfies the following axioms :

- (i) On the unit disc Δ , the Kobayashi pseudometric k_Δ coincides with the Poincaré metric;
- (ii) every holomorphic map $\varphi : \Delta \rightarrow X$ is a contraction: $\varphi^*(k_X) \leq k_\Delta$;
- (iii) k_X is the maximal pseudometric on X satisfying (i) and (ii).

REMARK Every holomorphic map $\varphi : X \rightarrow Y$ is a contraction: $\varphi^*(k_Y) \leq k_X$.

DEFINITION X is called *Kobayashi hyperbolic* if k_X is non-degenerate :

$$k_X(p, q) = 0 \iff p = q.$$

EXAMPLES $k_{\mathbb{C}^n} \equiv 0$, $k_{\mathbb{P}^n} \equiv 0$, $k_{\mathbb{T}^n} \equiv 0$, where $\mathbb{T}^n = \mathbb{C}^n/\Lambda$ is a complex torus, whereas $\mathbb{C} \setminus \{0, 1\}$ is hyperbolic (the Schottky-Landau Theorem.)

1.2. Classical theorems.

According to the above definition and to Royden’s Theorem, X is hyperbolic iff natural analogs of the classical Schottky and Landau Theorems hold for X .

Brody-Kiernan-Kobayashi-Kwack THEOREM

For a compact complex space X the following conditions are equivalent :

- X is Kobayashi hyperbolic;

- *Little Picard Theorem holds for X :*

$$\forall f : \mathbb{C} \rightarrow X, \quad f = \text{const};$$

- *Big Picard Theorem holds for X :*

$$\forall f : \Delta \setminus \{0\} \rightarrow X \quad \exists \bar{f} : \Delta \rightarrow X, : \bar{f}|(\Delta \setminus \{0\}) = f;$$

- *Montel Theorem holds for X : the space $HOL(\Delta, X)$ is compact.*

REMARK If X is hyperbolic then $\forall Y$, the space $HOL(Y, X)$ is compact.

DEFINITION Let M be a hermitian compact complex manifold. An entire curve $\varphi : \mathbb{C} \rightarrow M$ is called a *Brody curve* if

$$\|\varphi'(z)\| \leq 1 = \|\varphi'(0)\| \quad \forall z \in \mathbb{C}.$$

Brody's THEOREM M as above is hyperbolic iff it does not possess any Brody entire curve.

Brody's STABILITY THEOREM

Every compact hyperbolic subspace X of a complex space Z admits a hyperbolic neighborhood. Consequently, every compact subspace $X' \subseteq Z$ sufficiently close to X is hyperbolic. In particular, if $X \subseteq \mathbb{P}^n$ is a hyperbolic hypersurface then every hypersurface $X' \subseteq \mathbb{P}^n$ sufficiently close to X is hyperbolic too.

1.3. Hyperbolicity of hypersurfaces in \mathbb{P}^n .

Kobayashi Problem ('70)

Is it true that a (very) general hypersurface X of degree $d \geq 2n - 1$ in \mathbb{P}^n is Kobayashi hyperbolic? In particular, is this true for a (very) general surface X in \mathbb{P}^3 of degree $d \geq 5$?

Hyperbolic surfaces in \mathbb{P}^3

THEOREM (McQuillen [9], Demailly-El Goul [3])

A very general surface X in \mathbb{P}^3 of degree $d \geq 21$ is Kobayashi hyperbolic.

For some recent advances in higher dimensions, see Y.-T. Siu [15] and E. Rousseau [10].

EXAMPLES of small degree hyperbolic surfaces in \mathbb{P}^3

Concrete examples were found by

Brody-Green '77, $d = 2k \geq 50$,

Masuda-Noguchi '96, $d = 3e \geq 24$,

Khoai '96, $d \geq 22$,

Nadel '89, $d \geq 21$,
 Shiffman-Z' '00, $d \geq 16$,
 El Goul '96, $d \geq 14$,
 Siu-Yeung '96, Demailly-El Goul '97, $d \geq 11$,
 J. Duval '99 [5], Shirosaki-Fujimoto '00 [6], $d = 2k \geq 8$:

$$Q(X_0, X_1, X_2)^2 - P(X_2, X_3) = 0, \quad (1)$$

where Q, P are generic homogeneous forms of degree k and $d = 2k$, respectively,
 Shiffman-Z' '02 [11], $d = 8$,
 Shiffman-Z' '05 [12], $d \geq 8$,
 J. Duval '04 [4], $d = 6$.

Algebraic families of hyperbolic hypersurfaces $X_n \subseteq \mathbb{P}^n$ for any $n \geq 3$ were constructed e.g., by

Masuda-Noguchi '96,
 Siu-Yeung '97,
 Shiffman-Z' '02 [13].

In these examples $\deg X_n$ grows quadratically with n , for instance, $\deg X_n = 4(n - 1)^2$ [13]. Whereas the Kobayashi Conjecture suggests a linear growth of the minimal such degree. This leads to the following problem.

PROBLEM *Find a sequence of hyperbolic hypersurfaces $X_n \subseteq \mathbb{P}^n$ with $\deg X_n \leq Cn$ for some positive constant C .*

2. PROJECTION METHOD

2.1. Symmetric powers of curves as hyperbolic hypersurfaces.

PROPOSITION (Shiffman-Z' '00 [14]) *The n th symmetric power $C^{(n)}$ of a generic smooth projective curve C of genus $g \geq 3$ is hyperbolic iff $g \geq 2n - 1$. In particular, the symmetric square $C^{(2)}$ is always hyperbolic.*

THEOREM (Shiffman-Z' '00 [14]) *With C as before, let us consider an embedding $C^{(2)} \hookrightarrow \mathbb{P}^5$. Then a general projection S of $C^{(2)}$ to \mathbb{P}^3 is hyperbolic. The minimal degree of such a hyperbolic surface $S \subseteq \mathbb{P}^3$ is equal 16.*

EXAMPLE of degree 16: Let $C \subseteq \mathbb{P}^2 : x^4 - xz^3 - y^3z = 0$, and let $C^{(2)} \hookrightarrow \mathbb{P}^5$ be embedded via the natural embedding of the symmetric square of \mathbb{P}^2 in \mathbb{P}^5 . Then a general projection of $C^{(2)}$ to \mathbb{P}^3 is a singular hyperbolic surface $S \subseteq \mathbb{P}^3$ of degree 16, with the double curve D of genus 142.

Let us explain in brief our methods. Let $V \hookrightarrow \mathbb{P}^5$ be a smooth hyperbolic surface, and let $\pi : V \rightarrow S \hookrightarrow \mathbb{P}^3$ be a projection. Then S has self-intersection along a double curve $D \subseteq S$. By the universal property of the normalization, there is a commutative diagram

$$\begin{array}{ccc}
 V & \xrightarrow{\psi} & S_{\text{norm}} \\
 & \searrow \pi & \swarrow \nu \\
 & & S
 \end{array}$$

where $\nu : S_{\text{norm}} \rightarrow S$ is the normalization. By Zariski's Main Theorem, $\psi : V \rightarrow S_{\text{norm}}$ is an isomorphism. Hence any entire curve $\varphi : \mathbb{C} \rightarrow S$ can be lifted to $V = S_{\text{norm}}$:

$$\begin{array}{ccc}
 & & V \\
 & \nearrow \tilde{\varphi} & \downarrow \nu \\
 \mathbb{C} & \xrightarrow{\varphi} & S
 \end{array}$$

unless $\varphi(\mathbb{C}) \subseteq D$. Since V is hyperbolic, $\tilde{\varphi} = \text{cst}$. Thus S is hyperbolic iff D is. A similar argument shows that S is always hyperbolic modulo D . In the proof of the above theorem we show that, for a general projection, D is hyperbolic indeed and so S is. Similarly, for the Cartesian square of a curve the following holds.

PROPOSITION (Shiffman-Z' '00 [14]) *Let C be a smooth projective curve of genus $g \geq 2$. Let us fix an embedding $V = C \times C \hookrightarrow \mathbb{P}^n$. Then the double curve $D \subseteq S$ of a general projection $V \rightarrow S \subseteq \mathbb{P}^3$ is irreducible of genus $g(D) \geq 225$, and S is a singular hyperbolic surface of degree ≥ 32 .*

However for a non-generic projection, the double curve of the image surface can be neither irreducible nor hyperbolic.

EXAMPLE (Kaliman-Z' '01 [8]) Consider the smooth Fermat quartic

$$C : x^4 + y^4 + z^4 = 0 \quad \text{in } \mathbb{P}^2.$$

Then the product $V = C \times C$ admits a projective embedding and a projection to \mathbb{P}^3 such that the double curve D of the image surface $S \subseteq \mathbb{P}^3$ consists of 4 disjoint projective lines. Thus S is not hyperbolic whereas its normalization V is.

For 3-folds in \mathbb{P}^4 we have the following result.

THEOREM (Ciliberto-Z' '03 [2]) *For a general projective curve C of genus $g \geq 7$, we fix an embedding $C^{(3)} \hookrightarrow \mathbb{P}^7$. Then a general projection X of $C^{(3)}$ to \mathbb{P}^4 is a hyperbolic hypersurface in \mathbb{P}^4 . This is also true for a general quintic $C \subseteq \mathbb{P}^2$ ($g = 6$) and a certain special embedding $C^{(3)} \hookrightarrow \mathbb{P}^7$ of degree 125. The latter is the minimal degree which can be achieved via the projection method using the symmetric cubes $C^{(3)}$.*

The proof goes as follows. It is shown that

- $C^{(3)}$ does not contain any curve of genus $< g$; in particular, it is hyperbolic.
- $X \subseteq \mathbb{P}^4$ is hyperbolic iff the double surface $S = \text{sing}(X)$ is. This uses the above trick with lifting entire curves to the normalization $C^{(3)}$ of X .
- The irregularity $q(S) \geq g > 5$. This is based on the fact that for a curve C with general moduli, the Jacobian $J(C)$ is a simple abelian variety.
- S is hyperbolic iff it is algebraically hyperbolic that is, does not contain any rational or elliptic curve. This is based on the Bloch Conjecture.
- S is hyperbolic iff the triple curve $T \subseteq S$ of X is. Recall that in a general point of T , 3 smooth branches of X meet transversally. Actually T parameterizes the 3-secant lines of $C^{(3)} \subseteq \mathbb{P}^7$ parallel to the center of the projection $\mathbb{P}^7 \dashrightarrow \mathbb{P}^4$. The proof is based on Pirola's and Ciliberto-van der Geer's results on deformations of hyperelliptic and bielliptic curves on abelian varieties.
- Any irreducible component of the triple curve¹ T has genus ≥ 2 . The proof is rather involved.

3. DEFORMATION METHOD

Let $X_0 = f_0^*(0)$, $X_\infty = f_\infty^*(0)$ be two hypersurfaces of the same degree d in \mathbb{P}^n , and let

$$\{X_t\}_{t \in \mathbb{P}^1} = \langle X_0, X_\infty \rangle, \quad \text{where} \quad X_t = (f_0 + tf_\infty)^*(0),$$

be the pencil of hypersurfaces generated by X_0 and X_∞ . For small enough $|\varepsilon| \neq 0$ we call X_ε a small (linear) deformation of X_0 in direction of X_∞ .

DEFINITION We say that a (very) general small deformation of X_0 is hyperbolic if X_ε is for a (very) general X_∞ and for all sufficiently small $\varepsilon \neq 0$ (depending on X_∞).

Let us formulate the following

“Weak Kobayashi Conjecture” : *For every hypersurface $X \subseteq \mathbb{P}^n$ of degree $d \geq 2n - 1$, a (very) general small deformation of X is Kobayashi hyperbolic.*

By Brody's Theorem, the proof of hyperbolicity of X reduces to a certain degeneration principle for entire curves in X . The Green-Griffiths' 79' proof of Bloch's Conjecture [7] provides a kind of such degeneration principle. It was shown by McQuillen [9] and, independently, by Demailly-El Goul [3] (according with this principle) that every entire curve $\varphi : \mathbb{C} \rightarrow X$ in a very general surface $X \subseteq \mathbb{P}^3$ of degree $d \geq 36$ ($d \geq 21$, respectively) satisfies a certain algebraic differential equation.

¹Presumably T is irreducible, but we don't dispose a proof of this.

Consider again a pencil (X_t) . Assuming that for a sequence $\varepsilon_n \rightarrow 0$ the hypersurfaces X_{ε_n} are not hyperbolic, one can find a sequence of Brody entire curves $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$ which converges to a (non-constant) Brody curve $\varphi : \mathbb{C} \rightarrow X_0$.

Suppose in addition that X_0 admits a rational map $\pi : X_0 \dashrightarrow Y_0$ to a hyperbolic variety Y_0 (to a curve Y_0 of genus ≥ 2 in case $\dim X_0 = 2$). Then necessarily $\pi \circ \varphi = \text{cst}$, provided that the composition $\pi \circ \varphi$ is well defined. Anyhow the limiting Brody curve $\varphi : \mathbb{C} \rightarrow X_0$ degenerates. This degeneration however is not related to any specific property of the configuration $X_0 \cup X_\infty$, but of X_0 alone. Here is another degeneration principle which involves both X_0 and X_∞ .

PROPOSITION 1 (Shiffman-Z' '05 [11], Z' '07 [16]) Consider a pencil of degree d hypersurfaces $X_\varepsilon \subseteq \mathbb{P}^{n+1}$ generated by $X_0 = X'_0 \cup X''_0$ and X_∞ . Let $D = X'_0 \cap X''_0$. Then for any sequence of entire curves $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$ which converges to $\varphi : \mathbb{C} \rightarrow X'_0$ the following alternative holds:

- Either $\varphi(\mathbb{C}) \subseteq D$, or
- $\varphi(\mathbb{C}) \cap D \subseteq D \cap X_\infty$ and $d\varphi(t) \in T_P X'_0 \cap T_P X_\infty \quad \forall P = \varphi(t) \in D \cap X_\infty$.

THEOREM 1 (Z' '07 [16]) Let Y_0 be a Kobayashi hyperbolic hypersurface of degree d in \mathbb{P}^n ($n \geq 2$), where \mathbb{P}^n is realized as the hyperplane $H = \{z_{n+1} = 0\}$ in \mathbb{P}^{n+1} . Then a general small deformation $X_\varepsilon \subseteq \mathbb{P}^{n+1}$ of the double cone $2X_0$ over Y_0 is Kobayashi hyperbolic.

The proof is based on Proposition 1 and on the following lemma.

LEMMA 1 Let $\hat{Y} \subseteq \mathbb{P}^{n+1}$ be a cone over a projective variety $Y \subseteq \mathbb{P}^n$, and let $X' \subseteq \mathbb{P}^{n+1}$ be a general hypersurface of degree $e \geq 2 \dim Y$. Then X' meets every generator l of \hat{Y} in at least $k = e - 2 \dim Y$ points transversally.

Proof of Theorem 1. Suppose the contrary. Then we can find a sequence $\varepsilon_n \rightarrow 0$ and a sequence of Brody curves $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$ such that $\varphi_n \rightarrow \varphi$, where $\varphi : \mathbb{C} \rightarrow X_0$ is non-constant. We let $\pi : X_0 \dashrightarrow Y_0$ be the cone projection. Since Y_0 is assumed to be hyperbolic we have $\pi \circ \varphi = \text{cst}$. In other words $\varphi(\mathbb{C}) \subseteq l$, where $l \cong \mathbb{P}^1$ is a generator of the cone X_0 .

We note that $\nabla f_0^2|_{X_0} = 0$. If l and X_∞ meet transversally in a point $\varphi(t) \in l \cap X_\infty$ then $d\varphi(t) = 0$ by virtue of Proposition 1.

Since $Y_0 \subseteq \mathbb{P}^n$ is hyperbolic and $n \geq 2$ we have $d \geq n + 2$. In particular

$$\deg X_\infty = 2d \geq 2n + 4 \geq 2 \dim Y + 5.$$

By Lemma 1, l and X_∞ meet transversally in at least 5 points. Hence the nonconstant meromorphic function $\varphi : \mathbb{C} \rightarrow l \cong \mathbb{P}^1$ possesses at least 5 multiple values. Since the defect of a multiple value is $\geq 1/2$, this contradicts the Defect Relation. \square

REMARK Given a hyperbolic hypersurface $Y \subseteq \mathbb{P}^n$ of degree d , Theorem 1 provides a hyperbolic hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $2d$. Iterating the construction yields hyperbolic hypersurfaces in $\mathbb{P}^n \forall n \geq 3$ of degree that grows exponentially with n .

EXAMPLE (Z' '07 [16]) Let $C \subseteq \mathbb{P}^2$ be a hyperbolic curve of degree $d \geq 4$, and let $X_0 \subseteq \mathbb{P}^3$ be a cone over C . Then a general small deformation of the double cone $2X_0$ is a Kobayashi hyperbolic surface in \mathbb{P}^3 of even degree $2d \geq 8$.

The following example combines the projection and deformation methods.

EXAMPLE (Shiffman-Z' '03 [12]) There is a singular octic $X_0 \subseteq \mathbb{P}^3$ whose normalization is a simple abelian surface. Moreover, a general small deformation of X_0 is Kobayashi hyperbolic.

EXAMPLE (Shiffman-Z' '05 [11]) Let $X_0 = X'_0 \cup X''_0$ be the union of two cones in general position in \mathbb{P}^3 over smooth plane quartics $C', C'' \subseteq \mathbb{P}^2$, respectively. Then a general small deformation of X_0 is Kobayashi hyperbolic.

Sketch of the proof. Suppose that for a sequence $\varepsilon_n \rightarrow 0$, X_{ε_n} is not hyperbolic. Then we can find a sequence of Brody curves $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$ which converges to a Brody curve $\varphi : \mathbb{C} \rightarrow X_0$. We may assume that $\varphi(\mathbb{C}) \subseteq X'_0$.

Since C' has genus 3, $\pi' \circ \varphi : \mathbb{C} \rightarrow C'$ is constant, where $\pi' : X'_0 \dashrightarrow C'$ is the cone projection. Thus $\varphi(\mathbb{C}) \subseteq l$, where l is a generator of the cone X'_0 .

By Proposition 1, $\varphi(\mathbb{C})$ meets the double curve $D = X'_0 \cap X''_0$ of X_0 only in points of $D \cap X_\infty$. The projection $\pi' : D \rightarrow C'$ has degree $d'' = 4$ and simple ramifications. Hence every fiber of $\pi'|D$ contains at least 3 points. A general octic X_∞ does not meet the ramification fibers of $\pi' : D \rightarrow C'$ and crosses D passing through just one point of the corresponding fiber of $\pi'|D$. Therefore $D \setminus X_\infty$ contains at least 3 points of l . According to the Little Picard Theorem, $\varphi : \mathbb{C} \rightarrow l \setminus (D \setminus X_\infty)$ is constant, a contradiction.

The Degeneration Principle of Proposition 1 can be combined with the following one.

PROPOSITION 2 (Z' '07' [16]) *Let $(X_t)_{t \in \mathbb{P}^1}$ be a pencil of hypersurfaces in \mathbb{P}^{n+1} generated by two hypersurfaces X_0 and X_∞ of the same degree $d \geq 5$, where $X_0 = kQ$ with $k \geq 2$ for some hypersurface $Q \subseteq \mathbb{P}^{n+1}$, and $X_\infty = \bigcup_{i=1}^d H_{a_i}$, $a_1, \dots, a_d \in \mathbb{P}^1$, is a union of d distinct hyperplanes from a pencil $(H_a)_{a \in \mathbb{P}^1}$. If a sequence of Brody curves $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$, where $\varepsilon_n \rightarrow 0$, converges to a Brody curve $\varphi : \mathbb{C} \rightarrow X_0$, then $\varphi(\mathbb{C}) \subseteq X_0 \cap H_a$ for some $a \in \mathbb{P}^1$.*

EXAMPLES Given a pencil of planes (H_a) in \mathbb{P}^3 , using Proposition 2 one can deform

- $X_0 = 5Q$, where $Q \subseteq \mathbb{P}^3$ is a plane,
- a triple quadric $X_0 = 3Q \subseteq \mathbb{P}^3$, or
- a double cubic, quartic, etc. $X_0 = 2Q \subseteq \mathbb{P}^3$

to an irreducible surface $X_\varepsilon \in \langle X_0, X_\infty \rangle$ of the same degree d , where as before $X_\infty = \bigcup_{i=1}^d H_{a_i}$, so that every limiting Brody curve $\varphi : \mathbb{C} \rightarrow X_0$ is contained in a section $X_0 \cap H_a$ for some $a \in \mathbb{P}^1$.

The famous Bogomolov-Green-Griffiths-Lang Conjecture on strong algebraic degeneracy (see e.g., [1, 7]) suggests that every surface S of general type possesses only finite number of rational and elliptic curves and, moreover, the image of any nonconstant entire curve $\varphi : \mathbb{C} \rightarrow S$ is contained in one of them. In particular, this should hold for any smooth surface $S \subseteq \mathbb{P}^3$ of degree ≥ 5 , which fits the Kobayashi Conjecture. Indeed, by Clemens-Xu-Voisin's Theorem, a general smooth surface $S \subseteq \mathbb{P}^3$ of degree ≥ 5 does not contain rational or elliptic curves, hence should be hyperbolic. Anyhow, the deformation method leads to the following result, which is an immediate consequence of Proposition 2.

COROLLARY *Let $S \subseteq \mathbb{P}^3$ be a surface and $Z \subset S$ be a curve such that the image of any nonconstant entire curve $\varphi : \mathbb{C} \rightarrow S$ is contained in Z ². Let X_∞ be the union of $d = 2 \deg S$ planes from a general pencil of planes in \mathbb{P}^3 . Then any small enough linear deformation X_ε of $X_0 = 2S$ in direction of X_∞ is hyperbolic.*

Along the same lines, Proposition 2 applies in the following setting.

EXAMPLE Let us take for X_0 a double cone in \mathbb{P}^3 over a plane hyperbolic curve of degree $d \geq 4$, and for X_∞ a union of $2d$ distinct planes from a general pencil (H_a). Then small deformations X_ε of X_0 in direction of X_∞ provide examples of hyperbolic surfaces of any even degree $2d \geq 8$. For $d = 4$ the latter surfaces can be given by equation (1) in suitable coordinates. Hence these are actually the Duval-Fujimoto examples [5, 6].

A nice construction due to J. Duval '04 [4] of a hyperbolic sextic $X_\varepsilon \subseteq \mathbb{P}^3$ uses the deformation method iteratively in 5 steps, so that $\varepsilon = (\varepsilon_1, \dots, \varepsilon_5)$ has 5 subsequently small enough components. Hence X_ε belongs to a 5-dimensional linear system and the deformation of X_0 to X_ε neither is linear nor very generic. It was suggested in [12] that the union of 6 general planes in \mathbb{P}^3 admits a general small linear deformation to an irreducible hyperbolic sextic surface. Let us consider this conjectural example in more details.

EXAMPLE Let $X_0 = \bigcup_{i=1}^6 L_i$ be a union of 6 planes in \mathbb{P}^3 in general position, and let $X_\infty \subseteq \mathbb{P}^3$ be a general sextic surface. By virtue of Proposition 1, for any Brody curve $\varphi : \mathbb{C} \rightarrow X_0$ which is the limit of Brody curves $\varphi_n : \mathbb{C} \rightarrow X_{\varepsilon_n}$ ($\varepsilon_n \rightarrow 0$, $\varepsilon_n \neq 0$), the following hold.

- The entire curve $\varphi(\mathbb{C})$ is contained in one of the planes, say, L_i but in none of the intersection lines $l_{ij} := L_i \cap L_j$ ($i \neq j$), neither in the smooth sextic $q_i = L_i \cap X_\infty$.
- $\varphi(\mathbb{C})$ can meet a line l_{ij} only in the 6 intersection points of l_{ij} with q_i .
- $d\varphi(t) \in T_P q_i$ for any point $P = \varphi(t) \in l_{ij} \cap q_i$. Hence $(f_i \circ \varphi)'(t) = 0$, where $f_i = 0$ is an affine equation of q_i .

Consequently, the general small linear deformations X_ε of X_0 are hyperbolic provided that the following question can be answered affirmatively.

²The latter holds, for instance, if S is hyperbolic modulo Z .

QUESTION 1 Consider the union $l = \bigcup_{i=1}^5 l_i$ of 5 lines l_1, \dots, l_5 in general position in \mathbb{P}^2 , and let $q \subseteq \mathbb{P}^2$ be a general plane sextic. Let in a suitable affine chart in \mathbb{P}^2 , q be given by equation $f = 0$, where f is a polynomial of degree 6. Consider further an entire curve $\varphi : \mathbb{C} \rightarrow \mathbb{P}^2$ whose image is not contained in l . Is it true that $\varphi = \text{cst}$ provided that $(f \circ \varphi)'(t) = 0$ for every point $t \in \mathbb{C}$ such that $\varphi(t) \in l$? Is this true under the additional assumption that the entire curve $\varphi(\mathbb{C})$ does not meet the configuration l outside the intersection $l \cap q$ that is, $\varphi^{-1}(l) \subseteq \varphi^{-1}(q)$?

In other words, we are seeking to strengthen the Borel Lemma, or else the classical Ramification Theorem by replacing the 5 multiple values of $f \circ \varphi$ with the l_i -values of φ , $i = 1, \dots, 5$.

Let us specify further this conjectural example in the spirit of Proposition 2.

EXAMPLE Let again $X_0 = \bigcup_{i=1}^6 L_i$ be the union of 6 planes in \mathbb{P}^3 in general position, and let $X_\infty = \bigcup_{j=1}^6 H_{a_j}$ be a union of 6 planes from a pencil $(H_a = f^*(a))_{a \in \mathbb{P}^1}$ in \mathbb{P}^3 in general position with respect to X_0 . Let $(X_t)_{t \in \mathbb{P}^1}$ be the pencil generated by X_0 and X_∞ . Note that the surface X_t is not hyperbolic since it contains the union of lines $\Gamma = X_0 \cap X_\infty$. We suggest however that X_ε is hyperbolic modulo Γ for all small enough $\varepsilon \neq 0$. This leads to the following uniqueness problem for line configurations.

QUESTION 2 Consider as before the union $l = \bigcup_{i=1}^5 l_i$ of 5 lines in general position in \mathbb{P}^2 , and let $q = \bigcup_{j=1}^6 h_j$ be the union of 6 distinct lines $h_i = f^*(a_i)$, $i = 1 \dots, 6$, in \mathbb{P}^2 through a common point, where f is a (general) linear function in a suitable affine chart. Let an entire curve $\varphi : \mathbb{C} \rightarrow \mathbb{P}^2$ satisfies the following conditions:

- $\varphi(\mathbb{C}) \not\subseteq l$,
- $\varphi^{-1}(l) \subseteq \varphi^{-1}(q)$,
- $(f \circ \varphi)'(t) = 0 \ \forall t \in \varphi^{-1}(l)$.

Is then necessarily $f \circ \varphi = a_i$ for some $i \in \{1, \dots, 6\}$?

Let us finally mention some analogous problems concerning hyperbolicity of complements. Using the Borel Lemma, P. Kiernan '73, P. Kiernan and S. Kobayashi '73, and M. Green '77 showed that the complement $\mathbb{P}^n \setminus L$ of the union $L = \bigcup_{i=1}^{2n+1} L_i$ of $2n + 1$ hyperplanes in \mathbb{P}^n in general position is Kobayashi hyperbolic. In particular, this applies to the union l of 5 lines in \mathbb{P}^2 in general position. It was shown in [17] that, moreover, l can be deformed via a small deformation to a smooth quintic curve, while preserving the hyperbolicity of the complement. However, this deformation proceeds in 5 steps and neither is linear nor very generic. So let us ask the following question.

QUESTION 3 Let as before L stands for the union of $2n + 1$ hyperplanes in \mathbb{P}^n in general position. Is the complement of a general small linear deformation of L Kobayashi hyperbolic? In particular, does the union of 5 lines in \mathbb{P}^2 in general position admit a general small linear deformation to an irreducible quintic curve with hyperbolic complement?

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UNIVERSITÉ GRENOBLE I, INSTITUT FOURIER, UMR 5582 CNRS-UJF, BP 74, 38402 ST. MARTIN D'HÈRES CÉDEX, FRANCE

E-mail address: zaidenbe@ujf-grenoble.fr