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# VALUE DISTRIBUTION THEORY AND ITS APPLICATIONS TO THE UNIQUENESS PROBLEM

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## 1 The Second Main Theorem in Value Distribution Theory

The Value Distribution Theory of meromorphic functions on $C$ was established by Nevanlinna in 1925. The core of this theory consits of two Main Theorems, the first and the second. There are not many cases where the Second Main Theorem is established. The following are main results for the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ :

1) $m = n = 1$, for nonconstant meromorphic functions and fix points, we have the S. M. T with multiplicities are truncated by 1 (1925, Nevanlinna).

2) $m = 1, n \geqslant 1$ for linearly nondegenerate mappings and fixed hyperplanes in general position, we have the S. M. T with multiplicities are truncated by $n$ (1933, Cartan).

3) $m \geqslant 1, n \geqslant 1$ for linearly nondegenerate and fixed hyperplanes in

general position, we have the S. M. T with multiplicities are truncated by $n$ (1953, Stoll).

4) $m \geqslant 1, n \geqslant 1$ for linearly nondegenerate and moving hyperplanes in general position, we have the S. M. T with multiplicities are not truncated (1991, Stoll-Ru).

5) $m = n = 1$, for nonconstant meromorphic functions and moving points, we have the S. M. T with multiplicities are truncated by 1 (2002, Yamanoi).

6) $m = 1, n \geqslant 1$ for linearly nondegenerate and fixed and moving hypersurfaces in general position, we have the S. M. T with multiplicities are not truncated, the defect relation is bounded by $2n$ (1992, Eremenko- Sodin).

7) $m = 1, n \geqslant 1$ for algebracally nondegenerate and fixed hypersurfaces in general position, we have the S. M. T with multiplicities are not truncated, the defect relation is bounded by $n + 1$ (2004, Ru).

In this part we give a Second Main Theorem for the case where $m \geqslant 1, n \geqslant 1$, algebracally nondegenerate meromorphic mappings and moving hypersurfaces in general position (in the second main theorem, multiplicities are truncated by a positive integer $L$, the defect relation is bounded by $n+1$.)

We set $\|z\| := \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define $B(r) := \big\{z \in \mathbb{C}^m : |z| < r\big\}$, $S(r) := \big\{z \in \mathbb{C}^m : |z| = r\big\}$ for all $0 < r < \infty$.

Define

$$d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \quad \nu := \big(dd^c\|z\|^2\big)^{m-1} \quad \text{and}$$
$$\sigma := d^c \log\|z\|^2 \wedge \big(dd^c \log\|z\|^2\big)^{m-1}.$$

Let $F$ be a nonzero holomorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, expanding $F$ as $F = \sum P_i(z-a)$ with homogeneous polynomials $P_i$

of degree $i$ around $a$, we define

$$v_F(a) := \min\big\{i : P_i \not\equiv 0\big\}.$$

Let $\varphi$ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the map $v_\varphi$ as follows: For each $z \in \mathbb{C}^m$, we choose nonzero holomorphic functions $F$ and $G$ on a neighborhood $U$ of $z$ such that $\varphi = \dfrac{F}{G}$ on $U$ and $\dim\big(F^{-1}(0)\cap G^{-1}(0)\big) \leqslant m-2$ and then we put $v_\varphi(z) := v_F(z)$. Set $|v_\varphi| := \overline{\{z \in \mathbb{C}^m : v_\varphi(z) \neq 0\}}$.

Let $k$ be positive integer or $+\infty$. Set $v_\varphi^{[k]}(z) := \min\{v_\varphi(z), k\}$, and the counting function (with multiplicities are truncated by $k$)

$$N_\varphi^{[k]}(r) := \int\limits_1^r \frac{n^{[k]}(t)}{t^{2m-1}}dt \quad (1 < r < +\infty)$$

where

$$n^{[k]}(t) = \begin{cases} \displaystyle\int\limits_{|v_\varphi| \cap B(t)} v_\varphi^{[k]} \cdot \nu & \text{for } m \geqslant 2, \\ \displaystyle\sum_{|z| \leqslant t} v_\varphi^{[k]}(z) & \text{for } m = 1. \end{cases}$$

Let $f$ be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$. The characteristic function of $f$ is defined by

$$T_f(r) := \int\limits_{S(r)} \log\|f\|\sigma - \int\limits_{S(1)} \log\|f\|\sigma, \quad 1 < r < +\infty.$$

For a meromorphic function $\varphi$ on $\mathbb{C}^m$, the characteristic function $T_\varphi(r)$ of $\varphi$ is defined as $\varphi$ is a meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

We say that a meromorphic function $\varphi$ on $\mathbb{C}^m$ is "small" with respect to $f$ if $T_\varphi(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finte Lebesgue measure).

Denote by $\mathcal{K}_f$ the field of all "small" (with respect to $f$) meromorphic functions on $\mathbb{C}^m$.

For a homogeneous polynomial $Q \in \mathcal{K}_f[x_0, \ldots, x_n]$ of degree $d \geqslant 1$ with $Q(f_0, \ldots, f_n) \not\equiv 0$, we define

$$N_f^{[k]}(r, Q) := N_{Q(f_0,\ldots,f_n)}^{[k]}(r) \quad \text{and}$$

$$\delta_f(Q) = \liminf_{r\to\infty} \Big(1 - \frac{N_f^{[+\infty]}(r, Q)}{d \cdot T_f(r)}\Big).$$

We say that homogeneous polynomials $\{Q_j\}_{j=1}^q$ $(q \geqslant n+1)$ in $\mathcal{K}_f[x_0, \ldots, x_n]$ are general position if there exists $z \in \mathbb{C}^m$ such that for any $1 \leqslant j_0 < \cdots < j_n \leqslant q$ the system of equations

$$\begin{cases} Q_{j_i}(z)(w_0, \ldots, w_n) = 0 \\ 0 \leqslant i \leqslant n \end{cases}$$

has only the trivial solution $w = (0, \ldots, 0)$ in $\mathbb{C}^{n+1}$.

**Theorem 1 (Dethloff- Tan).** *Let $f$ be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Let homogeneuos polynomials $\{Q_j\}_{j=1}^q$ in $\mathcal{K}_f[x_0, \ldots, x_n]$ be in general position with $deg Q_j = d_j \geqslant 1$. Assume that $f$ is algebraically nondegenerate over $\mathcal{K}_f$. Then for any $\varepsilon > 0$, there exist a positive integer $L$, depending only on the $Q_j$ and $\varepsilon$, such that*

$$\|(q - n - 1 - \varepsilon)T_f(r) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^{[L]}(r, Q_j).$$

*for all* $r \in [1, +\infty)$ *except for a subset* $E$ *of* $(1, +\infty)$ *of finite Lebesgue measure.*

For the case of fix hypersurfaces and multiplicities are not truncated, the above Theorem was proved by Ru in 2004 . In the proof, we use again some techniques of Ru, Ru-Stoll, Fujimoto, Shiroshaki, and our idea mainly appear in two points: truncating multiplicities and to overcome the difficulties which come from the case of moving hypersurfaces ($\mathcal{K}_f$ is not algebraically closed in general,...)

**Corollary.** (B. Shiffman conjecture for moving hypersurfaces) *Under the same assumption in the above theorem, we have*

$$\sum_{j=1}^{q} \delta_f(Q_j) \leqslant n+1.$$

## 2 Uniqueness problem of meromorphic maps

In 1929, Cartan declared that there are at most two meromorphic functions on $\mathbb{C}$ which have the same inverse images ( multiplicities are truncated by 1) for four distinct values. However in 1988, Steinmetz gave examples which showed that Cartan's declaration is false. On the other hand, in 1998, Fujimoto showed that Cartan's declaration is true if we assume that meromorphic functions on $\mathbb{C}$ share four distinct values counted with multiplicities truncated by 2. Furthermore, He extended this result to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ which have the same inverse images ( multiplicities are truncated by 2) for $q = 3n + 1$ hyperplanes. He also proposed an open problem asking if the number of hyperplanes in his result can be replaced by a smaller one. We note that the number of hyperplanes $q = 3n + 1$ has been motivated from the observation of the case of one dimension.

In 2004 Dethloff and Tan gave an answer for the above Fujimoto's question. We showed that the Fujimoto's result can be extended to the case of $3n-1$ hyperplanes, futhermore for $n \geq 7$, the multiplicities are truncated by 1 is enough. Since the above result and some other ones, It seems to us that we did not find any essential relation between the case of one dimension and the case of high dimension when giving the number of hyperplanes for uniqueness theorems.

We now give an other extension of Fujimoto's result to the case of few hyperplanes.

Let $f$ be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with reduced representation $f = (f_0 : \cdots : f_n)$. Let $d$ be a positive integer and let $H_1, \cdots, H_q$ be $q$ hyperplanes in $\mathbb{C}P^n$ in general position with dim $\{z \in \mathbb{C}^m : \nu_{(f,H_i)}(z) > 0$ and

$$\nu_{(f,H_j)}(z) > 0\} \leqslant m-2 \; (1 \leqslant i < j \leqslant q).$$

Consider the set $\mathcal{F}(f, \{H_j\}_{j=1}^q, d)$ of all linearly nondegenerate meromorphic mappings $g : \mathbb{C}^m \to \mathbb{C}P^n$ with reducded representation $g = (g_0 : \cdots : g_n)$ satisfying the conditions:

(a) $\min(\nu_{(f,H_i)}, d) = \min(\nu_{(g,H_i)}, d)$ $(1 \leqslant i \leqslant q)$,

(b) $Zero(f_j) \cap f^{-1}(H_i) = Zero(g_j) \cap f^{-1}(H_i)$, for all $1 \leqslant i \leqslant q,\ 0 \leqslant j \leqslant n$,

(c) $\mathcal{D}^\alpha\big(\frac{f_k}{f_s}\big) = \mathcal{D}^\alpha\big(\frac{g_k}{g_s}\big)$ on $\big(\cup_{i=1}^q f^{-1}(H_i)\big)\backslash\big(Zero(f_s)\big)$, for all $|\alpha| < d,\ 0 \leqslant k \neq s \leqslant n$.

**Theorem 2 (Quang-Tan).** *If*

$$q > \max\{\frac{7(n+1)}{4}, \frac{\sqrt{17n^2+16n}+3n+4}{4}\}$$

*then* $\mathcal{F}(f, \{H_i\}_{i=1}^q, 2)$ *contains at most two mappings.*