GEOMETRY OF DOMAINS IN $\mathbb{C}^n$ WITH NONCOMPACT AUTOMORPHISM GROUPS

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1 Characterization of domains in $\mathbb{C}^n$ by their noncompact automorphism groups

Let Ω be a domain, i.e. connected open subset, in a complex manifold M. Let the *automorphism group* of Ω (denoted $Aut(\Omega)$) be the collection of biholomorphic self-maps of Ω with composition of mappings as its binary operation. The topology on $Aut(\Omega)$ is that of uniform convergence on compact sets (i.e., the compact-open topology).

One of the important problems in several complex variables is to study the interplay between the geometry of a domain and the structure of its automorphism group. More precisely, we wish to see to what extent a domain is determined by its automorphism group.

It is a standard and classical result of H. Cartan that if Ω is a bounded domain in $\mathbb{C}^n$ and the automorphism group of Ω is noncompact then there exist a point $x \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\varphi_j \in Aut(\Omega)$ such that $\varphi_j(x) \to p$. In this circumstance we call p a *boundary orbit accumulation point.*

Work in the past twenty years has suggested that the local geometry of the so-called "boundary orbit accumulation point" p in turn gives global information about the characterization of model of the domain. For instance, B. Wong and J. P. Rosay proved the following theorem.

Wong-Rosay theorem. *Any bounded domain* $\Omega \Subset \mathbb{C}^n$ *with a* C^2 *strongly pseudoconvex boundary orbit accumulation point is biholomorphic to the unit ball in* $\mathbb{C}^n$*.*

We now recall the definition of finite type in the sense of J. P. D'Angelo.

Let $D \subset \mathbb{C}^n$ be a domain with $C^\infty-$smooth boundary and let $p \in \partial D$. Then the type $\tau(p)$ of ∂D at p is defined as

$$\tau(p) = \sup_F \frac{\nu(\tau \circ F)}{\nu(F)},$$

where ρ is a defining function of D near p, the supremum is taken over all holomorphic mappings F defined in a neighbourhood of $0 \in \mathbb{C}$ into $\mathbb{C}^n$ such that $F(0) = p$, and $\nu(\phi)$ is the order of vanishing of a function ϕ at the origin.

The boundary ∂D is said to be of finite type at p if $\tau(p) < \infty$.

The domain D is a domain of finite type if ∂D is of finite type at every point.

By using the scaling technique, introduced by S. Pinchuk, in 1991 E. Bedford and S. Pinchuk proved the theorem about the characterization of the complex ellipsoids.

Bedford-Pinchuk theorem. *Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded pseudoconvex domain of finite type whose boundary is smooth of class C^∞, and suppose that the Levi form has rank at least $n-1$ at each point of the boundary. If $Aut(\Omega)$ is noncompact, then Ω is biholomorphically equivalent to the domain*

$$E_m = \{(w, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} : |w|^2 + |z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2 < 1\},$$

for some integer $m \geq 1$.

The approach of Bedford-Pinchuk involves two steps. In the first step they use the method of scaling to show that the domain Ω in consideration is holomorphically equivalent to a domain D of the form

$$D = \{(z_1, \tilde{z}) \in \mathbb{C}^{n+1} : \operatorname{Re} z_1 + Q(\tilde{z}, \bar{\tilde{z}}) < 0\},$$

where Q is a polynomial. The domain D has a non-trivial holomorphic vector field. In the second step this vector field is transported back to Ω, the result is analyzed at the parabolic fixed point, and this information is used to determine the original domain.

There has been also certain progress, by other authors, on the first step of the above procedure of Bedford-Pinchuk. The following completely local result for domains (not necessary bounded) in $\mathbb{C}^2$ was obtained by F. Berteloot in 1994.

Berteloot theorem. *Let Ω be a domain in $\mathbb{C}^2$ and let $\xi_0 \in \partial\Omega$. Assume that there exists a sequence (φ_p) in $Aut(\Omega)$ and a point $a \in \Omega$ such that $\lim \varphi_p(a) = \xi_0$. If $\partial\Omega$ is pseudoconvex and of finite type near ξ_0 then Ω is biholomorphically equivalent to $\{(w, z) \in \mathbb{C}^2 : \operatorname{Re} w + H(z, \bar{z}) < 0\}$, where H is a homogeneous subharmonic polynomial on $\mathbb{C}$ with degree $2m$.*

The first aim in this talk is to show a completely local result on the first step of the above procedure of Bedford-Pinchuk for domains (not necessary bounded) in $\mathbb{C}^n$. Namely, we prove the following.

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$ and let $\xi_0 \in \partial\Omega$. Assume that*

(a) $\partial\Omega$ is pseudoconvex, of finite type and smooth of class C^∞ in some neighbourhood of $\xi_0 \in \partial\Omega$.

(b) The Levi form has rank at least $n-2$ at ξ_0.

(c) There exists a sequence (φ_p) in $Aut(\Omega)$ such that $\lim \varphi_p(a) = \xi_0$ for some $a \in \Omega$.

Then Ω is biholomorphically equivalent to a domain of the form

$$M_H = \{(w_1, \cdots, w_n) \in \mathbb{C}^n : Re w_n + H(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0\},$$

where H is a homogeneous subharmonic polynomial with $\Delta H \not\equiv 0$.

Without the assumption (b) we show that Theorem 1.1 also is true for linearly convex domains in $\mathbb{C}^n$. On the other hand, the Berteloot theorem holds for linearly convex domains in $\mathbb{C}^n$.

Theorem 1.2. *Let Ω be a domain in $\mathbb{C}^n$, and let p_∞ be a point of $\partial\Omega$. Assume that p_∞ is an accumulating point for a sequence of automorphisms of Ω. If $\partial\Omega$ is smooth, linearly convex, and of finite type $2m$ near p_∞, then Ω is biholomorphically equivalent to a rigid polynomial domain*

$$D = \{z \in \mathbb{C}^n : \operatorname{Re} z_1 + P(z') < 0\},$$

where P is a real nondegenerate plurisubharmonic polynomial of degree less than or equal to $2m$.

The nondegeneracy of P is given by condition "$\{P = 0\}$ without nontrivial analytic set".

Open questions.

1. We would like to emphasize here that the assumption on boundedness of domains in the Bedford-Pinchuk theorem is essential in their proofs. It seems to us that some key techniques in their proofs could not use for unbounded domains in $\mathbb{C}^n$. Thus, the first natural question that whether the Bedford-Pinchuk theorem is true for any domain in $\mathbb{C}^n$.

2. Is it true that the theorems on characterization of smoothly bounded domains in $\mathbb{C}^n$ with noncompact automorphism groups holds without extra assumption such as the finiteness of type or pseudoconvexity?

Many experts believe that the answers are positive.

2 Green-Krantz-Conjecture

In 1993, R. E. Green and S. G. Krantz introduced the following.

Green-Krantz-Conjecture. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain with noncompact automorphism group. Then $\partial\Omega$ is of finite type at any boundary orbit accumulation point.*

The conjecture in its full generality is open.

We now recall some definitions.

For a domain $\Omega \subseteq \mathbb{C}^n$, we let

$$A(\Omega) = \{f \in C(\overline{\Omega}) : f \text{ is holomorphic on } \Omega\}.$$

A point $q \in \partial\Omega$ is called a *peak point* for $A(\Omega)$ if there is a function $f \in A(\Omega)$ such that

(i) $f(q) = 1$;

(ii) $|f(z)| < 1$ for $z \in \overline{\Omega} \setminus \{q\}$.

We call a boundary point $p \in \partial\Omega$ a *hyperbolic orbit accumulation point* if it admits another boundary point $q \in \partial\Omega \setminus \{p\}$, a sequence $\{\varphi_j\}_{j=1}^{\infty} \subset Aut(\Omega)$, φ_j's extend to a diffeomorphisms of the closure $\overline{\Omega}$ and an interior point $x_0 \in \Omega$ satisfying the following properties:

(1) $\varphi_j(p) = p$ and $\varphi_j(q) = q$ for every $j = 1, 2, \ldots$.

(2) $\lim_{j\to\infty} \varphi_j(x_0) = p$ and $\lim_{j\to\infty} \varphi_j^{-1}(x_0) = q$.

Remark.

1. If our domain has a globally finite type boundary then our φ_j's extend to a diffeomorphisms of the closure $\overline{\Omega}$ by the extension theorem of Bell-Ligocka.

2. K-T. Kim and S. G. Krant showed that if $\Omega \subset \mathbb{C}^n$ is a bounded domain with a finite type boundary in the sense of D'Angelo, then every hyperbolic orbit accumulation boundary point is a peak point.

Let Ω be a domain in $\mathbb{C}^n$. A boundary point $p \in \partial\Omega$ is called a *parabolic orbit accumulation point* if there is a one-parameter subgroup

$$\{\psi_t \in Aut(\Omega), -\infty < t < \infty\}$$

of automorphisms such that

$$\lim_{t \to \pm\infty} \psi_t(x_0) = p$$

for some $x_0 \in \Omega$.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a C^∞ smooth boundary. We say that Ω satisfies Bell's condition R if the Bergman projection $P : C^\infty(\Omega) \to C^\infty(\Omega)$ extends to a map $C^\infty(\bar{\Omega}) \to C^\infty(\bar{\Omega})$.

Theorem of Kim-Krantz (2006) *Let* $\Omega \subset \mathbb{C}^2$ *be a pseudoconvex domain with a* C^∞ *smooth boundary satisfying Bell's condition R. Assume also that* $\partial\Omega$ *does not contain any non-trivial analytic variety. Then every parabolic orbit accumulation boundary point is of finite D'Angelo type.*

This theorem provides a proof of an important special case of the Greene-Krantz-Conjecture. Unfortunately, their proof is incorrect.

In fact, we show gaps in their proof.

Let $p \in \partial\Omega$ be a parabolic orbit accumulation point of infinite D'Angelo type. Choose a holomorphic local coordinate system at p so that p now becomes the origin and the local defining function of Ω takes the form

$$\rho(z) = \text{Re } z_1 + \Psi(z_2, \text{Im } z_1).$$

Then they pointed out that Ψ vanishes to infinite order at the origin. But, in general, it is not true, e.g., $\psi(z_2, \operatorname{Im} z_1) = e^{-1/|z_2|^2} + |z_2|^4.|\operatorname{Im} z_1|^2$.

By an another approach, we proved the following.

Theorem 2.3. *Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain in $\mathbb{C}^2$ and $0 \in \partial\Omega$. Assume that*

(1) $\partial\Omega$ is $C^\infty-$ smooth satisfying the Bell's condition (R),

(2) $\partial\Omega$ does not contain any non-trivial analytic variety,

(3) There exists a neighborhood U of $0 \in \partial\Omega$ such that

$$\Omega \cap U = \{(z_1, z_2) \in \mathbb{C}^2 : \rho = \operatorname{Re} z_1 + P(z_2) + Q(z_2, \operatorname{Im} z_1) < 0\},$$

where P and Q satisfy the following

(i) $\lim_{z_2 \to 0} \dfrac{P(z_2)}{|z_2|^N} = 0, \ N = 0, 1, 2, \cdots,$

(ii) $Q(0, \operatorname{Im} z_1) = Q(z_2, 0) = 0,$

$\dfrac{\partial}{\partial z_1} Q(z_2, 0) = \dfrac{\partial}{\partial z_2} Q(z_2, 0) = 0,$

$\dfrac{\partial^2}{\partial z_2 \partial \bar{z}_2} Q(0, \operatorname{Im} z_1) = 0$ *and*

$\dfrac{\partial^N}{\partial z_2^N} Q(0, \operatorname{Im} z_1) = 0, \ N = 0, 1, 2, \cdots.$

Then, $(0, 0)$ *is not a parabolic orbit accumulation point.*

Remark. By a simple computation, we see that

• The funtions $P(z_2) = e^{-1/|z_2|^2}$ and $Q(z_2, \operatorname{Im} z_1) = |z_2|^4.|\operatorname{Im} z_1|^2$ satisfy the above conditions.

•$(0, 0)$ is of infinite type. Hence every parabolic orbit accumulation boundary point of the above-mentioned domain is of finite D'Angelo type.