A fundamental inequality for holomorphic curves into projective varieties

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Then the First Main Theorem implies that

$$T_f(r) = m_f(r, D) + N_f(r, D) + O(1).$$

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Remark: By considering $\phi : \mathbb{P}^n \to \mathbb{P}^{q-1}$ defined $\phi(x) = [L_1(x) : \cdots : L_q(x)]$, we can assume that $H_1, \ldots, H_q$ are coordinate hyperplanes.

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where $x^{a_0}, \ldots, x^{a_{q_m}}$ are the monomials of degree m. Denote by X_m the smallest linear sub-variety of $\mathbb{P}^{q_m}$ containing $\phi_m(X)$.

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$$\mathbb{C} \to^{f} X \subset \mathbb{P}^N \hookrightarrow^{\phi_m} X_m \cong^{\psi_m^{-1}} \mathbb{P}^{n_m}.$$

We want to apply the (general) H. Cartan's theorem to $F = \psi_m^{-1} \circ \phi_m \circ f : \mathbb{C} \to \mathbb{P}^{n_m}$ with linear forms $L_0, \ldots, L_{q_m}$ on $\mathbb{P}^{n_m}$

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$$\log \prod_{i \in J} \frac{1}{|L_i(F)(z)|} = \log \prod_{i \in J} \frac{1}{|f_0(z)|^{a_{i,0}} \cdots |f_N(z)|^{a_{i,N}}} + O(1).$$

Recall that the Hilbert Weight of X w.r.r. the weight c is defined by

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$$S_X(m, c) = \max \left(\sum_{i=1}^{H_I(m)} a_i \cdot c \right),$$

where the maximum is taken over all sets of monomials $x^{a_1}, \ldots, x^{a_{H_I(m)}}$ whose residue classes modulo I_X form a basis of $\mathbb{C}[x_0, \ldots, x_N]_m/(I_X)m$.

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$$\max_J \log \prod_{i \in J} \frac{\|F(x)\| \|L_i\|}{|L_i(F)(z)|} = S_X(m, c(z)) - mH_X(m) \log \|f(z)\|$$
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Applying the (general) cartan's theorem, we get

$$\int_0^{2\pi} \frac{1}{mH_X(m)} S_X(m, c(re^{i\theta})) \frac{d\theta}{2\pi}. \leq .(1 + \epsilon)T_f(r).$$

Now using (modified) Munford's result:

$$\frac{1}{mH_X(m)} S_X(m, c) \geq \frac{1}{(n+1)d} e_X(c) - \frac{(2n+1)d}{m} \left(\max_{0 \leq i \leq N} c_i \right),$$

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we obtain our Main Theorem.