On the Kobayashi conjecture

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Hyperbolicity

Definition

- Let X be a complex manifold. Fix a point $x \in X$ and a tangent vector $\xi \in T_{X,x}$. Then we define the infinitesimal Kobayashi length of ξ at x to be

$$k_X(\xi) = \inf\{\lambda > 0 \quad , \quad \text{such that there exists a holomorphic map } f : \Delta \to X \text{ with } f(0) = x \text{ and } \lambda f'(0) = \xi\}.$$

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- The Kobayashi pseudo-distance d_X, is the geodesic pseudodistance obtained by integrating the Kobayashi-Royden infinitesimal pseudometric.
- The manifold X is hyperbolic in the sense of S. Kobayashi if d_X is a distance.

Examples and basic properties

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Proposition

Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds then it is distance decreasing i.e for $x, x' \in X$ we have

$$d_Y(f(x), f(x')) \leq d_X(x, x').$$

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$$d_Y(f(x), f(x')) \leq d_X(x, x').$$

Corollary

If X is Kobayashi hyperbolic then every entire map $f : \mathbb{C} \to X$ is constant i.e X is Brody hyperbolic.

Examples and basic properties

Theorem (Brody)

If X is compact then X is Kobayashi hyperbolic iff X is Brody hyperbolic.

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Theorem

Let X be a compact manifold and $Y \subset X$ an hypersurface. If Y and $X \setminus Y$ are Brody hyperbolic then $X \setminus Y$ is hyperbolic.

Conjectures

Conjecture (Kobayashi)

- *A general hypersurface* $X \subset \mathbb{P}^n$ $(n \geq 3)$ *of degree* $\deg X \geq 2n - 1$ *is hyperbolic.*

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A projective manifold X *is hyperbolic iff* X *and all its subvarieties are of general type.*

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Conjecture (Lang)

A projective manifold X *is hyperbolic iff* X *and all its subvarieties are of general type.*

Conjecture (Green-Griffiths, Lang)

X *a projective manifold of general type. Then there exists a subvariety* $Y \subsetneq X$ *such that for every non constant entire map* $f : \mathbb{C} \to X$, $f(\mathbb{C}) \subset Y$.

Algebraic hyperbolicity: the compact case

Definition (Demailly)

X compact manifold, ω a hermitian metric. X is algebraically hyperbolic if $\exists \varepsilon > 0, \forall C \subset X$ compact irreducible curve

$$2g(\widetilde{C}) - 2 \geq \varepsilon \deg_\omega(C)$$

where $g(\widetilde{C})$ is the genus of the normalization of C and $\deg_\omega(C) = \int_C \omega$.

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Algebraic hyperbolicity: the logarithmic case

Definition (Chen)

- Let (X, D) be a log-manifold. For each reduced curve $C \subset X$ that meets D properly, let $\nu : \widetilde{C} \to C$ be the normalization of C. Then $i(C, D)$ is the number of distinct points in the set $\nu^{-1}(D) \subset \widetilde{C}$.

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- A logarithmic variety (X, D) is algebraically hyperbolic if there exists a positive number ε such that

$$2g(\widetilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_\omega(C)$$

for all reduced and irreducible curves $C \subset X$ meeting D properly where $\widetilde{C}$ is the normalization of C, $g(\widetilde{C})$ its genus and $\deg_\omega(C) = \int_C \omega$ with ω a hermitian metric on X.

Algebraic hyperbolicity: the logarithmic case

Definition

Let Y be a relatively compact complex submanifold of a complex manifold Z. Y is hyperbolically imbedded in Z if $\exists \varepsilon > 0, \forall x \in Y, \forall \xi \in T_{Y,x}, k_Y(\xi) \geq \varepsilon \left\|\xi\right\|_{\omega}$ where ω is a hermitian metric on Z.

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Proposition (Pacienza,Rousseau)

Let X be a projective manifold and D an effective divisor on X such that $X \backslash D$ is hyperbolic and hyperbolically imbedded. Then (X, D) is algebraically hyperbolic.

Lang's characterization: the compact case

Theorem (Clemens, Voisin, Pacienza)

All subvarieties of a very generic hypersurface $X_d \subset \mathbb{P}^n$ *for* $d \geq 2n-1, n \geq 4$ *and* $d \geq 6, n = 3$, *are of general type.*

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A very generic hypersurface $X_d \subset \mathbb{P}^n$ *for* $d \geq 2n-1, n \geq 4$ *and* $d \geq 6, n = 3,$ *is algebraically hyperbolic.*

Lang's characterization: the logarithmic case

Theorem (Pacienza, Rousseau)

Let X be a very general hypersurface of arbitrary degree d in $\mathbb{P}^n$. Let Y be a k-dimensional subvariety in $\mathbb{P}^n$ meeting X properly, $D := Y \cap X$ the induced divisor and $\nu : \widetilde{Y} \to Y$ a log-resolution of (Y, D) i.e $\widetilde{Y}$ is smooth, ν is a projective birational morphism and $\nu^{-1}(D) + Exc(\nu)$ is a normal crossing divisor. Then

$$h^0(\widetilde{Y}, \overline{K}_{\widetilde{Y}} \otimes \nu^* \mathcal{O}_{\mathbb{P}^n}(2n+1-k-d)) \neq 0,$$

where $\overline{K}_{\widetilde{Y}}$ denotes the log-canonical bundle of the log-variety $(\widetilde{Y}, \nu^{-1}(D))$.

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Corollary

Let $X \subset \mathbb{P}^n$ be a very general hypersurface of degree $d \geq 2n + 2 - k$, $k \geq 1$. Then any k-dimensional log-subvariety (Y, D) of $(\mathbb{P}^n, X)$, for Y not contained in X, is of log-general type.

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- Then

$$0 \leq \deg(K_{\widetilde{C}}(\widetilde{D}) \otimes f^* \mathcal{O}_{\mathbb{P}^n}(2n-d)) = 2g(C) - 2 + i(C, X) - (d-2n) \deg C.$$

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- For general $F \in U$, and *arbitrary* degree d we want to produce a non zero element in $H^0(\widetilde{Y}_F, \overline{K}_{\widetilde{Y}_F}(2n+1-k-d))$, where $\overline{K}_{\widetilde{Y}_F} = K_{\widetilde{Y}_F}(\widetilde{D}_F)$.

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$$\left(\bigwedge^{n-k} T_{\mathbb{P}^n \times \mathbb{P}^N}(-\log \mathcal{X})\right)\Bigg|_{\mathbb{P}^n_F} \otimes \overline{K}_{\mathbb{P}^n_F} \simeq \Omega^{N+k}_{\mathbb{P}^n \times \mathbb{P}^N}(\log \mathcal{X})\Big|_{\mathbb{P}^n_F}$$

Proof of theorem

- For F general in U we have a non zero map

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Proposition

The twisted logarithmic tangent bundle

$$T_{\mathbb{P}^n\times\mathbb{P}^{N_d}}(-\log\mathcal{X})(1,0):=T_{\mathbb{P}^n\times\mathbb{P}^{N_d}}(-\log\mathcal{X})\otimes p^*\mathcal{O}_{\mathbb{P}^n}(1)$$

is generated by its global sections.

Weak version of Kobayashi conjectures

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Theorem (Debarre, Pacienza, Paun)

Let $U \to \mathbb{P}^{N_d} := \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be an étale cover of an open subset of $\mathbb{P}^{N_d}$, and let $\Phi : \mathbb{C} \times U \to \mathbb{P}^n \times U$ be a holomorphic map such that $\Phi(\mathbb{C} \times \{t\}) \subset X_t$ for all $t \in U$. If $d \geq 2n$, the rank of Φ cannot be maximal anywhere.

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Theorem

Let $X \subset \mathbb{P}^4_{\mathbb{C}}$ be a generic hypersurface such that $d = deg(X) \geq 593$. Then every entire curve $f : \mathbb{C} \to X$ is algebraically degenerated, i.e there exists a proper subvariety $Y \subset X$ such that $f(\mathbb{C}) \subset Y$.

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For $X \subset \mathbb{P}^3_{\mathbb{C}}$ a generic hypersurface such that $d = deg(X) \geq 586$, every entire curve $f : \mathbb{C} \to \mathbb{P}^3_{\mathbb{C}} \backslash X$ is algebraically degenerated i.e there exists a proper subvariety $Y \subset \mathbb{P}^3_{\mathbb{C}}$ such that $f(\mathbb{C}) \subset Y$.

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Let $C = C_1 \cup C_2$ *where* C_i , $i = 1, 2,$ *are very generic irreducible smooth curves of* $\mathbb{P}^2$ *of degree* $d_1 \leq d_2$. *Then for*

$$\begin{aligned} d_1 &\geq 4, \\ d_1 &= 3, d_2 \geq 5; \\ d_1 &= 2, d_2 \geq 8; \\ d_1 &= 1, d_2 \geq 11 \end{aligned}$$

$\mathbb{P}^2 \backslash C$ *is hyperbolic and hyperbolically imbedded.*

Jet differentials: the compact case

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- If $f : (\mathbb{C}, 0) \to (X, x)$ is a germ of holomorphic curve tangent to V then it can be lifted to X_1 as $f_{[1]}$.
- By induction, we obtain a tower of varieties (X_k, V_k). $\pi_k : X_k \to X$ is the natural projection.

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- By induction, we obtain a tower of varieties (X_k, V_k). $\pi_k : X_k \to X$ is the natural projection.
- We have a tautological line bundle $\mathcal{O}_{X_k}(1)$ and we denote $u_k := c_1(\mathcal{O}_{X_k}(1))$.

Jet differentials: the compact case

- Take $V := T_X$. The direct image $\pi_{k*}(\mathcal{O}_{X_k}(m))$ is a vector bundle over X which can be described with local coordinates.

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- Let $z = (z_1, ..., z_n)$ be local coordinates centered in $x \in X$. A local section of $\pi_{k*}(\mathcal{O}_{X_k}(m))$ is a polynomial

$$P = \sum_{|\alpha_1|+2|\alpha_2|+...+k|\alpha_k|=m} R_\alpha(z) dz^{\alpha_1} ... d^k z^{\alpha_k}$$

which acts naturally on the fibers of the bundle $J_k X \to X$ of k-jets of germs of curves in X, and which is invariant under reparametrization i.e

$$P((f \circ \phi)', ..., (f \circ \phi)^{(k)})_t = \phi'(t)^m P(f', ..., f^{(k)})_{\phi(t)}$$

for every $\phi \in \mathbb{G}_k$, the group of k-jets of biholomorphisms of $(\mathbb{C}, 0)$.

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- This bundle of invariant jet differentials is a subbundle of the bundle of jet differentials, of order k and degree m, $E_{k,m}^{GG}T_X^* \to X$ whose fibres are complex-valued polynomials $Q(f', f'', ..., f^{(k)})$ on the fibers of J_kX, of weight m under the action of $\mathbb{C}^*$:

$$Q(\lambda f', \lambda^2 f'', ..., \lambda^k f^{(k)}) = \lambda^m Q(f', f'', ..., f^{(k)})$$

for any $\lambda \in \mathbb{C}^*$ and $(f', f'', ..., f^{(k)}) \in J_kX$.

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- For $k = 1$, $E_{1,m}T_X^* = S^m T_X^*$.
- If X is a surface we have the following description of $E_{2,m}T_X^*$. Let W be the wronskian, $W = dz_1 d^2 z_2 - dz_2 d^2 z_1$, then every invariant differential operator of order 2 and degree m can be written

$$P = \sum_{|\alpha|+3k=m} R_{\alpha,k}(z) dz^\alpha W^k.$$

Jet differentials: the compact case

Theorem (Green-Griffiths, Demailly, Siu)

Assume that there exist integers $k, m > 0$ and an ample line bundle L on X such that

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^* L^{-1}) \simeq H^0(X, E_{k,m} T_X^* \otimes L^{-1})$$

has non zero sections $\sigma_1, ..., \sigma_N$. Let $Z \subset X_k$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X$ is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k-$ invariant polynomial differential operator P with values in L^{-1}, every entire curve $f : \mathbb{C} \to X$ must satisfy the algebraic differential equation $P(f) = 0$.

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- Start with the log directed manifold (X, D, V) where $V \subset \overline{T}_X$ and $\overline{T}_X = T_X(-\log D)$.

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Jet differentials: the logarithmic case

- The direct image $\pi_{k*}(\mathcal{O}_{X_k}(m))$ is a locally free sheaf denoted $E_{k,m}\overline{T}^*_X$ generated by all polynomial operators in the derivatives of order $1, 2, ..., k$ of f, together with the extra function $\log s_j(f)$ along the $j-th$ component of D, which are moreover invariant under arbitrary changes of parametrization: a germ of operator $Q \in E_{k,m}\overline{T}^*_X$ is characterized by the condition that, for every germ in $X \backslash D$ and every germ $\phi \in \mathbb{G}_k$ of k-jet biholomorphisms of $(\mathbb{C}, 0)$,

$$Q(f \circ \phi) = \phi'^m Q(f) \circ \phi.$$

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has non zero sections $\sigma_1, ..., \sigma_N$. Let $Z \subset X_k$ be the base locus of these sections. Then every entire curve $f : \mathbb{C} \to X \backslash D$ is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global $\mathbb{G}_k-$ invariant polynomial differential operator P with values in L^{-1}, every entire curve $f : \mathbb{C} \to X \backslash D$ must satisfy the algebraic differential equation $P(f) = 0$.

Global jet differentials in dim 2

- Take (X, D) a log surface, we have the following filtration of log-jet differentials of order 2:

$$Gr^\bullet E_{2,m}\overline{T}^*_X = \bigoplus_{0 \leq j \leq m/3} S^{m-3j}\overline{T}^*_X \otimes \overline{K}_X^{\otimes j}$$

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$$\chi(X, E_{2,m} \overline{T}^*_X) = \frac{m^4}{648} (13 \overline{c}_1^2 - 9 \overline{c}_2) + O(m^3)$$

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Theorem (El Goul)

If (X, D) is an algebraic log surface of log general type and A an ample line bundle over X, then

$$h^0(X, E_{2,m}\overline{T}_X^* \otimes \mathcal{O}(-A)) \geq \frac{m^4}{648}(13\overline{c}_1^2 - 9\overline{c}_2) + O(m^3)$$

Global jet differentials in dim 2

Corollary (El Goul)

Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d \geq 11$. Then $h^0(X, E_{2,m}\overline{T}^*_X \otimes \mathcal{O}(-A)) \neq 0$ for m large enough.

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Corollary

Let $C = C_1 \cup C_2$ where $C_i, i = 1, 2,$ are irreducible smooth curves of $\mathbb{P}^2$ of degree $d_1 \leq d_2$. Then for

$$\begin{aligned} d_1 &\geq 3; \\ d_1 &= 2, \ d_2 \geq 5; \\ d_1 &= 1, \ d_2 \geq 7; \end{aligned}$$

$h^0(X, E_{2,m}\overline{T}^*_X \otimes \mathcal{O}(-A)) \neq 0$ for m large enough.

Logarithmic vector fields and hyperbolicity

- $\mathcal{X} \subset \mathbb{P}^2 \times \mathbb{P}^{N_d}$ the universal curve given by the equation

$$\sum_{|\alpha|=d} a_\alpha Z^\alpha = 0, \text{ where } [a] \in \mathbb{P}^{N_d} \text{ and } [Z] \in \mathbb{P}^2.$$

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Proposition

The vector bundle $T_{\overline{J_2^v}(\mathbb{P}^3 \times \mathbb{P}^{N_d})} \otimes \mathcal{O}_{\mathbb{P}^2}(7) \otimes \mathcal{O}_{\mathbb{P}^{N_d}}(*)$ *is generated by its global sections on* $\overline{J_2^v}(\mathbb{P}^3 \times \mathbb{P}^{N_d}) \backslash \Sigma$.

Proof

- Consider an entire curve $f : \mathbb{C} \to \mathbb{P}^2 \backslash C$ for a generic curve in $\mathbb{P}^2$ of degree $d \geq 14$ such that the projectivized first derivative $f_{[1]} : \mathbb{C} \to X_1$ is Zariski dense.

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- We have a section

$$\sigma \in H^0(\mathbb{P}^2, E_{2,m}\overline{T}^*_{\mathbb{P}^2} \otimes \overline{K}_{\mathbb{P}^2}^{-t}) \simeq H^0((\mathbb{P}^2)_2, \mathcal{O}_{(\mathbb{P}^2)_2}(m) \otimes \pi_2^* \overline{K}_{\mathbb{P}^2}^{-t}).$$

with zero set Z and vanishing order $t(d-3)$.

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- Consider the family

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of curves of degree d in $\mathbb{P}^2$.
- General semicontinuity arguments concerning the cohomology groups show the existence of a Zariski open set $U_d \subset \mathbb{P}^{N_d}$ such that for any $a \in U_d$, there exists an irreducible and reduced divisor

$$Z_a = (P_a = 0) \subset (\mathbb{P}^2_a)_2$$

where

$$P_a \in H^0((\mathbb{P}^2_a)_2, \mathcal{O}_{(\mathbb{P}^2_a)_2}(m) \otimes \pi_2^* \overline{K}_{(\mathbb{P}^2_a)}^{-t})$$

such that the family $(P_a)_{a \in U_d}$ varies holomorphically.

Proof

Proposition (El Goul)

Let (X, D) be a log surface of log general type with $Pic(X) = \mathbb{Z}$. Suppose that

$$m(13\overline{c}_1^2 - 9\overline{c}_2) > 12t\overline{c}_1^2$$

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- We can find a vector field v such that $dP(v)$ is a holomorphic jet differential vanishing on ample divisor and algebraically independant of P provided that

$$\frac{m(13\overline{c}_1^2 - 9\overline{c}_2)}{12\overline{c}_1^2}(d-3) > 7$$

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- The previous numerical condition is verifed for $d \geq 14$.
- We get the algebraic degeneracy of the curve by a theorem of McQuillan-El Goul.