Real singular Del Pezzo surfaces and threefolds fibred by rational curves

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Algebraic varieties defined over $\mathbb{R}$

Let X be a projective algebraic variety defined over $\mathbb{R}$ (possibly singular)
Notation $X := X(\mathbb{C})$ the set of complex points and $X(\mathbb{R})$ the set of real points, $X(\mathbb{R})$ is the real part of X

If X nonsingular and $X(\mathbb{R}) \neq \emptyset$

$$\Rightarrow \begin{cases} X \quad \text{and} \quad X(\mathbb{R}) \quad \text{compact smooth manifolds} \\ \dim_{\mathbb{R}}(X(\mathbb{R})) = \dim_{\mathbb{C}}(X) \end{cases}$$

Kod(X) imposes restrictions on the topology of $X(\mathbb{R})$

A famous example in dimension 2

Theorem (Comessatti, 1914)

Suppose X be a nonsingular rational surface, then any orientable connected component of $X(\mathbb{R})$ is a sphere or a torus.

Du Val surfaces

X real projective surface with Du Val singularities.
$x \in X(\mathbb{R})$ a singular point.
Classification over $\mathbb{C}$ = Dynkin diagrams A_μ, $\mu \geq 1$, D_μ, $\mu \geq 4$, E_6, E_7, E_8.
Over $\mathbb{R}$, more possibilities.
Focus on two of them:
x is A_μ^+ $\Leftrightarrow$ P real analytically equivalent to

$$x^2 + y^2 - z^{\mu+1} = 0, \ \mu \geq 1 \ ;$$

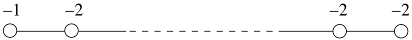
x is A_μ^- $\Leftrightarrow$ P real analytically equivalent to

$$x^2 - y^2 - z^{\mu+1} = 0, \ \mu \geq 1 \ .$$

A_1^+ real analytically isomorphic to A_1^-;

Weighted blow-up

Suppose X rational ($=$ birational to $\mathbb{P}^2$).
No bound on the number of singular points on $X(\mathbb{R})$.
Example: We can produce an arbitrarily high number of A_μ^- points
$x^2 - y^2 - z^{\mu+1} = 0, \ \mu \geq 1$.
Blow-up a smooth point $\mu + 1$ times (the last blow-up gives the (-1)-curve)



Contracting the μ (-2)-curves, we obtain a rational surface with a A_μ^- real singular point

Topological normalization

Definition

Let V simplicial complex, $Sing(V)$ finite
$\overline{n}\colon \overline{V} \to V$ *= unique proper continuous map s.t.*

- $\overline{n}$ *homeomorphism over* $V \setminus Sing(V)$
- $x \in Sing(V) \Rightarrow \overline{n}^{-1}(x)$ *one-to-one with connected components of some punctured neighborhood of x in V.*

If V is pure of dimension 2, then $\overline{V}$ is a topological manifold.

Definition

Let $x \in X(\mathbb{R})$ be a singular point of type $A_{\mu}^{\pm}$ with μ odd.
$\overline{X(\mathbb{R})}$ *has two connected components locally near x.*
x is globally separating if these two local components are on different connected components of $\overline{X(\mathbb{R})}$ and globally nonseparating otherwise.

Main theorem

A weighted blow-up produces a globally nonseparating singular A_μ^- point. Define

$$\mathcal{P}_X := \operatorname{Sing} X \setminus \{x \text{ of type } A_\mu^-,\ \mu \text{ even}\}$$
$$\setminus \{x \text{ of type } A_\mu^-,\ \mu \text{ odd and } x \text{ is globally nonseparating}\}$$

Theorem (Catanese and M–, 2007)

X rational Du Val surface defined over $\mathbb{R}$, $M \subset \overline{X(\mathbb{R})}$ connected component, then $\#(\overline{n}^{-1}(\mathcal{P}_X) \cap M) \leq 4$.

- This result improves the bound given by Kollár in 1999: $\#(\overline{n}^{-1}(\mathcal{P}_X) \cap M) \leq 6$
- There are examples with $\#(\overline{n}^{-1}(\mathcal{P}_X) \cap M) = 4$.

Application on the topology of 3-folds

N 3-dimensional compact topological manifold without boundary

- $N :=$ Lens space $\Leftrightarrow N = S^3/\mathbb{Z}_m$
- $N :=$ Seifert fibred manifold $\Leftrightarrow \exists g : N \to F$, C^∞-S^1-fibration locally trivial except near a finite number of multiple fibres

$W \to X$ real smooth projective threefold fibred by rational curves. Suppose $W(\mathbb{R})$ orientable.

Theorem (Kollár, 99)

1. *A connected component $N \subset W(\mathbb{R})$ is essentially a Seifert fibred manifold, or a connected sum of lens spaces.*
2. *Let $k := k(N)$ be the number of multiple fibres in case Seifert or the number of lens spaces. When X is rational, then $k \leq 6$.*

Real projective 3-fold fibred by rational curves

Theorem (Huisman and M–, 2005)

All the manifolds N as above do indeed occur as connected component of the real part of a real smooth projective threefold fibred by rational curves.

Answers two conjectures of Kollár.

Real rationally connected Threefolds

Theorem (Catanese and M–, 2007)

Suppose X rational.

- *For each connected component $N \subset W(\mathbb{R})$, $k(N) \leq 4$.*
- *if N admits a Seifert fibration over the torus $S^1 \times S^1$. Then $k(N) = 0$. Furthermore, X is then rational over $\mathbb{R}$ and $W(\mathbb{R})$ is connected.*

Answers two Kollár's conjectures.

Comessatti in dimension 3 ?

What essentially means ?
Let N be a closed 3-dimensional manifold. Take a decomposition $N = N' \#^a \mathbb{P}^3(\mathbb{R}) \#^b (S^1 \times S^2)$ with $a + b$ maximal. This decomposition is unique by a theorem of Milnor.

Definition

N is essentially $\mathcal{P}$ if N' is $\mathcal{P}$.

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Conjecture (M–, fresh of the day)

Suppose X be a nonsingular rationally connected 3-dimensional projective variety defined over $\mathbb{R}$. Suppose that $X(\mathbb{R})$ is orientable. A connected component of $X(\mathbb{R})$ is essentially spherical or euclidean.