Schottky uniformization of real algebraic curves and an application to moduli

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Abstract

We show that a nonsingular compact connected real algebraic curve can be uniformized by a real Schottky group, i.e., a Schottky group in $\mathrm{PGL}_2(\mathbb{C})$ which is actually contained in $\mathrm{PGL}_2(\mathbb{R})$. As an application we show that the set $M^{\mathrm{rp}}_{g/\mathbb{R}}$ of isomorphism classes of nonsingular compact connected real algebraic curves of genus g having real points, has a structure of a semianalytic variety. We show that this structure coincides with the semianalytic structure on $M^{\mathrm{rp}}_{g/\mathbb{R}}$ defined via real Teichmüller spaces.

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1 Introduction

Let X be a nonsingular compact connected real algebraic curve. In a previous paper we have shown that X can be uniformized by an open subset Ω of $\mathbb{P}^1(\mathbb{C})$ such that the set of real points of X is uniformized by the set $\Omega\cap\mathbb{P}^1(\mathbb{R})$ of real points of Ω [7]. Such a uniformization is called a strict uniformization of the real algebraic curve X. In fact, such a uniformization has been considered by P. Koebe for the first time [8].

Let $p\colon \Omega \to X$ be a strict uniformization of X. In the present paper we will show that, if X has real points, the group of automorphisms G of the covering map p is a real Schottky group, i.e., a Schottky group in $\mathrm{PGL}_2(\mathbb{C})$ which is actually contained in $\mathrm{PGL}_2(\mathbb{R})$. This statement has been treated in two separated cases: the case of X being dividing, i.e. the complement $X(\mathbb{C})\backslash X(\mathbb{R})$ consists of 2 connected components, and the case of X being nondividing. The former case has been treated in the papers [13, 1, 2]. The latter case has been treated in the papers [11, 12] but I cannot follow the proof. Recently, the paper [10] handles both cases, however, still seperately. In the present paper, we will neatly handle both cases at the same time by attacking the problem in a conceptual way.

We apply the fact that a real algebraic curve having real points can be uniformized by a real Schottky group to moduli of such curves. Before explaining this application, we need to introduce some notation.

Let g be an integer satisfying $g \geq 2$. Let $M^{\mathrm{rp}}_{g/\mathbb{R}}$ be the set of isomorphism classes of nonsingular compact connected real algebraic curves of genus g. Note that two real algebraic curves X and Y are said to be isomorphic when they are isomorphic *as real algebraic curves.*

Let F be the free group on the symbols $\gamma_1, \ldots, \gamma_g$. A *normalized embedded Schottky group of rank* g is a morphism of groups $\iota\colon F \to \mathrm{PGL}_2(\mathbb{C})$ such that $\iota(F)$ is a Schottky group of rank g, 0 and ∞ are attracting and repelling fixed points of $\iota(\gamma_1)$, respectively, and 1 is an attracting fixed point of $\iota(\gamma_2)$. Such a normalized embedded Schottky group is said to be *real* when $\iota(F)$ is contained in $\mathrm{PGL}_2(\mathbb{R})$. Let $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ be the set of real normalized embedded Schottky groups.

Let

$$\psi\colon S^{\mathrm{norm}}_{g/\mathbb{R}}(F) \longrightarrow M^{\mathrm{rp}}_{g/\mathbb{R}}$$

be the map defined by letting $\psi(\iota)$ be the isomorphism class of the real algebraic curve Ω/G, where G is the real Schottky group $\iota(F)$ and Ω is its domain of discontinuity. The fact that any nonsingular compact connected real algebraic curve having real points can be uniformized by a real Schottky group implies then that ψ is surjective. From the interpretation of a real Schottky uniformization as a universal strict covering, it will follow that ψ is a quotient map for the natural action on $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ of the group $\mathrm{Out}(F)$ of outer automorphism of F.

Since the set $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ has a natural structure of a real analytic variety and since the action of $\mathrm{Out}(F)$ on $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ is properly discontinuous, the quotient $M^{\mathrm{rp}}_{g/\mathbb{R}}$ acquires the structure of a semianalytic variety. We will show that this structure coincides with the structure of the semianalytic coarse moduli space on $M^{\mathrm{rp}}_{g/\mathbb{R}}$ [6]. In fact, we will show that any connected component C of $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ is real bianalytically isomorphic to a real Teichmüller space T, and that this isomorphism can be chosen so that the actions of the real modular group on T and the stabilizer subgroup of $\mathrm{Out}(F)$ of the connected component C correspond (see Theorem 5.1).

One should compare with the situation for complex algebraic curves. Let $M_{g/\mathbb{C}}$ be the set of isomorphism classes of nonsingular compact connected complex algebraic curves of genus g. Let $S^{\mathrm{norm}}_{g/\mathbb{C}}(F)$ be the set of normalized embedded Schottky groups. Then, one defines a map

$$\psi' \colon S_{g/\mathbb{C}}^{\mathrm{norm}}(F) \longrightarrow M_{g/\mathbb{C}}$$

by letting $\psi'(\iota)$ be the isomorphism class of the complex algebraic curve Ω/G, where Ω is the domain of discontinuity of $G = \iota(F)$. This map is surjective since any nonsingular compact connected complex algebraic curve can be uniformized by a Schottky group. However, $S_{g/\mathbb{C}}^{\mathrm{norm}}(F)$ is nontrivially universally covered by the Teichmüller space of complex algebraic curves of genus g. In particular, $S_{g/\mathbb{C}}^{\mathrm{norm}}(F)$ is not biholomorphic to the Teichmüller space of complex algebraic curves of genus g. Note also that, although ψ' factorizes through the quotient map of $S_{g/\mathbb{C}}^{\mathrm{norm}}(F)$ by $\mathrm{Out}(F)$, the map ψ' is not a quotient map for the action of $\mathrm{Out}(F)$ on $S_{g/\mathbb{C}}^{\mathrm{norm}}(F)$.

An interesting feature of the present construction of the semianalytic structure on $M_{g/\mathbb{R}}^{\mathrm{rp}}$ is that it realizes $M_{g/\mathbb{R}}^{\mathrm{rp}}$ as a semianalytic subset of the set of real points of the connected complex analytic variety

$$Z = S_{g/\mathbb{C}}^{\mathrm{norm}}(F)/\mathrm{Out}(F)$$

which is naturally defined over the reals. It seems that one can partly compactify Z—by allowing real Schottky groups to have parabolic elements as in [4]—in order to study analytic moduli of stable real algebraic curves having only real singularities (cf. [14]).

Another interesting feature of the construction of the semianalytic structure on $M_{g/\mathbb{R}}^{\mathrm{rp}}$ through real Schottky groups is that it is so explicit that it seems to allow to study the topology of the moduli space $M_{g/\mathbb{R}}^{\mathrm{rp}}$.

The paper is written in such a way that Sections 2, 3 and 4 can be read as an elementary construction of a structure of a semianalytic variety on the set $M_{g/\mathbb{R}}^{\mathrm{rp}}$ of isomorphism classes of real algebraic curves of genus g having real points. Section 5 can be considered as an appendix where it is shown that this structure is the right structure. The proof relies on the universal family of marked real algebraic curves over the real Teichmüller space as constructed in [6].

Conventions and notation. We denote by Σ the Galois group of $\mathbb{C}$ over $\mathbb{R}$. The nontrivial element of Σ is denoted by σ. Let X be a set on which Σ acts. A subset Y of X is stable if $\sigma \cdot Y \subseteq Y$. The subset of X of fixed points for the action of Σ is denoted by X^{Σ}. When a map is said to be equivariant it is understood to be so with respect to actions of Σ, unless otherwise stated.

A Riemann surface is not necessarily connected or compact. When Σ is said to act on a Riemann surface, σ is understood to act antiholomorphically.

A compact connected Riemann surface endowed with an action of Σ is called a real algebraic curve. When X is a real algebraic curve, the set X^Σ is called the set of real points of X.

2 Strict uniformization of real algebraic curves

A Σ-*space* is a topological space X endowed with an action of Σ such that every element of Σ acts continuously. When X and Y are Σ-spaces, an equivariant map $p\colon Y \to X$ is said to be *strict* if it maps every orbit bijectively onto its image. It is then clear what a universal strict equivariant covering is. Any reasonable Σ-space X has a universal strict equivariant covering $p\colon \tilde{X} \to X$. One defines the *strict fundamental group* $\sigma_1(X, \Sigma)$ of X to be the group $\mathrm{Aut}(\tilde{X}/X)$ of automorphisms of the equivariant covering p (see [7] for details).

Let X be a Σ-space. There is an equivalence between the category of strict equivariant coverings of X and the category of ordinary coverings of the quotient space X/Σ. Indeed, if $p\colon Y \to X$ is a strict covering of X then the induced map

$$\overline{p}\colon Y/\Sigma \longrightarrow X/\Sigma$$

is a covering of X/Σ. It is easily verified that this defines an equivalence between the two fore-mentioned categories. It follows that a strict equivariant map $p\colon Y \to X$ is a universal strict equivariant covering of X if and only if the induced map $\overline{p}$ is an ordinary universal covering. In particular, the strict fundamental group $\sigma_1(X, \Sigma)$ is isomorphic to the ordinary fundamental group $\pi_1(X/\Sigma)$ of X/Σ.

Theorem (Strict uniformization of real algebraic curves). *Let X be a real algebraic curve of genus g. Suppose that X has real points if $g = 0$ or 1. Then, there are a stable open subset Ω of $\mathbb{P}^1(\mathbb{C})$ containing the double half-plane $\mathbb{D}$ and a universal strict equivariant holomorphic covering $p\colon \Omega \to X$ of X by Ω.*

Sketch of proof. Let Y be the Riemann surface $X \backslash X^\Sigma$. Let Y_1 be a connected component of Y. By the hypothesis on X, the connected Riemann surface Y_1 is hyperbolic; i.e., there is a universal holomorphic covering $q_1\colon \mathbb{U} \to Y_1$ of Y_1 by the upper half-plane $\mathbb{U}$. Put $Y_\sigma = \sigma \cdot Y_1$ and let $q_\sigma\colon \mathbb{L} \to Y_\sigma$ be the map defined by $q_\sigma(z) = \sigma \cdot q_1(\sigma(z))$ for all z in the lower half-plane $\mathbb{L}$. Then, let $q\colon \mathbb{D} \to Y$ be the map defined on the double half-plane $\mathbb{D} = \mathbb{U} \cup \mathbb{L}$ whose restriction to $\mathbb{U}$ is equal to q_1 and whose restriction to $\mathbb{L}$ is equal to q_σ. One can show that q extends to a map $p\colon \Omega \to X$ for some open subset Ω of $\mathbb{P}^1(\mathbb{C})$ and that this map has the required properties (see [7] for details). $\square$

Any universal strict equivariant holomorphic covering $p\colon \Omega \to X$ of a real algebraic curve X by a stable open subset Ω of $\mathbb{P}^1(\mathbb{C})$ will be called a *strict uniformization* of X.

A Kleinian subgroup of $\mathrm{PGL}_2(\mathbb{C})$ is called a *real Kleinian group* if it is contained in $\mathrm{PGL}_2(\mathbb{R})$ (we refer to [9] for definitions and facts concerning Kleinian groups).

The interest of the strict fundamental group now becomes clear; a strict uniformization of a real algebraic curve X realizes the strict fundamental group $\sigma_1(X,\Sigma)$ of X as a real Kleinian group.

Let X be a real algebraic curve. Let $g = g(X)$ be the genus of X. The number of connected components of X^Σ will be denoted by $s = s(X)$. The real algebraic curve is said to be *dividing* if $X \setminus X^\Sigma$ is not connected. It is well known that $s \equiv g+1 \mod 2$ and $1 \le s \le g+1$ if X is dividing, and that $0 \le s \le g$ if X is nondividing [15].

Proposition 2.1. *Let X be a real algebraic curve. Let $g = g(X)$ and $s = s(X)$. Then, the strict equivariant fundamental group $\sigma_1(X,\Sigma)$ is isomorphic to the group generated by elements $\gamma_1, \ldots, \gamma_{g+1}$ subject to the following relation.*

1. *If X is dividing then*

$$\gamma_1 \cdots \gamma_s \cdot [\gamma_{s+1}, \gamma_{s+2}] \cdots [\gamma_g, \gamma_{g+1}] = 1.$$

2. *If X is nondividing then*

$$\gamma_1 \cdots \gamma_s \cdot \gamma_{s+1}^2 \cdots \gamma_{g+1}^2 = 1.$$

In particular, the group $\sigma_1(X,\Sigma)$ is a free group on g generators if X has real points.

Proof. Consider the quotient space X/Σ. This topological space is a compact connected topological surface with s connected boundary components. Since the Euler characteristic $\chi(X^\Sigma)$ of the set of fixed points X^Σ of X is equal to 0, one has $\chi(X/\Sigma) = \frac{1}{2}\chi(X) = 1-g$. Let S be a compact connected topological surface such that X/Σ is homeomorphic to the complement of the union of s disjoint open discs in S. Then, $\chi(S) = 1-g+s$.

If X is dividing then X/Σ is orientable. The same then holds for S. It follows that S is an orientable surface of genus $\frac{1}{2}(g-s+1)$. Then, the group $\sigma_1(X,\Sigma)$, being isomorphic to the fundamental group of X/Σ, is generated by elements $\gamma_1, \ldots, \gamma_{g+1}$, subject to relation 1.

If X is nondividing then X/Σ is nonorientable, and so is S. It follows that S is the connected sum of $g+1-s$ real projective planes. In this case, the group $\sigma_1(X, \Sigma)$ is generated by elements $\gamma_1, \dots, \gamma_{g+1}$ subject to relation 2. □

Using strict uniformization of a real algebraic curve, the preceding proposition and standard facts on Kleinian groups, the following statement can easily be proven.

Proposition 2.2. *Let g be a nonnegative integer. Let X be a real algebraic curve of genus g having real points. Let $p\colon \Omega \to X$ be a strict uniformization of X by some stable open subset Ω of $\mathbb{P}^1(\mathbb{C})$. Let G be the group of automorphisms of the equivariant covering p. Then, G is a loxodromic real Kleinian group, free on g generators. Its domain of discontinuity is equal to Ω. Moreover, the map p is a quotient map for the action of G on Ω, in particular, Ω/G is equivariantly biholomorphic to X.* □

Any finitely generated free loxodromic Kleinian group is known to be a Schottky group [9]. Therefore, Proposition 2.2 already implies that a real algebraic curve having real points can be uniformized by a real Schottky group. However, for the application we have in mind, we will need the stronger statement that a real algebraic curve having real points can be uniformized by a real classical Schottky group.

One has the following converse to Proposition 2.2:

Proposition 2.3. *Let g be a nonnegative integer. Let G be a loxodromic real Kleinian group which is free on g generators and let Ω be its domain of discontinuity. Let X be the quotient Riemann surface Ω/Σ and let $p\colon \Omega \to X$ be the quotient map. Then, Ω is stable for the action of Σ on $\mathbb{P}^1(\mathbb{C})$, X is a real algebraic curve of genus g having real points and p is a strict uniformization of X.*

Proof. Of course, we may suppose that $g \neq 0$. By definition of the induced Σ-action on X, the map p is equivariant. Since G is loxodromic, G acts fixed point-freely on Ω. Hence, p is a covering map.

Let us show that p is a strict equivariant map; i.e., $p^{-1}(X^\Sigma) = \Omega^\Sigma$. Clearly, $p(\Omega^\Sigma)$ is contained in X^Σ, so that Ω^Σ is contained in $p^{-1}(X^\Sigma)$. To show the reverse inclusion, suppose $x \in \Omega$ has its image $p(x)$ in X^Σ. This means that there is $\alpha \in G$ such that $\alpha(x) = \sigma(x)$. Then, x is a fixed point of α^2. Since G is loxodromic, $\alpha^2 = 1$ and hence, $\alpha = 1$ and $x \in \Omega^\Sigma$. This proves that $p^{-1}(X^\Sigma) = \Omega^\Sigma$; i.e., p is a strict equivariant map. Hence, p is a strict equivariant covering map.

Since G is a real Kleinian group, G acts discontinuously on the double half-plane $\mathbb{D}$, so that $\mathbb{D} \subseteq \Omega$. In particular, the quotient Ω/Σ is simply connected. It then follows that $p\colon \Omega \to X$ is a universal strict equivariant covering of X.

Since G is loxodromic, the quotient Riemann surface $X = \Omega/G$ is compact. If X were not connected, Ω would have been equal to $\mathbb{D}$ and the upper half-plane $\mathbb{U}$ would have been stable for the action of G. The group G would then have been isomorphic to the ordinary fundamental group of a compact connected surface. This contradicts the fact that G is free on g generators and $g \neq 0$. It follows that X is a real algebraic curve having real points.

Since G is isomorphic to the strict fundamental group of X, the genus of X is equal to g by Proposition 2.1. □

Corollary 2.4. *Any discrete finitely generated free loxodromic subgroup of* $\mathrm{PGL}_2(\mathbb{R})$ *acts discontinuously on some open dense subset of* $\mathbb{P}^1(\mathbb{R})$.

3 Schottky uniformization of real algebraic curves having real points

In the Introduction of the paper, a real Schottky group was defined as a Schottky group in $\mathrm{PGL}_2(\mathbb{C})$ which was contained in $\mathrm{PGL}_2(\mathbb{R})$. In this section we define the notion of a real classical Schottky group as a classical Schottky group in $\mathrm{PGL}_2(\mathbb{C})$ such that all the data defining it is "defined over $\mathbb{R}$." In the end, it will then turn out that a classical Schottky that is contained in $\mathrm{PGL}_2(\mathbb{R})$ actually is a real classical Schottky group (Corollary 3.6). So that the terminology "real classical Schottky group" will be justified.

Let g be any nonnegative integer. Let $C_1, \ldots, C_{2g}$ be circles on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ having real centers and which are such that, for all i, the circles C_j, for $j \neq i$, are all entirely contained in the same connected component of $\mathbb{P}^1(\mathbb{C}) \setminus C_i$. Let D be the closed connected subset of $\mathbb{P}^1(\mathbb{C})$ such that

$$\partial D = \bigcup_{i=1}^{2g} C_i.$$

Choose for each odd integer i, $1 \leq i \leq 2g$, an element α_i of $\mathrm{PGL}_2(\mathbb{R})$ such that $\alpha_i(C_i) = C_{i+1}$ and $\alpha_i(D) \cap D = C_{i+1}$. Then, the subgroup G of $\mathrm{PGL}_2(\mathbb{R})$ generated by α_i, i odd and $1 \leq i \leq 2g$, is a real Kleinian group. We will call any Kleinian group constructed in this way, a *real classical Schottky group*.

Let Ω be the domain of discontinuity of G. Let X be the quotient Riemann surface Ω/G. Since G is a real Kleinian group, X comes with an action of Σ. Moreover, $D \subseteq \Omega$ is a fundamental domain for the action of G on Ω. Hence, X is connected and compact; i.e., X is a real algebraic curve of genus g. Note that X has real points.

Let X be any real algebraic curve having real points. A *real Schottky uniformization* of the real algebraic curve X is an equivariant holomorphic covering $p\colon \Omega \to X$ of X by some stable open subset Ω of $\mathbb{P}^1(\mathbb{C})$, such that the automorphism group G of this equivariant covering is a real classical Schottky group and such that p induces a biholomorphic map from Ω/G onto X.

Let $p\colon \Omega \to X$ be a real Schottky uniformization of a real algebraic curve X. Observe that p is then a strict uniformization of X by Proposition 2.3. In this section we are going to prove the converse: a strict uniformization of a real algebraic curve X having real points is a real Schottky uniformization of X. First we need some preparation.

An *embedding* of a manifold with boundary N into a manifold with boundary M is a homeomorphism $f\colon N \to M$ of N onto $f(N)$ such that $f^{-1}(\partial M) = \partial N$. We denote by I the closed unit interval considered as a manifold with boundary.

Lemma 3.1. *Let M be a compact connected surface with nonempty boundary. Let $g = 1 - \chi(M)$. Then there are embeddings $e_i\colon I \to M$, $i = 1, \ldots, g$, such that*

1. *$e_i(I) \cap e_j(I) = \emptyset$ whenever $i \neq j$, and*
2. *$M \backslash \bigcup e_i(I)$ is simply connected.*

Proof. We proceed by induction on g. For $g = 0$ the statement evidently holds. Suppose that the statement holds for some $g \in \mathbb{N}$. Let M be a compact connected manifold with nonempty boundary such that $g + 1 = 1 - \chi(M)$. Since $\chi(M) \neq 1$, the surface M is not simply connected. Therefore, there is an embedding e_{g+1} of I into M such that $M \backslash e_{g+1}(I)$ is connected. Let M' be the compact connected surface obtained by gluing in two copies of I along the "open boundary" of $M \backslash e_{g+1}(I)$. Since the Euler characteristic of I is equal to 1, one has

$$1 - \chi(M') = 1 - (\chi(M) + 1) = g.$$

By induction, there are embeddings e_i of I in M', for $i = 1, \ldots, g$, satisfying conditions 1 and 2 of the statement for the surface M'. Observe that we may assume that $e_i(I)$ is entirely contained in the open subset $M \backslash e_{g+1}(I)$ of M'. It then follows that the embeddings e_i, for $i = 1, \ldots, g+1$, satisfy the required conditions for M. ☐

We consider the unit circle $S^1 \subseteq \mathbb{C}$ as a manifold endowed with an action of Σ induced by its action on $\mathbb{C}$.

Applying the preceding lemma to the quotient X/Σ of a real algebraic curve X having real points, one immediately gets the following corollary.

Corollary 3.2. *Let X be a real algebraic curve having real points. Then there are strict equivariant embeddings $e_i\colon S^1 \to X$, $i = 1, \ldots, g$, such that*

1. *$e_i(S^1) \cap e_j(S^1) = \emptyset$ whenever $i \neq j$, and*
2. *$X \setminus \bigcup e_i(S^1)$ has a trivial strict fundamental group.*

Theorem (Schottky uniformization of real algebraic curves). *Let X be a real algebraic curve having real points. Let $\Omega \subseteq \mathbb{P}^1(\mathbb{C})$ be an open stable subset and let $p\colon \Omega \to X$ be an equivariant holomorphic map. Then, the following two conditions are equivalent.*

1. *p is a strict uniformization of X.*
2. *p is a real Schottky uniformization of X.*

In particular, any real algebraic curve having real points has a real Schottky uniformization.

Proof. We have already observed that a real Schottky uniformization of X is necessarily a strict uniformization.

Suppose therefore that p is a strict uniformization of the real algebraic curve X. Let G be the group of automorphisms of the equivariant covering p. We need to show that G is a real classical Schottky group.

Let g be the genus of X and let e_i, $i = 1, \ldots, g$ be embeddings of S^1 into X as in the statement of Corollary 3.2. Let S_i be the image of e_i, for $i = 1, \ldots, g$, and put

$$F = X \setminus \bigcup_{i=1}^{g} S_i.$$

Then, according to Corollary 3.2, the strict fundamental group of F is trivial. Since the restriction of p to $p^{-1}(F)$ is a strict equivariant covering of F, this covering is trivial. Hence there is a connected subset F' of Ω such that the restriction of p to F' is a homeomorphism onto F. Since the covering is strict and X has real points, F' is a stable subset of Ω. Then, of course, the restriction of p to F' is equivariant.

Let $\overline{F'}$ be the closure of F' in Ω. Let B be the boundary of $\overline{F'}$. Since F' is stable, B is stable too and the restriction of p to B is a strict equivariant covering of $\bigcup S_i$. But each S_i has a trivial strict fundamental group. Hence, p maps each connected component of B homeomorphically onto some S_i.

Since X is a 2-manifold without boundary, there are, for each i, exactly 2 connected components of B that are mapped onto S_i.

Let T_i, $i = 1, \ldots, 2g$, be the connected components of the boundary B of $\overline{F'}$ numbered in such a way that $p(T_{2i-1}) = p(T_{2i}) = S_i$ for all $i = 1, \ldots, g$. As we observed already, the restriction of p to B is a strict equivariant map onto $\bigcup S_i$. Since $S_i^\Sigma \neq \emptyset$ for all $i = 1, \ldots, g$, each connected component T_i of B is stable. It follows that the restriction of p to T_{2i-1} (resp. T_{2i}) is an equivariant homeomorphism, for $i = 1, \ldots, g$. In particular, T_i^Σ consists of 2 points, for $i = 1, \ldots, 2g$.

Since the universal strict covering $p\colon \Omega \to X$ is Galois, there are, for each odd i, $1 \leq i \leq 2g$, elements $\alpha_i \in G$ such that $\alpha_i(T_i) = T_{i+1}$.

Let for each $i = 1, \ldots, 2g$, C_i be the circle on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ with real center and such that $C_i \cap \mathbb{P}^1(\mathbb{R}) = T_i^\Sigma$. Since $T_i \cap T_j = \emptyset$ whenever $i \neq j$, one also has $C_i \cap C_j = \emptyset$ whenever $i \neq j$. Since F' is connected, one has that, for each i, the subsets T_j, for $j \neq i$, are all entirely contained in the same connected component of $\mathbb{P}^1(\mathbb{C}) \backslash T_i$. Therefore, the same holds for the circles C_i. Since $\alpha_i(T_i^\Sigma) = T_{i+1}^\Sigma$ for all odd i, $\alpha_i(C_i) = C_{i+1}$, for all odd i. Let D be the unique closed connected subset of $\mathbb{P}^1(\mathbb{C})$ having the circles C_i as its boundary components. Since $\alpha_i(\overline{F'}) \cap \overline{F'} = T_{i+1}$, one has $\alpha_i(D) \cap D = C_{i+1}$.

Let $H \subseteq G$ be the subgroup generated by α_i, i odd and $1 \leq i \leq 2g$. Then, H is a real classical Schottky group. Let $\Omega' \subseteq \Omega$ be the union of the sets $\alpha \cdot \overline{F'}$, $\alpha \in H$. Then, the restriction of p to Ω' is a strict equivariant covering of X. Since p is universal, $\Omega' = \Omega$. It follows that $H = G$. Hence, G is a real classical Schottky group. □

The preceding theorem, combined with Proposition 2.3, has the following corollaries, each of which is a consequence of the first. In order to state them in a convenient way, let us call a Schottky group in $\mathrm{PGL}_2(\mathbb{C})$ a *complex Schottky group*, and a classical Schottky group in $\mathrm{PGL}_2(\mathbb{C})$ a *complex classical Schottky group.*

Corollary 3.3. *Let G be a complex Schottky group which is conjugate to a real Kleinian group. Then, G is conjugate to a real classical Schottky group. In particular, G is a complex classical Schottky group.*

Corollary 3.4. *A complex Schottky group whose limit set lies on a circle in $\mathbb{P}^1(\mathbb{C})$ is a complex classical Schottky group.*

Corollary 3.5. *Any real Schottky group is a real classical Schottky group.*

Corollary 3.6. *A complex classical Schottky group is a real classical Schottky group if and only if it is contained in* $\mathrm{PGL}_2(\mathbb{R})$.

According to Corollary 3.5, there is no need to distinguish between classical or nonclassical real Schottky groups. Therefore, in the sequel, we suppress mention of "classical" for real Schottky groups.

4 The moduli space of real algebraic curves having real points

Let g be a nonnegative integer. Let $M^{\mathrm{rp}}_{g/\mathbb{R}}$ be the set of isomorphism classes of compact connected real algebraic curves of genus g having real points. Note that two real algebraic curves are isomorphic when they are equivariantly biholomorphic. When X is a compact connected real algebraic curve of genus g, its isomorphism class is denoted by $[X]$.

Let $S_{g/\mathbb{R}}$ be the subset of real Schottky groups of rank g. According to Proposition 2.3, we can define a map

$$\pi\colon S_{g/\mathbb{R}} \longrightarrow M^{\mathrm{rp}}_{g/\mathbb{R}}$$

by letting $\pi(G)$ be the isomorphism class $[X]$ of the real algebraic curve $X = \Omega/G$, where Ω is the domain of discontinuity of G. According to Proposition 2.2, the map π is surjective.

The group $\mathrm{PGL}_2(\mathbb{R})$ acts on the set $S_{g/R}$ by conjugation. It is clear that the map π is constant on the orbits of $S_{g/\mathbb{R}}$ for this action. In fact, more is true:

Lemma 4.1. *Let G and H be in $S_{g/\mathbb{R}}$. Then, $\pi(G) = \pi(H)$ if and only if G and H belong to the same orbit of $S_{g/\mathbb{R}}$.*

Proof. We have already noted that if G and H belong to the same orbit of $S_{g/\mathbb{R}}$, then $\pi(G) = \pi(H)$.

Conversely, suppose that $\pi(G) = \pi(H)$ for some real Schottky groups G and H in $S_{g/\mathbb{R}}$. This means that the real algebraic curves $X = \Omega/G$ and $Y = \Gamma/H$ are isomorphic; here Ω and Γ are the domains of discontinuity of G and H, respectively. Let

$$f\colon X \longrightarrow Y$$

be an isomorphism. By Proposition 2.3, the quotient maps

$$p\colon \Omega \longrightarrow X \quad \text{and} \quad q\colon \Gamma \longrightarrow Y$$

are strict universal coverings of X and Y, respectively. By the universal property satisfied by these coverings, there is an equivariant biholomorphic map

$$\tilde{f}\colon \Omega \longrightarrow \Gamma$$

such that $q \circ \tilde{f} = f \circ p$. Since Ω and Γ both contain the double half-plane $\mathbb{D}$, the restriction to $\mathbb{D}$ of the map $\tilde{f}$ is an equivariant automorphism of $\mathbb{D}$. It follows that $\tilde{f} \in \mathrm{PGL}_2(\mathbb{R})$. Then, $\tilde{f} H \tilde{f}^{-1} = G$; i.e., the real Schottky groups G and H belong to the same orbit in $S_{g/\mathbb{R}}$. □

To put the statement of Lemma 4.1 differently, the map

$$\pi\colon S_{g/\mathbb{R}} \longrightarrow M^{\mathrm{rp}}_{g/\mathbb{R}}$$

is a quotient map for the action of $\mathrm{PGL}_2(\mathbb{R})$ on the set $S_{g/\mathbb{R}}$.

Let F be the free group on the symbols $\gamma_1, \dots, \gamma_g$. Define

$$S_{g/\mathbb{R}}(F) = \{\iota\colon F \to \mathrm{PGL}_2(\mathbb{R}) \mid \iota(F) \in S_{g/\mathbb{R}}\}.$$

Of course, the maps ι in this definition are supposed to be homomorphisms of groups. An element of $S_{g/\mathbb{R}}(F)$ will be called an *embedded real Schottky group*. We have a forgetful map

$$\varphi\colon S_{g/\mathbb{R}}(F) \longrightarrow S_{g/\mathbb{R}}$$

defined by forgetting the embedding; i.e., $\varphi(\iota) = \iota(F)$ for all $\iota \in S_{g/\mathbb{R}}(F)$.

Let $\mathrm{Aut}(F)$ be the group of automorphisms of F. Then, the assignment $(\iota, f) \mapsto \iota \circ f$ defines a right action of the group $\mathrm{Aut}(F)$ on the set $S_{g/\mathbb{R}}(F)$. It is obvious that the map φ is a quotient map for this action.

The group $\mathrm{PGL}_2(\mathbb{R})$ also acts on the set $S_{g/\mathbb{R}}(F)$ by conjugation and the map φ is equivariant with respect to this action. Note that the left action of $\mathrm{PGL}_2(\mathbb{R})$ and the right action of $\mathrm{Aut}(F)$ on $S_{g/\mathbb{R}}(F)$ commute.

Let $\overline{S}_{g/\mathbb{R}}(F)$ be the quotient $\mathrm{PGL}_2(\mathbb{R})\backslash S_{g/\mathbb{R}}(F)$ and let

$$\rho\colon S_{g/\mathbb{R}}(F) \longrightarrow \overline{S}_{g/\mathbb{R}}(F)$$

be the quotient map.

Clearly, we have an induced action of the group $\mathrm{Aut}(F)$ on $\overline{S}_{g/\mathbb{R}}(F)$. Observe that its subgroup $\mathrm{Inn}(F)$ of inner automorphisms acts trivially. Hence, we get an action of the group $\mathrm{Out}(F)$ of outer automorphisms on the set $\overline{S}_{g/\mathbb{R}}(F)$. We also get an induced map

$$\psi\colon \overline{S}_{g/\mathbb{R}}(F) \longrightarrow M^{\mathrm{rp}}_{g/\mathbb{R}}$$

such that the diagram

$$\begin{array}{ccc} S_{g/\mathbb{R}}(F) & \xrightarrow{\varphi} & S_{g/\mathbb{R}} \\ \rho\downarrow & & \downarrow\pi \\ \overline{S}_{g/\mathbb{R}}(F) & \xrightarrow[\psi]{} & M^{\mathrm{rp}}_{g/\mathbb{R}} \end{array}$$

commutes. In fact since all other maps in the diagram are quotient maps, the map ψ is a quotient map for the action of $\mathrm{Out}(F)$ on $\overline{S}_{g/\mathbb{R}}(F)$.

From now on we suppose that $g \geq 2$. Then, the action of $\mathrm{PGL}_2(\mathbb{R})$ on $S_{g/\mathbb{R}}(F)$ is free.

An embedded Schottky group $\iota \in S_{g/\mathbb{R}}(F)$ is *normalized* if 0 is an attracting and ∞ a repelling fixed point of $\iota(\gamma_1)$ and if 1 is an attracting fixed point of $\iota(\gamma_2)$. Let $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ be the subset of $S_{g/\mathbb{R}}(F)$ of normalized embedded Schottky groups. Then, it is clear that for every embedded Schottky group $\iota \in S_{g/\mathbb{R}}(F)$, there is a unique normalized embedded Schottky group $\kappa \in S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ that belongs to the orbit of ι for the action of $\mathrm{PGL}_2(\mathbb{R})$. To put it otherwise, the restriction ρ' of ρ to the subset $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ is a bijection onto the set $\overline{S}_{g/\mathbb{R}}(F)$. Therefore, we may think of the set $\overline{S}_{g/\mathbb{R}}(F)$ as the set of normalized embedded Schottky groups.

Note that one can identify the set $S_{g/\mathbb{R}}(F)$ with an open subset of $\mathrm{PGL}_2(\mathbb{R})^g$ by means of the injective map

$$\begin{aligned} S_{g/\mathbb{R}}(F) &\longrightarrow \mathrm{PGL}_2(\mathbb{R})^g \\ \iota &\longmapsto (\iota(\gamma_1), \ldots, \iota(\gamma_g)). \end{aligned}$$

The set $S_{g/\mathbb{R}}(F)$ acquires thus the structure of a real analytic manifold. It is of dimension $3g$. The subset $S^{\mathrm{norm}}_{g/\mathbb{R}}(F)$ is then a real analytic submanifold of dimension $3g-3$. By transport of structure we get the structure of a $(3g-3)$-dimensional real analytic variety on $\overline{S}_{g/\mathbb{R}}(F)$. It is clear that $\mathrm{Out}(F)$ acts on $\overline{S}_{g/\mathbb{R}}(F)$ by real analytic automorphisms.

Lemma 4.2. *Let g be an integer satisfying $g \geq 2$. The action of the group* $\mathrm{Out}(F)$ *on* $\overline{S}_{g/\mathbb{R}}(F)$ *is properly discontinuous.*

Proof. This can be proved directly. We omit a proof since the statement is also a consequence of Theorem 5.1 and of the fact that the real modular group of a real algebraic curve X acts properly discontinuously on the real Teichmüller space of X. ☐

The *topological type* of a real algebraic curve X is the homeomorphism class of the topological pair (X, X^Σ). In fact, it is easy to see that two real algebraic curves X and Y are of the same topological type if and only if $g(X) = g(Y)$, $s(X) = s(Y)$ and X and Y are both dividing or both nondividing.

Let X be a real algebraic curve and let g be its genus. One defines $M_{g/\mathbb{R}}(X)$ to be the set of isomorphism classes of all real algebraic curves Y having the same topological type as X. Then, $M_{g/\mathbb{R}}(X)$ is a subset of $M^{\mathrm{rp}}_{g/\mathbb{R}}$. In fact, the subsets $M_{g/\mathbb{R}}(X)$ of $M^{\mathrm{rp}}_{g/\mathbb{R}}$ constitute a partition of $M^{\mathrm{rp}}_{g/\mathbb{R}}$ into

$[\frac{1}{2}(3g+2)]$ disjoint subsets. This is because a real algebraic curve of genus g having real points can have exactly $[\frac{1}{2}(3g+2)]$ different topological types [15].

Proposition 4.3. *Let g be an integer satisfying $g \geq 2$. The set $M^{rp}_{g/\mathbb{R}}$ has a unique structure of a semianalytic variety such that the map*

$$\psi\colon \overline{S}_{g/\mathbb{R}}(F) \longrightarrow M^{rp}_{g/\mathbb{R}}$$

is a quotient map in the category of semianalytic varieties. Let X be a real algebraic curve of genus g. Then the connected component of $M^{rp}_{g/\mathbb{R}}$ containing the isomorphism class $[X]$ of X is equal to the subset $M_{g/\mathbb{R}}(X)$ of $M^{rp}_{g/\mathbb{R}}$.

Proof. The statement about the semianalytic structure follows from Lemma 4.2.

Let X be a real algebraic curve of genus g having real points. Let $p\colon \Omega \to X$ be a real Schottky uniformization of X. Let G be the group of automorphisms of the equivariant covering p. Since G is a real Schottky group, there are circles $C_i \subseteq \mathbb{P}^1(\mathbb{C})$, $i = 1 \ldots , 2g$, with real centers and elements $\alpha_i \in G$, $1 \leq i \leq 2g$ and i odd, as in the definition of a real classical Schottky group. Let again $D \subseteq \mathbb{P}^1(\mathbb{C})$ denote the closed connected subset having as boundary the union of the circles C_i, $i = 1, \ldots , 2g$. Let $\iota\colon F \to \mathrm{PGL}_2(\mathbb{R})$ be defined by $\iota(\gamma_i) = \alpha_{2i-1}$, for $i = 1, \ldots , g$. Let $S_{g/\mathbb{R}}(\iota)$ be the connected component of $\overline{S}_{g/\mathbb{R}}(F)$ containing ι. We show that

$$\psi(S_{g/\mathbb{R}}(\iota)) = M_{g/\mathbb{R}}(X)$$

in order to conclude that the connected component of $M^{\mathrm{rp}}_{g/\mathbb{R}}$ containing $[X]$ is equal to $M_{g/\mathbb{R}}(X)$.

Let κ be an element of $\overline{S}_{g/\mathbb{R}}(\iota)$. Let Y be the real algebraic curve $\psi(\kappa)$. Since κ belongs to the connected component of ι, there is a continuous path

$$\lambda\colon [0,1] \longrightarrow \overline{S}_{g/\mathbb{R}}(F)$$

such that $\lambda_0 = \iota$ and $\lambda_1 = \kappa$. Then, one has a continuous family

$$\mathcal{C} = \{\mathcal{C}_t\}_{t\in[0,1]}$$

of real algebraic curves over $[0,1]$ defined by letting $\mathcal{C}_t$ be the real algebraic curve $\psi(\lambda_t)$. Since $[0,1]$ is connected, $s(X) = s(Y)$ and X and Y are either both dividing or both nondividing. Hence, $[Y] \in M_{g/\mathbb{R}}(X)$.

In order to show the reverse inclusion, let Y be a real algebraic curve of the same topological type as X. Let $q\colon \Gamma \to Y$ be a real Schottky uniformization of Y. Since the topological pairs (X, X^Σ) and (Y, Y^Σ) are homeomorphic, there is an orientation-preserving equivariant homeomorphism $f\colon X \to Y$. Let $\tilde{f}\colon \Omega \to \Gamma$ be any lift of f. Replacing Γ by $-\Gamma$, we may assume that $\tilde{f}$ maps the upper half-plane $\mathbb{U}$ into itself. Define

$$\kappa\colon F \longrightarrow \mathrm{PGL}_2(\mathbb{R})$$

by requiring $\tilde{f} \circ \iota(\gamma) = \kappa(\gamma) \circ \tilde{f}$ for all $\gamma \in F$. Then, κ is an embedded real Schottky group and $\psi(\kappa) = [Y]$. We show that κ is in the connected component $\overline{S}_{g/\mathbb{R}}(\iota)$ containing ι, by constructing a continuous path

$$\lambda\colon [0,1] \longrightarrow \overline{S}_{g/\mathbb{R}}(F)$$

such that $\lambda_0 = \iota$ and $\lambda_1 = \kappa$.

Since $\tilde{f}$ is orientation-preserving and maps $\mathbb{U}$ into itself, there is an isotopy

$$h\colon [0,1] \times D^\Sigma \longrightarrow \mathbb{P}^1(\mathbb{R})$$

such that h_0 is the inclusion of D^Σ into $\mathbb{P}^1(\mathbb{R})$ and such that h_1 is equal to the restriction to D^Σ of $\tilde{f}$. Let, for i odd with $1 \le i \le 2g$, P_i and Q_i be the 2 intersection points of C_i with $\mathbb{P}^1(\mathbb{R})$. Let P_{i+1} and Q_{i+1} be the images by α_i of P_i and Q_i, respectively. Then, P_{i+1} and Q_{i+1} are the intersection points of C_{i+1} with $\mathbb{P}^1(\mathbb{R})$, for odd i, $1 \le i \le 2g$.

Let, for any $t \in [0,1]$ and for any i with $1 \le i \le 2g$, the points $h(t, P_i)$ and $h(t, Q_i)$ be denoted by P_i^t and Q_i^t, respectively. One can choose, for any $t \in [0,1]$ and for any odd i with $1 \le i \le 2g$, elements β_i^t of $\mathrm{PGL}_2(\mathbb{R})$ such that β_i^t maps P_i^t onto P_{i+1}^t and Q_i^t onto Q_{i+1}^t. Moreover, one can assume that each β_i^t depends continuously on t.

Since β_i^0 and α_i both map P_i^0 onto P_{i+1}^0 and Q_i^0 onto Q_{i+1}^0, we may suppose that β_i^0 is equal to α_i, for all odd i, $1 \le i \le 2g$.

Let $\alpha'_{2i-1} = \kappa(\gamma_i)$ for $i = 1, \ldots, g$. Since $\tilde{f} \circ \alpha_i = \alpha'_i \circ \tilde{f}$, the element α'_i of $\mathrm{PGL}_2(\mathbb{R})$ maps $\tilde{f}(P_i)$ onto $\tilde{f}(P_{i+1})$ and $\tilde{f}(Q_i)$ onto $\tilde{f}(Q_{i+1})$. Since β_i^1 has the same property, we may suppose that β_i^1 is equal to α'_i, for all odd i, $1 \le i \le 2g$.

Now, one defines a continuous path $\lambda\colon [0,1] \to \overline{S}_{g/\mathbb{R}}(F)$ by defining

$$\lambda_t(\gamma_i) = \beta_{2i-1}^t$$

for $i = 1, \ldots, g$ and $t \in [0,1]$. Then, $\lambda_0 = \iota$ and $\lambda_1 = \kappa$. □

We will show that the structure on $M_{g/\mathbb{R}}^{\mathrm{rp}}$ of a semianalytic variety that makes $\psi\colon \overline{S}_{g/\mathbb{R}}(F) \to M_{g/\mathbb{R}}^{\mathrm{rp}}$ a quotient map in the category of semianalytic varieties, coincides with the structure on $M_{g/\mathbb{R}}^{\mathrm{rp}}$ of a semianalytic variety defined via real Teichmüller spaces (cf. Corollary 5.3). The semianalytic variety

$M_{g/\mathbb{R}}^{\mathrm{rp}}$ is called the *coarse moduli space* of compact connected real algebraic curves of genus g having real points. It should be noted that the semianalytic structure on $M_{g/\mathbb{R}}^{\mathrm{rp}}$ is a true semianalytic structure. In fact none of the connected components of $M_{g/\mathbb{R}}^{\mathrm{rp}}$ is a real analytic variety [5].

We conclude this section with two remarks.

Remark 4.4. As is observed in the Introduction, the above construction gives $M_{g/\mathbb{R}}^{\mathrm{rp}}$ as a semianalytic subset of the set of real points of a connected complex analytic variety endowed with an action of Σ. Indeed, let $S_{g/\mathbb{C}}(F)$ be the set of embedded complex Schottky groups of $\mathrm{PGL}_2(\mathbb{C})$. This set has a natural structure of a connected complex analytic manifold of dimension $3g$. Moreover, it comes also naturally with an action of Σ. Then, as for $S_{g/\mathbb{R}}(F)$, the groups $\mathrm{PGL}_2(\mathbb{R})$ and $\mathrm{Aut}(F)$ act on $S_{g/\mathbb{C}}(F)$ to the left and the right, respectively. These actions are equivariant with respect to the action of Σ. Since $\mathrm{PGL}_2(\mathbb{R})$ acts fixed point-free on $S_{g/\mathbb{C}}(F)$, the quotient

$$\overline{S}_{g/\mathbb{C}}(F) = \mathrm{PGL}_2(\mathbb{R})\backslash S_{g/\mathbb{C}}(F)$$

has a natural structure of a connected complex analytic manifold of dimension $3g-3$. The action of Σ on $S_{g/\mathbb{C}}(F)$ induces an action of Σ on $\overline{S}_{g/\mathbb{C}}(F)$. As before, we also have an induced action of the group $\mathrm{Out}(F)$ on $\overline{S}_{g/\mathbb{C}}(F)$. It is clear that this action is equivariant with respect to the action of Σ. Moreover, $\mathrm{Out}(F)$ acts properly discontinuously. Hence, the quotient

$$Z = \overline{S}_{g/\mathbb{C}}(F)/\mathrm{Out}(F)$$

has a natural structure of a connected complex analytic variety of dimension $3g-3$. It has an induced action of Σ. Then, Z^{Σ} is a real analytic variety of dimension $3g-3$ that contains $M_{g/\mathbb{R}}^{\mathrm{rp}}$ as a semianalytic subset.

Remark 4.5. A regretful event is that the moduli space $M_{g/\mathbb{R}}^{\mathrm{rp}}$ does not take into account real algebraic curves not having real points. This is due to the fact that such curves do not admit a strict uniformization by a real Schottky group.

5 Real Teichmüller spaces

Let X be a real algebraic curve and let g be its genus. Let $T_{g/\mathbb{R}}(X)$ be the *real Teichmüller space of* X. Its elements are *marked real algebraic curves modeled on* X, i.e., pairs (Y, f), where Y is a real algebraic curve and $f\colon X \to Y$ is an orientation-preserving quasiconformal equivariant homeomorphism. Two such marked real algebraic curves (Y, f) and (Z, h) represent the same element of $T_{g/\mathbb{R}}(X)$ if and only if there is an isomorphism $k\colon Y \to Z$ of real algebraic curves such that $k \circ f$ is equivariantly homotopic to h. It is known that $T_{g/\mathbb{R}}(X)$ admits a natural structure of a real analytic manifold. For that structure, $T_{g/\mathbb{R}}(X)$ is connected and of dimension $3g-3$ if $g \geq 2$ [3].

Let $Q^+_{g/\mathbb{R}}(X)$ be the group of orientation-preserving quasiconformal equivariant selfhomeomorphisms of X. One has a right action of $Q^+_{g/\mathbb{R}}(X)$ on $T_{g/\mathbb{R}}(X)$ defined by $(Y,f)\cdot h = (Y, f\circ h)$. It is clear that the stabilizer $Q^{+,0}_{g/\mathbb{R}}(X)$ of the base point (X, id) of $T_{g/\mathbb{R}}(X)$ is the subgroup of orientation-preserving quasiconformal equivariant selfhomeomorphisms of X that are equivariantly homotopic to the identity. Being a stabilizer, $Q^{+,0}_{g/\mathbb{R}}(X)$ is a normal subgroup of $Q^+_{g/\mathbb{R}}(X)$. The quotient group

$$\mathrm{Mod}_{g/\mathbb{R}}(X) = Q^+_{g/\mathbb{R}}(X)/Q^{+,0}_{g/\mathbb{R}}(X)$$

is the *real modular group* of X. Since $Q^{+,0}_{g/\mathbb{R}}(X)$ acts trivially on $T_{g/\mathbb{R}}(X)$, the real modular group of X acts on $T_{g/\mathbb{R}}(X)$. This action is a properly discontinuous group action if $g \geq 2$ [3].

Recall that $M_{g/\mathbb{R}}(X)$ is the set of isomorphism classes of real algebraic curves Y having the same topological type as X. It is clear that a real algebraic curve Y has the same topological type as X if and only if there is an orientation-preserving quasiconformal equivariant homeomorphism $f\colon X \to Y$. Then, define a map

$$p\colon T_{g/\mathbb{R}}(X) \longrightarrow M_{g/\mathbb{R}}(X)$$

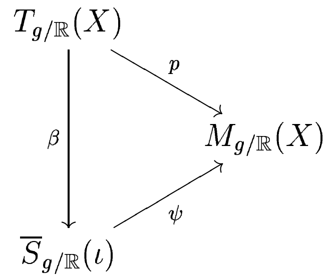
by letting $p(Y,f)$ be the isomorphism class $[Y]$ of the real algebraic curve Y. Clearly, p is a quotient map for the action of $\mathrm{Mod}_{g/\mathbb{R}}(X)$. Since this action is properly discontinuous if $g \geq 2$, there is a unique structure of a semianalytic variety on $M_{g/\mathbb{R}}(X)$ such that p is a quotient map in the category of semianalytic varieties. This semianalytic structure on $M_{g/\mathbb{R}}(X)$ is natural; i.e., $M_{g/\mathbb{R}}(X)$ equipped with this structure is a coarse semianalytic moduli space of real algebraic curves having the same topological type as X [6].

Now, suppose that g satisfies $g \geq 2$ and let F again be the free group on the symbols $\gamma_1, \ldots, \gamma_g$. Let X be any real algebraic curve of genus g having real points. Choose any embedded Schottky group $\iota \in \overline{S}_{g/\mathbb{R}}(F)$ such that $\psi(\iota) = [X]$; i.e., the real algebraic curve Ω/G is isomorphic to X, where Ω is the domain of discontinuity of G and $G = \iota(F)$. Let $\overline{S}_{g/\mathbb{R}}(\iota)$ be the connected component of $\overline{S}_{g/\mathbb{R}}(F)$ containing ι. Then clearly, ψ maps this connected component into $M_{g/\mathbb{R}}(X)$. We denote its restriction to $\overline{S}_{g/\mathbb{R}}(\iota)$ again by ψ. Let $\mathrm{Out}(F)(\iota)$ be the stabilizer subgroup of the connected component $\overline{S}_{g/\mathbb{R}}(\iota)$. Then, the map ψ from $\overline{S}_{g/\mathbb{R}}(\iota)$ into $M_{g/\mathbb{R}}(X)$ is a quotient map for the action of $\mathrm{Out}(F)(\iota)$.

Theorem 5.1. *Let g be an integer satisfying $g \geq 2$. Let X be a real algebraic curve of genus g having real points. Then, with notation as above, there is a real bianalytic map*

$$\beta\colon T_{g/\mathbb{R}}(X) \longrightarrow \overline{S}_{g/\mathbb{R}}(\iota)$$

making the diagram



commutative. In particular, there is an isomorphism

$$\lambda\colon \mathrm{Mod}_{g/\mathbb{R}}(X) \longrightarrow \mathrm{Out}(F)(\iota)$$

such that $\beta((Y,f)\cdot h) = (\beta(Y,f))\cdot\lambda(h)$ for all $(Y,f) \in T_{g/\mathbb{R}}(X)$ and for all $h \in \mathrm{Mod}_{g/\mathbb{R}}(X)$.

Proof. Define the map β as follows. Let (Y,f) be in $T_{g/\mathbb{R}}(X)$. The quasiconformal homeomorphism f gives rise to a Beltrami coefficient μ with support on Ω for the real Kleinian group G. Let w^μ be the unique orientation-preserving quasiconformal selfhomeomorphism of $\mathbb{P}^1(\mathbb{C})$ having 0, 1 and ∞ as fixed points and such that its complex dilation is equal to μ. Since f is equivariant, μ is a real Beltrami coefficient; i.e., $\mu(\sigma(z)) = \sigma(\mu(z))$ almost everywhere. It follows that w^μ is equivariant. Then, the quasiconformal deformation

$$\iota^\mu\colon F \longrightarrow \mathrm{PGL}_2(\mathbb{R})$$

of G defined by letting $\iota^\mu(\gamma) = w^\mu \circ \iota(\gamma) \circ (w^\mu)^{-1}$ is in the connected component $\overline{S}_{g/\mathbb{R}}(\iota)$ containing ι. One defines the map β by $\beta(Y,f) = \iota^\mu$.

It follows from quasiconformal deformation theory that β is real analytic. It is also easy to see that β is surjective. It follows from the construction of β that $\psi \circ \beta = p$.

In order to show that β is a real bianalytic map, we construct a real analytic map

$$\delta\colon \overline{S}_{g/\mathbb{R}}(\iota) \longrightarrow T_{g/\mathbb{R}}(X)$$

such that the composition $\delta \circ \beta$ is equal to the identity on $T_{g/\mathbb{R}}(X)$. That will suffice to prove that β is bianalytic.

Observe that there is a canonical real analytic family $\mathcal{C}$ of marked real algebraic curves modeled on X over the real analytic manifold $\overline{S}_{g/\mathbb{R}}(\iota)$. Let

$\mathcal{U}$ be the universal real analytic family over $T_{g/\mathbb{R}}(X)$ of marked real algebraic curves modeled on X [6]. Since this family is universal, there is a real analytic map δ from $\overline{S}_{g/\mathbb{R}}(\iota)$ into $T_{g/\mathbb{R}}(X)$ such that the two real analytic families of marked real algebraic curves $\delta^{\star}\mathcal{U}$ and $\mathcal{C}$ are isomorphic as families of curves modeled on X. It is then clear that the map $\delta \circ \beta$ is equal to the identity. $\square$

Corollary 5.2. *Let g be an integer satisfying $g \geq 2$. Let X be a compact connected real algebraic curve of genus g having real points. Let $M_{g/\mathbb{R}}(X)$ be the set of isomorphism classes of real algebraic curves Y having the same topological type as X. Consider the following two structures of a semianalytic variety on $M_{g/\mathbb{R}}(X)$:*

1. *the semianalytic structure such that the map $p\colon T_{g/\mathbb{R}}(X) \to M_{g/\mathbb{R}}(X)$ is a quotient map in the category of semianalytic varieties;*
2. *the semianalytic structure that $M_{g/\mathbb{R}}(X)$ acquires as a connected component of the semianalytic variety $M^{rp}_{g/\mathbb{R}}$ defined in the preceding section.*

Then, the semianalytic structures 1 and 2 on the set $M_{g/\mathbb{R}}(X)$ coincide.

Corollary 5.3. *Let g be an integer satisfying $g \geq 2$. Let X_i, $i = 1, \dots, [\frac{1}{2}(3g+2)]$, be real algebraic curves of genus g having real points, such that X_i and X_j are of different topological type whenever $i \neq j$. Let $M^{rp}_{g/\mathbb{R}}$ be the set of isomorphism classes of real algebraic curves of genus g having real points. Consider the following two structures of a semianalytic variety on $M^{rp}_{g/\mathbb{R}}$:*

1. *the semianalytic structure which is the disjoint sum of the semianalytic varieties $M_{g/\mathbb{R}}(X_i)$ each of which is given the semianalytic structure making the map $p_i\colon T_{g/\mathbb{R}}(X_i) \to M_{g/\mathbb{R}}(X_i)$ a quotient map in the category of semianalytic varieties.*
2. *the semianalytic structure such that the map $\psi\colon \overline{S}_{g/\mathbb{R}}(F) \to M^{rp}_{g/\mathbb{R}}$ is a quotient map in the category of semianalytic varieties.*

Then, the semianalytic structures 1 and 2 on the set $M_{g/\mathbb{R}}(X)$ coincide.

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References

[1] Bobenko, A. I. : Schottky uniformization and finite-gap integration. *Soviet Math. Dokl.* 36 (1988), 38–42

[2] Button, J. : All Fuchsian Schottky groups are classical Schottky groups. *Geometry and Topology Monographs* 1 (1998), 117-125. http://www.maths.warwick.ac.uk/gt/gtmcontents1.html

[3] Earle, C. J.: Moduli of surfaces with symmetries. *Advances in the theory of Riemann surfaces.* Ahlfors, L.V. et. al. (eds) Princeton University Press and University of Tokyo Press, 1971

[4] Hidalgo, R. A. : The noded Schottky space. *Proc. London Math. Soc.* (3) 73 (1996), 385–403

[5] Huisman, J.: Real quotient singularieties and nonsingular real algebraic curves in the boundary of the moduli space. (to appear in *Compositio Mathematica*)

[6] Huisman, J.: Real Teichmüller spaces and moduli of real algebraic curves. (submitted)

[7] Huisman, J.: Strict uniformization of real algebraic curves and global real analytic coordinates on real Teichmüller spaces. *Rev. Mat. Complut.* 12, 1 (1999)

[8] Koebe, P. : Ueber die Uniformisierung reeller algebraischer Kurven. *Nachr. Akad. Wiss. Goettingen* (1907), 177-190

[9] Maskit, B.: *Kleinian groups.* Springer Verlag, Berlin, 1988

[10] Maskit, B. : Special uniformizations of symmetric Riemann surfaces. (unpublished)

[11] Natanzon, S. M. : Moduli spaces of real curves. *Trans. Moscow Math. Soc.* 1 (1980), 233–272

[12] Natanzon, S. M. : Klein surfaces. *Uspekhi Mat. Nauk.* 45:6 (1990), 47–90

[13] Purzitsky, N. : A cutting and pasting of noncompact polygons with applications to Fuchsian groups. *Acta Math.* 143, 3-4 (1979), 233–250

[14] Seppälä, M. : Moduli spaces of stable real algebraic curves. *Ann. Sci. Éc. Norm. Sup.* 4e série t. 24 (1991), 519–544

[15] Weichold, G.: Über symmetrische Riemann'sche Flächen und die Periodicitätsmoduln der zugehörigen Abel'schen Normalintegrale erster Gattung. *Zeit. f. Math. u. Phys.* 28 (1883), 321–351

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