On the space of real line arrangements

J. Huisman

Abstract

The set of all real line arrangements of given degree in the real projective plane is known to have a natural semialgebraic structure. The nonreduced arrangements are singular points of this structure. We show that the set of all real line arrangements of given degree has also a natural structure of a smooth compact connected affine real algebraic variety. As a consequence, we get a projectively linear structure on the set of all real line arrangements of given degree.

MSC 2000: 52C35

Keywords: real line arrangement, configuration of real lines

1 Introduction

Let K be any field. A *line arrangement* over K is a closed subscheme [2] of the projective plane $\mathbb{P}^2 = \mathbb{P}^2_K$, defined by a nonzero homogeneous polynomial $F \in K[X, Y, Z]$ that is equal to the product of its linear factors in $K[X, Y, Z]$. Equivalently, a line arrangement over K is a proper closed subscheme of $\mathbb{P}^2$ that is the scheme-theoretic union of finitely many projective lines in $\mathbb{P}^2$. Note that, with the current definition, line arrangements are not necessarily reduced or nonempty (cf. [3]).

Since the set of projective lines in $\mathbb{P}^2$ is parametrized by the set $(\mathbb{P}^2)^\curlyvee(K)$ of all K-rational points of the dual projective plane $(\mathbb{P}^2)^\curlyvee$, the set of all line arrangements of degree d over K is naturally parametrized by the symmetric power

$$\mathcal{A}_d = ((\mathbb{P}^2)^\curlyvee(K))^{(d)},$$

where d is a natural integer.

Now, the set $\mathcal{A}_d$ has two bad properties:

1. $\mathcal{A}_d$ is not, in a natural way, the set of K-rational points of an algebraic variety over K, and

2. $\mathcal{A}_d$ contains singularities, as a subset of the algebraic variety $((\mathbb{P}^2)^\curlyvee)^{(d)}$.

Indeed, as for property 1, $\mathcal{A}_d$ is a subset of the set $((\mathbb{P}^2)^\curlyvee)^{(d)}(K)$ of all K-rational points of the symmetric power $((\mathbb{P}^2)^\curlyvee)^{(d)}$. One has a strict inclusion

$$\mathcal{A}_d \subsetneq ((\mathbb{P}^2)^\curlyvee)^{(d)}(K)$$

if and only if the field K admits a nontrivial extension of degree $\leq d$. For example, if K is algebraically closed then $\mathcal{A}_d$ *is* equal to the set of K-rational points of $((\mathbb{P}^2)^\curlyvee)^{(d)}$. However, if K is not algebraically closed, $\mathcal{A}_d$ is strictly contained in $((\mathbb{P}^2)^\curlyvee)^{(d)}(K)$ for all $d \geq d_0$, for some natural integer d_0. For example, if K is the field $\mathbb{R}$ of real numbers then $\mathcal{A}_d$ is a strict semialgebraic subset of $((\mathbb{P}^2)^\curlyvee)^{(d)}(\mathbb{R})$ for all $d \geq 2$.

As for property 2, since $(\mathbb{P}^2)^\curlyvee$ is 2-dimensional, the symmetric power $((\mathbb{P}^2)^\curlyvee)^{(d)}$ is singular along the so-called big diagonal Δ for all $d \geq 2$.

While seemingly nothing can be done to resolve property 1, one can resolve property 2 by resolution of singularities. This has, however, the following drawback. Let $\tilde{\mathcal{A}}_d$ be a resolution of singularities of $\mathcal{A}_d$. Let $\mathcal{A}$ be the disjoint union of $\mathcal{A}_d$, for $d \in \mathbb{N}$ and let, similarly, $\tilde{\mathcal{A}}$ be the disjoint union of $\tilde{\mathcal{A}}_d$, for $d \in \mathbb{N}$. Then, the scheme-theoretic union of line arrangements over K, which is a monoid law

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

on $\mathcal{A}$, does not extend to a map

$$\tilde{\mathcal{A}} \times \tilde{\mathcal{A}} \longrightarrow \tilde{\mathcal{A}}.$$

Therefore, even if K is algebraically closed, property 2 erects serious obstacles.

The object of this paper is to show that, when K is the field $\mathbb{R}$ of real numbers, both bad properties 1 and 2 can be resolved. More precisely, we show that $\mathcal{A}_d$ can be identified, in a natural way, with the *whole* set of real points of a proper *smooth* algebraic variety over $\mathbb{R}$ (see Corollary 2.2). In particular, $\mathcal{A}_d$ has a natural structure of a smooth compact connected affine real algebraic variety in the sense of [1]. Moreover, with respect to this structure, the scheme-theoretic union of real line arrangements

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

is an algebraic map (see Corollary 2.3).

The idea is that the set $\mathcal{A}_d$ is parametrized by the set of all effective divisors of degree $2d$ on a certain real algebraic curve Q (see Section 2). In Section 3, we determine more explicitly this parametrization as a parametrization of $\mathcal{A}_d$ by Laurent polynomials.

2 A real algebraic structure on the space of real line arrangements

Let Q be the anisotropic real conic in $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{R}}$ defined by the equation

$$X^2 + Y^2 + Z^2 = 0.$$

To say that Q is anisotropic means that Q has no real points, i.e., $Q(\mathbb{R}) = \emptyset$.

Define, for any natural integer d,

$$\mathrm{Div}^{2d}_{\geq 0}(Q)$$

to be the set of effective divisors on Q of degree $2d$. Let

$$\mathrm{Div}_{\geq 0}(Q) = \coprod_{d \in \mathbb{N}} \mathrm{Div}^{2d}_{\geq 0}(Q)$$

to be the disjoint union of $\mathrm{Div}^{2d}_{\geq 0}(Q)$, for $d \in \mathbb{N}$. Since all divisors on Q are of even degree, the set $\mathrm{Div}_{\geq 0}(Q)$ is nothing but the set of all effective divisors on Q, as is suggested by its notation. The addition of effective divisors defines a monoid law

$$\mathrm{Div}_{\geq 0}(Q) \times \mathrm{Div}_{\geq 0}(Q) \longrightarrow \mathrm{Div}_{\geq 0}(Q)$$

on $\mathrm{Div}_{\geq 0}(Q)$.

Define a map

$$\varphi \colon \mathcal{A} \longrightarrow \mathrm{Div}_{\geq 0}(Q)$$

as follows. Let A be a real line arrangement in $\mathbb{P}^2$. Then, the intersection product $A \cdot Q$ is well defined since no irreducible component of A is contained in Q. Therefore, the intersection product $A \cdot Q$ is a well defined effective divisor on Q. Define

$$\varphi(A) = A \cdot Q.$$

Theorem 2.1. *The map φ is an isomorphism of graded monoids, i.e., φ is a bijective morphism of monoids such that*

$$\varphi(\mathcal{A}_d) = \mathrm{Div}^{2d}_{\geq 0}(Q)$$

for all $d \in \mathbb{N}$.

Proof. Let A and B be two real line arrangements. Denote by $A + B$ the scheme-theoretic union of A and B. One has

$$\varphi(A + B) = (A + B) \cdot Q = A \cdot Q + B \cdot Q = \varphi(A) + \varphi(B).$$

Therefore, φ is a morphism of monoids. Moreover, φ is a morphism of graded monoids since

$$\varphi(\mathcal{A}_d) \subseteq \mathrm{Div}^{2d}_{\geq 0}(Q),$$

by Bezout's Theorem.

In order to show that φ is an isomorphism, it suffices to show that the restriction φ_d of φ to $\mathcal{A}_d$ is bijective onto $\mathrm{Div}^{2d}_{\geq 0}(Q)$, for any $d \in \mathbb{N}$. Choose $D \in \mathrm{Div}^{2d}_{\geq 0}(Q)$. There are distinct closed points $P_1, \dots, P_n$ of Q and nonzero natural integers $m_1, \dots, m_n$ such that

$$D = \sum_{i=1}^{n} m_i P_i.$$

For each $i \in \{1, \dots, n\}$, there is exactly one real projective line L_i in $\mathbb{P}^2$ such that $L_i \cdot Q = P_i$. Indeed, a closed point P_i of Q corresponds to a pair of distinct complex conjugate points $\{Q_i, \overline{Q}_i\}$ of the complexification $Q_{\mathbb{C}} = Q \times_{\mathbb{R}} \mathbb{C}$ of Q. The real projective line L_i is the unique real projective line whose complexification passes through Q_i and $\overline{Q}_i$. Let A be the real line arrangement $\sum m_i L_i$. Then $A \in \mathcal{A}_d$ and $\varphi_d(A) = D$. This shows surjectivity of φ_d. Moreover, one easily sees that A is the unique real line arrangement in $\mathcal{A}_d$ satisfying $\varphi_d(A) = D$. Hence, φ_d is also injective. $\square$

Since Q is a rational curve over $\mathbb{R}$, the set $\mathrm{Div}^{2d}_{\geq 0}$ can be naturally identified with the set of real points of a real projective space. Indeed, let $\mathcal{L}(d)$ be the restriction to Q of the invertible sheaf $\mathcal{O}(d)$ on $\mathbb{P}^2$, for any $d \in \mathbb{N}$. The map

$$\psi_d \colon \mathbb{P}(H^0(Q, \mathcal{L}(d))) \longrightarrow \mathrm{Div}^{2d}_{\geq 0}(Q)$$

that associates to a nonzero global section s of $\mathcal{L}(d)$ its divisor $\mathrm{div}(s)$, is a bijection by the Riemann-Roch Theorem. Here, the notation $\mathbb{P}(V)$ denotes the real projective space of all 1-dimensional subspaces of the real vector space V.

Corollary 2.2. *Let $d \in \mathbb{N}$. The map*

$$\psi_d^{-1} \circ \varphi_d \colon \mathcal{A}_d \longrightarrow \mathbb{P}(H^0(Q, \mathcal{L}(d)))$$

is a bijection. In particular, the set $\mathcal{A}_d$ of all real line arrangements of degree d has a natural structure of a real algebraic variety in the sense of [1]. With respect to this structure, $\mathcal{A}_d$ is isomorphic to $\mathbb{P}^{2d}(\mathbb{R})$. In particular, $\mathcal{A}_d$ is a smooth compact connected affine real algebraic variety. $\square$

Let us make precise what is meant by a *natural* structure of a real algebraic variety on the set $\mathcal{A}_d$. Let $\{A_t\}_{t\in T}$ be an algebraic family of real line arrangements of degree d over a base T. More precisely, T is an affine real algebraic variety in the sense of [1], and the subset

$$A = \bigcup_{t\in T} A_t \times \{t\}$$

of $\mathbb{P}^2 \times T$ is an algebraic subset, each of whose fibers A_t over $t \in T$ is a real line arrangement of degree d. More concretely, A is defined by a homogeneous polynomial

$$F \in \mathcal{R}(T)[X, Y, Z]$$

of degree d with coefficients in the ring $\mathcal{R}(T)$ of all regular functions on T [1], such that for all $t \in T$, the evaluation of F at t defines a real line arrangement in $\mathbb{P}^2$ of degree d. To say that the above structure on $\mathcal{A}_d$ of a real algebraic is natural means that the map

$$f\colon T \to \mathcal{A}_d$$

defined by $f(t) = A_t$ is a real algebraic morphism.

Note that the situation is rather subtle; the universal family

$$U_d = \bigcup_{A\in\mathcal{A}_d} A \times \{A\} \subseteq \mathbb{P}^2 \times \mathcal{A}_d$$

of real line arrangements of degree d is *not* an algebraic family of real line arrangements over $\mathcal{A}_d$. In fact, the subset U_d is only semialgebraic. More precisely, U is defined by a homogeneous polynomial F with coefficients in the ring $\mathcal{S}(\mathcal{A}_d)$ of all semialgebraic functions on $\mathcal{A}_d$, and not with coefficients in $\mathcal{R}(\mathcal{A}_d)$. This will be proven in the next section (see Proposition 3.1).

Another observation we would like to make is that, by Corollary 2.2, $\mathcal{A}_d$ is isomorphic to $\mathbb{P}^{2d}(\mathbb{R})$, with respect to its natural real algebraic structure. In particular, one gets a projectively linear structure on the set $\mathcal{A}_d$. For example, given two distinct real line arrangements A and B of degree d, there is a unique real projective line of real line arrangements of degree d that contains A and B!

We conclude this section by a further consequence of Theorem 2.1. Put

$$\mathbb{P}(H^0(Q, \mathcal{L}(\star))) = \coprod_{d\in\mathbb{N}} \mathbb{P}(H^0(Q, \mathcal{L}(d))).$$

The tensor product of global sections endows $\mathbb{P}(H^0(Q, \mathcal{L}(\star))$ with the structure of a graded monoid. Let

$$\psi\colon \mathbb{P}(H^0(Q, \mathcal{L}(\star))) \longrightarrow \mathrm{Div}_{\geq 0}(Q)$$

be the map whose restriction to $\mathbb{P}(H^0(Q, \mathcal{L}(d)))$ is equal to ψ_d.

Corollary 2.3. *The map*

$$\psi^{-1} \circ \varphi \colon \mathcal{A} \longrightarrow \mathbb{P}(H^0(Q, \mathcal{L}(\star)))$$

is an isomorphism of graded monoids. In particular, $\mathcal{A}$ is a real algebraic monoid, i.e., the scheme-theoretic union on the set of all real line arrangements $\mathcal{A}$ is real algebraic with respect to the natural real algebraic structure on $\mathcal{A}$. □

3 An explicit description of the real algebraic structure on $\mathcal{A}_d$

As observed in Section 2, the real algebraic curve Q is rational. Hence, its complexification $Q_{\mathbb{C}}$ is isomorphic to the complex projective line $\mathbb{P}^1_{\mathbb{C}}$. Choose, once and for all, an isomorphism between $Q_{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}}$ having the following property. The action of complex conjugation on $Q_{\mathbb{C}}$ corresponds to the action of complex conjugation on $\mathbb{P}^1_{\mathbb{C}}$ defined by

$$z \mapsto -\frac{1}{\overline{z}}$$

for $z \in \mathbb{C}$, where $z \mapsto \overline{z}$ denotes the usual action of complex conjugation on $\mathbb{C}$. Such an isomorphism exists since the action of complex conjugation on $\mathbb{P}^1_{\mathbb{C}}$ defined above does not have any fixed points on $\mathbb{P}^1(\mathbb{C})$. With the point of view of $\mathbb{P}^1_{\mathbb{C}}$ as the Riemann sphere $S^2 = \mathbb{P}^1(\mathbb{C})$, the action of complex conjugation on $Q_{\mathbb{C}}$ corresponds to the antipodal action on S^2.

Let $d \in \mathbb{N}$. With respect to the isomorphism $Q_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$, the complexification $\mathcal{L}(d)_{\mathbb{C}}$ of the invertible sheaf $\mathcal{L}(d)$ on Q is isomorphic to the invertible sheaf $\mathcal{O}(d \cdot 0 + d \cdot \infty)$ on $\mathbb{P}^1_{\mathbb{C}}$. The complex vector space of global sections of $\mathcal{O}(d \cdot 0 + d \cdot \infty)$ is the complex vector space $L_{\mathbb{C}}(d)$ of all complex Laurent polynomials

$$\sum_{i=-d}^{d} a_i Z^i,$$

where $a_i \in \mathbb{C}$ for $i = -d, \dots, d$. The action of complex conjugation on the set of all global sections of $\mathcal{L}(d)_{\mathbb{C}}$ corresponds to the action of complex conjugation on $L_{\mathbb{C}}(d)$ defined by

$$\sum_{i=-d}^{d} a_i Z^i \mapsto \sum_{i=-d}^{d} (-1)^i \overline{a_{-i}} Z^i,$$

where $a_i \in \mathbb{C}$ for $i = -d, \dots, d$. Therefore, one can identify the real vector space of global sections of $\mathcal{L}(d)$ with the real vector space $L(d)$ of all complex

Laurent polynomials

$$\sum_{i=-d}^{d} a_i Z^i,$$

where the $a_i \in \mathbb{C}$ satisfy $a_{-i} = (-1)^i \overline{a_i}$ for all $i = 0, \dots, d$. In particular, we can identify $\mathbb{P}(H^0(Q, \mathcal{L}(d)))$ with the real projective space $\mathbb{P}(L_d)$.

The set of effective divisors $\mathrm{Div}^{2d}_{\geq 0}(Q)$ of degree $2d$ on Q can be identified with the set D^{2d} of effective divisors of degree $2d$ on $\mathbb{P}^1_{\mathbb{C}}$ that are stable for the action of complex conjugation on $\mathbb{P}^1_{\mathbb{C}}$ as defined above.

The map $\psi_d \colon \mathbb{P}(H^0(Q, \mathcal{L}(d))) \to \mathrm{Div}^{2d}_{\geq 0}(Q)$ then corresponds to the map

$$\chi_d \colon \mathbb{P}(L(d)) \longrightarrow D^{2d}$$

defined by letting $\chi(P)$ be the divisor $\mathrm{div}(P) + d \cdot 0 + d \cdot \infty$, for any Laurent polynomial $P \in L(d)$. Here, $\mathrm{div}(P)$ is the divisor of P as a rational function on $\mathbb{P}^1_{\mathbb{C}}$.

Next, we want to have a more concrete description of the map

$$\varphi_d^{-1} \colon \mathrm{Div}^{2d}_{\geq 0}(Q) \longrightarrow \mathcal{A}_d.$$

The set $\mathrm{Div}^{2d}_{\geq 0}(Q)$ has already been identified with D^{2d}. We define a map

$$\rho_d \colon D^{2d} \longrightarrow \mathcal{A}_d$$

as follows. An element D of D^{2d} is a divisor on the Riemann sphere S^2 of the form

$$\sum_{i=1}^{n} m_i (P_i + [-1]P_i),$$

where $[-1]$ is the antipodal map on S^2, the points $P_i, [-1]P_i$, for $i = 1, \dots, n$, are distinct, and the m_i are nonzero natural integers. For each $i \in \{1, \dots, n\}$ let $C_i \subseteq S^2$ be the great circle of points that are equidistant to P_i and $[-1]P_i$. Let $\pi \colon S^2 \to \mathbb{P}^2(\mathbb{R})$ be the quotient map for the antipodal action on S^2. Let L_i be the real projective line in $\mathbb{P}^2$ such that $L_i(\mathbb{R}) = \pi(C_i)$. Define

$$\rho_d(D) = \sum_{i=1}^{n} m_i L_i.$$

Then it is an easy matter to check that ρ corresponds to the map φ_d^{-1}, after a suitable change of coordinates on $\mathbb{P}^2$.

Resuming, the map

$$\rho_d \circ \chi_d \colon \mathbb{P}(L(d)) \longrightarrow \mathcal{A}_d$$

corresponds under the above identifications to the map $\varphi_d^{-1} \circ \psi_d$. In particular, $\rho_d \circ \chi_d$ is an isomorphism of real algebraic varieties with respect to the natural structure of a real algebraic variety on $\mathcal{A}_d$.

As an application we show the following statement.

Proposition 3.1. *Let $d \in \mathbb{N}$. The universal family U_d of real line arrangements of degree d is semialgebraic. If $d \geq 2$ then U_d is not algebraic.*

Proof. The universal family of effective divisors of degree $2d$ on Q is clearly algebraic. Therefore, the universal family of divisors in D^{2d} on the Riemann sphere S^2 is algebraic as well. Then the universal family of arrangements of great circles in S^2 of degree d is semialgebraic. Hence, the universal family U_d of real line arrangements of degree d is semialgebraic.

We show that U_d is truely semialgebraic, i.e., nonalgebraic if $d \geq 2$. By Corollary 2.3, it suffices to show this for $d = 2$.

Define a family of complex polynomials $P_t \in \mathbb{C}[Z]$, depending on a real parameter $t \in \mathbb{R}$, by

$$P_t = Z^2 + t.$$

Symmetrize P_t multiplicatively to a family of Laurent polynomials in $L(2)$:

$$L_t = tZ^{-2} + (t^2 + 1) + tZ^2.$$

for $t \in \mathbb{R}$.

Let C_t be the associated family of arrangements of great circles of degree 2 in S^2, and let A_t be the associated family of real line arrangements of degree 2.

For $t < 0$, the roots of P_t are on the real axis. Therefore, the divisor $\operatorname{div}(L_t) + 2 \cdot 0 + 2 \cdot \infty$ has its support on the real axis as well, for $t < 0$. It follows that the points $\pm\sqrt{-1}$ belong to C_t for $t < 0$. In fact, the points $\pm\sqrt{-1}$ are the intersection points of the two great circles of C_t, for $t < 0$. Hence, the point $\pi(\pm\sqrt{-1}) \in \mathbb{P}^2(\mathbb{R})$ belongs to A_t when $t < 0$.

Now suppose that U_2 is algebraic. Then A_t is algebraic too, and hence the point $\pi(\pm\sqrt{-1})$ belongs to A_t for $t \geq 0$ as well. But for $t = 1$, the roots of P_t are $\pm\sqrt{-1}$. Therefore,

$$\operatorname{div}(L_1) + 2 \cdot 0 + 2 \cdot \infty = 2 \cdot \sqrt{-1} + 2 \cdot (-\sqrt{-1}).$$

Then, C_1 is equal to $2 \cdot \mathbb{P}^1(\mathbb{R})$, where $\mathbb{P}^1(\mathbb{R})$ is considered as a subset of $\mathbb{P}^1(\mathbb{C}) = S^2$. It follows that C_1 does not contain any of the points $\pm\sqrt{-1}$. Then, A_1 does not contain the point $\pi(\pm\sqrt{-1})$ of $\mathbb{P}^2(\mathbb{R})$. Contradiction. $\square$

References

[1] J. Bochnak, M. Coste, M-F. Roy, Géométrie algébrique réelle. Ergeb. der Math. (3), Vol. 12, Springer Verlag, 1987.

[2] R. Hartshorne, Algebraic geometry. Grad. Texts in Math. Vol. 52, Springer Verlag, 1977.

[3] P. Orlik, H. Terao, Arrangements of hyperplanes. Grundlehren Math. Wiss. Vol. 300, Springer-Verlag, 1992.

Institut de Recherche Mathématique de Rennes
Université de Rennes 1
Campus de Beaulieu
35042 Rennes Cedex
France
E-mail: huisman@univ-rennes1.fr
Home page: http://www.maths.univ-rennes1.fr/~huisman/

Typeset by $\mathcal{AMS}$-LaTeX