Rino's flight around the world in circles

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Rino took off in his private jet for a trip around the world when he discovered that his rudder wouldn't steer right. In fact, the situation was worse, he couldn't even steer straight ahead. The only thing he could do was steer left, covering the whole range from unperceptibly left to sharp left. A routine flyer as he is, Rino kept calm. He reassured himself with the fact that he could always make a 270-degree left turn, which would amount to the same thing as a 90-degree right turn. So, nothing could stop Rino; he could fly over any place he wanted, and in any direction. Except that, in doing so, he would be flying around in circles (see Figure 1).

After Rino's safe return to his home base, what can be said about his circuit? Nothing special, one is tempted to say, except for the fact that he has been flying around making left turns all the time. Wrong! One can say something else: there is necessarily a place on earth that Rino has flown over, but over whose antipodal place he didn't fly! The article is devoted to the proof of this, in my opinion, amusing statement.

Mathematical translation and proof

First of all, let us translate the problem into a mathematical one. The earth will be mathematically represented by the unit sphere S^2 in 3-dimensional Euclidean space $\mathbb{R}^3$. Rino made a closed circuit over the surface of the earth, which gives us a continuous map

$$\gamma\colon [0,1] \longrightarrow S^2$$

such that $\gamma(1) = \gamma(0)$. The places on earth Rino has flown over are represented by the closed subset $C = \gamma([0,1])$ of S^2.

It will be convenient to extend γ over the whole real line $\mathbb{R}$ by periodicity, i.e., $\gamma(t) = \gamma(\langle t \rangle)$ for all $t \in \mathbb{R}$, where $\langle t \rangle$ denotes the fractional part of t. We assume that γ is of class C^2 on $\mathbb{R}$, i.e., the component functions of γ are

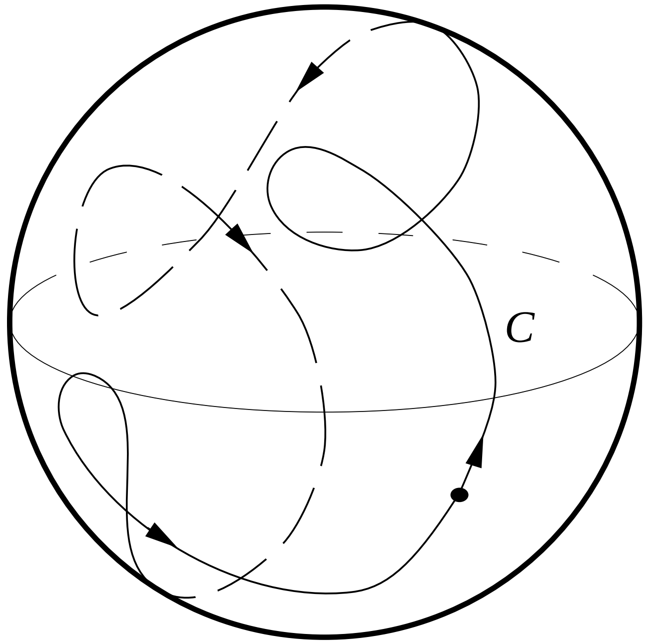

Figure 1: The circuit C of places on earth Rino has flown over. The dot on the circuit is Rino's home base.

twice continuously differentiable on $\mathbb{R}$. In particular, we assume that Rino returned home in the same direction from which he departed. Since a flying airplane has a strictly positive velocity, we may assume that the velocity vector $\gamma'(t)$ is nonzero for all $t \in \mathbb{R}$. We assume, moreover, that C has only transverse self-intersections, i.e., we assume that $\gamma'(s)$ and $\gamma'(t)$ are linearly independent if $\gamma(s) = \gamma(t)$ for some $s, t \in \mathbb{R}$ such that $\langle s \rangle \neq \langle t \rangle$. The fact that Rino could only steer left is expressed by the formula

$$\det(\gamma'(t), \gamma''(t), \gamma(t)) > 0$$

for all $t \in \mathbb{R}$ (see Figure 2).

Now, the claim that there is a place on earth over which Rino flew, but over whose antipodal place he didn't fly, finds its mathematical formulation in the following statement.

Theorem 1. *There is a point $q \in C$ such that $-q \notin C$.*

Before attacking its proof, we need some preparation. The following statement roughly says that C has only finitely many self-intersections.

Lemma 2. *There is a finite subset $\Sigma \subseteq S^2$ such that $C \setminus \Sigma$ is a closed C^2-submanifold of $S^2 \setminus \Sigma$.*

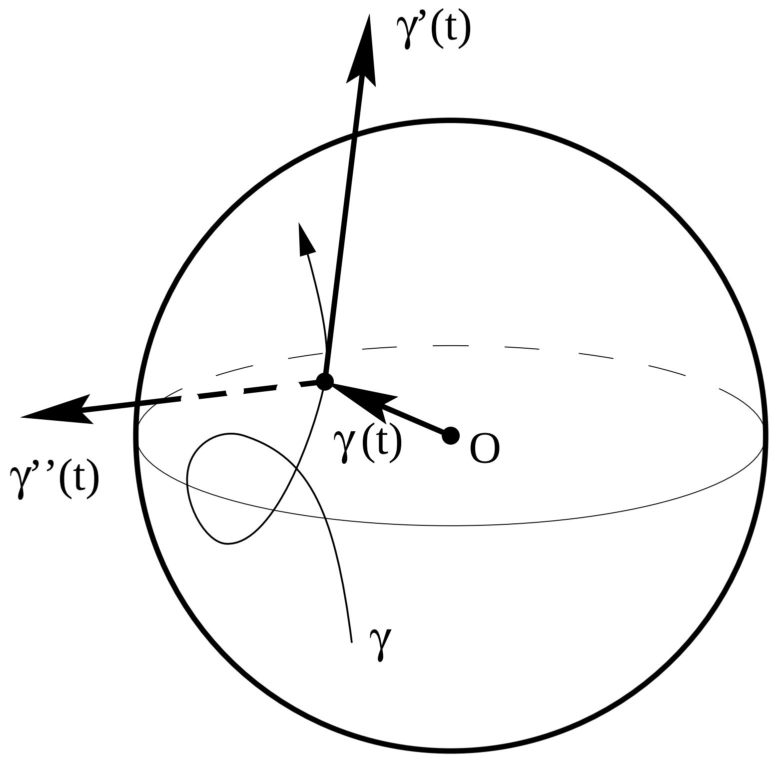

Figure 2: The path γ on S^2 with depicted the three vectors $\gamma(t)$, $\gamma'(t)$, and $\gamma''(t)$ for some $t \in \mathbb{R}$.

Proof. Denote by S^1 the quotient manifold $\mathbb{R}/\mathbb{Z}$. The C^2-map γ induces a C^2-map $\overline{\gamma}$ from S^1 into S^2. Let R be the equivalence relation on S^1 determined by $\overline{\gamma}$, i.e., $(p,q) \in R$ if and only if $\overline{\gamma}(p) = \overline{\gamma}(q)$ for $p, q \in S^1$. Denote by Δ the diagonal of $S^1 \times S^1$. Let $\Sigma' \subseteq S^1$ be the projection on the first factor of the subset $R \setminus \Delta$ of $S^1 \times S^1$. We show that Σ' is finite.

It follows from its definition that R is closed in $S^1 \times S^1$. Since $\overline{\gamma}$ is locally an embedding, Δ is open in R. Hence, $R \setminus \Delta$ is closed in $S^1 \times S^1$ as well. Since C has only transverse self-intersections, every element of $R \setminus \Delta$ is an isolated point [2, p. 38] of $R \setminus \Delta$. It follows that $R \setminus \Delta$ is a discrete subset of $S^1 \times S^1$. Since $S^1 \times S^1$ is compact, $R \setminus \Delta$ is finite. Therefore, Σ' is finite.

Let $\Sigma = \overline{\gamma}(\Sigma')$. Then $\overline{\gamma}^{-1}(\Sigma) = \Sigma'$, and the restriction of $\overline{\gamma}$ to $S^1 \setminus \Sigma'$ is a closed, injective C^2-immersion into $S^2 \setminus \Sigma$. Therefore, its image $C \setminus \Sigma$ is a closed C^2-submanifold of $S^2 \setminus \Sigma$ [4, Exercise 1.3.1]. $\square$

The next statement states that any other parametrization of C is equivalent to γ.

Lemma 3. *Suppose $\delta\colon \mathbb{R} \to S^2$ is another C^2-map such that*

1. *$\delta(\mathbb{R}) = C$,*
2. *$\delta'(t) \neq 0$ for all $t \in \mathbb{R}$, and*
3. *δ is periodic.*

Then there is a C^2-diffeomorphism τ of $\mathbb{R}$ such that $\delta = \gamma \circ \tau$.

Proof. By Lemma 2, there is a finite subset Σ of S^2 such that $C \setminus \Sigma$ is a closed C^2-submanifold of $S^2 \setminus \Sigma$. Let T be the manifold obtained by blowing up S^2 at the points of Σ [1, p. 70]. Let $\pi\colon T \to S^2$ be the natural map. The strict transform $\widetilde{C}$ of C is the closure in T of the subset $\pi^{-1}(C \setminus \Sigma)$. Since C has only transverse self intersections, $\widetilde{C}$ is a connected closed C^1-submanifold of T. In particular, $\widetilde{C}$ is C^1-diffeomorphic to S^1. Moreover, the maps γ and δ lift uniquely to C^1-maps $\widetilde{\gamma}$ and $\widetilde{\delta}$ from $\mathbb{R}$ into T. Then, $\widetilde{\gamma}$ and $\widetilde{\delta}$ are both universal C^1-covering maps of $\widetilde{C}$ [3, p. 23]. Hence, there is a C^1-diffeomorphism τ of $\mathbb{R}$ such that $\widetilde{\delta} = \widetilde{\gamma} \circ \tau$. It follows that $\delta = \gamma \circ \tau$. Since γ and δ are, locally, C^2-embeddings of $\mathbb{R}$ into S^2, the diffeomorphism τ is C^2 as well. $\square$

Proof of Theorem 1. Suppose that $-q \in C$ for all $q \in C$. Let $\delta\colon \mathbb{R} \to S^2$ be defined by $\delta(t) = -\gamma(t)$. Then, δ satisfies the conditions of Lemma 3. Hence, there is a C^2-diffeomorphism τ of $\mathbb{R}$ such that $\delta = \gamma \circ \tau$. For $t \in \mathbb{R}$, one has

$$-\gamma'(t) = \delta'(t) = \gamma'(\tau(t)) \cdot \tau'(t),$$

and

$$-\gamma''(t) = \delta''(t) = \gamma''(\tau(t)) \cdot (\tau'(t))^2 + \gamma'(\tau(t)) \cdot \tau''(t).$$

It follows that

$$\det(\gamma'(t), \gamma''(t), \gamma(t)) = -(\tau'(t))^3 \cdot \det(\gamma'(\tau(t)), \gamma''(\tau(t)), \gamma(\tau(t))).$$

Since both determinants are strictly positive, $\tau'(t) < 0$, i.e., τ is a strictly decreasing C^2-diffeomorphism of $\mathbb{R}$. Thus τ has a fixed point, i.e., there is $t \in \mathbb{R}$ such that $\tau(t) = t$. But then $-\gamma(t) = \delta(t) = \gamma(\tau(t)) = \gamma(t)$, which is a contradiction. □

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