

A Real Algebraic Vector Bundle is Strongly Algebraic whenever its Total Space is Affine

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ABSTRACT. The aim of this note is to prove the following statement. Let X be an affine real algebraic variety and let $\xi = (E, \pi)$ be a real algebraic vector bundle over X . Then, ξ is strongly algebraic if and only if E is an affine real algebraic variety.

The aim of this note is to prove the following statement in a more elementary way than that of [4].

THEOREM. *Let X be an affine real algebraic variety¹ and let $\xi = (E, \pi)$ be a real algebraic vector bundle over X . Then, ξ is strongly algebraic if and only if E is an affine real algebraic variety.*

The only-if part being trivial, the difficulty lies in showing that ξ is strongly algebraic if E is affine. Some effort has already been put into this problem, see [5], [4]. Both papers follow the idea of enlarging the scheme $\text{Spec } \mathcal{R}(E)$ as a scheme over $\text{Spec } \mathcal{R}(X)$ in order to make it a geometric vector bundle over $\text{Spec } \mathcal{R}(X)$. While the former paper uses arguments which I could not follow, the latter applies a generalisation, due to M. Artin, of a theorem of A. Weil on rational group laws [1].

We will however prove the Theorem in a more elementary way. The idea roughly is the following. The normal bundle of the zero section in E is strongly

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¹In this note definitions and notations from [2] are freely used.

algebraic², E being affine. This normal bundle should be isomorphic, at least morally, to E as a real algebraic vector bundle.

The actual proof turns out to be more agreeable if we work with the conormal bundle instead of the normal one, and with sheaves of sections instead of vector bundles. The idea of the proof remains the same, however.

DEFINITION. Let X be a closed real algebraic subvariety of the real algebraic variety Y . Then the conormal sheaf $\mathcal{C}_{X/Y}$ of X in Y is the sheaf of \mathcal{R}_X -modules $\mathcal{I}/\mathcal{I}^2$, where \mathcal{I} is the sheaf on Y of vanishing ideals of X , i.e.

$$\Gamma(U, \mathcal{I}) = \{f \in \Gamma(U, \mathcal{R}_Y) \mid \forall x \in U \cap X : f(x) = 0\},$$

where U is an open subset of Y .

Note that, a priori, $\mathcal{I}/\mathcal{I}^2$ only is a sheaf on Y . However, its support is obviously equal to X . Therefore, it might be considered to be a sheaf on X , and as such naturally has the structure of a sheaf of \mathcal{R}_X -modules.

Recall that a real algebraic vector bundle $\xi = (E, \pi)$ over an affine real algebraic variety X is strongly algebraic if and only if there exists a projective $\mathcal{R}(X)$ -module M of finite type such that the sheaf $\widetilde{M} = M \otimes_{\mathcal{R}(X)} \mathcal{R}_X$ is isomorphic to the sheaf of sections $\Gamma(\cdot, \xi)$ of ξ ([2], Théorème 12.1.7).

PROOF OF THE THEOREM. Suppose E is affine. Embed X in E via the zero section. Then, the conormal sheaf $\mathcal{C}_{X/E}$ of X in E is isomorphic to the dual of the sheaf of sections $\Gamma(\cdot, \xi)$ of ξ . Indeed, let ξ^\vee be the dual vector bundle of ξ . One constructs a morphism of sheaves of \mathcal{R}_X -modules

$$\varphi: \Gamma(\cdot, \xi^\vee) \longrightarrow \mathcal{C}_{X/E}$$

as follows. Let U be an open subset of X and s a section of ξ^\vee over U . Define a regular function f_s on $\pi^{-1}(U)$ by $f_s(x, v) = s(x)v$. Since f_s vanishes on U (remember that we have embedded X in E via the zero section), f_s determines a section $\varphi(s)$ of $\mathcal{C}_{X/E}$ over U . This defines the morphism φ . Local triviality of ξ makes it easy to see that φ is in fact an isomorphism. Therefore, the sheaf of sections of the real algebraic vector bundle ξ^\vee is isomorphic to $\mathcal{C}_{X/E}$. Since E is affine, the sheaf of vanishing ideals \mathcal{I} of X in E is isomorphic to $I \otimes_{\mathcal{R}(E)} \mathcal{R}_E$, where $I = \Gamma(E, \mathcal{I})$. Hence

$$\mathcal{C}_{X/E} \cong (I/I^2) \otimes_{\mathcal{R}(X)} \mathcal{R}_X.$$

Therefore, $\Gamma(\cdot, \xi^\vee)$ is isomorphic to $(I/I^2) \otimes_{\mathcal{R}(X)} \mathcal{R}_X$. In particular, for each maximal ideal \mathfrak{m} of $\mathcal{R}(X)$ the $\mathcal{R}(X)_{\mathfrak{m}}$ -module $(I/I^2)_{\mathfrak{m}}$ is free of rank n , where n is the rank of ξ . On the other hand, I/I^2 is of finite type as $\mathcal{R}(X)$ -module, $\mathcal{R}(E)$ being Noetherian. Hence ([3], Theorem II.5.2.2) the $\mathcal{R}(X)$ -module I/I^2 is projective and of finite type. This implies that ξ^\vee is strongly algebraic. It follows that ξ is strongly algebraic. \square

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