Real algebraic differential forms on complex algebraic varieties

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Abstract

We show that every de Rham cohomology class on a nonsingular quasiprojective complex algebraic variety can be realized by a real algebraic differential form.

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1 Introduction

In the transcendental theory of nonsingular complex algebraic varieties, one often considers the underlying differentiable structure on such varieties. When one wants to squeeze properties of the original algebraic structure out of this differentiable structure, it is to be expected that one runs into considerable technical complications. As an example one may think of the analytic proof of the Hodge decomposition of the de Rham cohomology of a nonsingular projective complex algebraic variety ([7], Theorem 5.5.1). Relatively recently, an elementary algebraic proof of the Hodge decomposition has been found [2]. To my opinion, it is somewhat unsatisfactory that this algebraic proof runs through algebraic geometry in positive characteristic. Therefore, it may be worth to investigate to what extend the underlying real algebraic structure—coarser than the complex algebraic structure but not as coarse as the differentiable structure—on a nonsingular complex algebraic variety would allow to avoid technical complications of the transcendental theory as well as arguments proper to algebraic geometry in positive characteristic.

This paper could be considered as a certificate of good character of the underlying real algebraic structure on a complex algebraic variety. Namely, we show that any de Rham cohomology class on a nonsingular quasiprojective complex algebraic variety can be realized by a real algebraic differential form.

The paper is organized as follows. In Section 2 we explain what it means for a de Rham cohomology class to be realizable by a real algebraic differential form. We also define the real algebraic de Rham cohomology of a nonsingular real algebraic variety. In Section 3 we recall the construction of restriction of scalars with respect to the field extension $\mathbb{C}/\mathbb{R}$ of a complex algebraic variety. Restriction of scalars of such a variety X is an algebraic variety over $\mathbb{R}$ whose set of real points *is* the underlying real algebraic structure on X. This is the key fact to the study of the underlying real algebraic structure on a complex algebraic variety [5]. Section 4 contains a statement that will be used in the proof of the main result presented in Section 5. We conclude the paper with a question in Section 6.

Conventions and notation An algebraic variety over $\mathbb{C}$, or a complex algebraic variety is an integral separated scheme of finite type over $\mathbb{C}$ [4]. We distinguish between real algebraic varieties and algebraic varieties over $\mathbb{R}$. A real algebraic variety is an irreducible separated real algebraic variety in the sense of [1], whereas an algebraic variety over $\mathbb{R}$ is an absolutely integral separated scheme of finite type over $\mathbb{R}$. Let X be an algebraic variety over $\mathbb{R}$. Then, we consider the set of real points $X(\mathbb{R})$ of X as a real algebraic variety in the natural way.

2 Real algebraic de Rham cohomology

Let M be a nonsingular real algebraic variety. Since M is a nonsingular real algebraic variety, we can consider M as a C^∞ differentiable manifold. Let $\mathcal{E}^i(M)$ be the vector space of complex-valued C^∞ differential i-forms on the differentiable manifold M. Let $d^i\colon \mathcal{E}^i(M) \to \mathcal{E}^{i+1}(M)$ be the exterior derivative. Then, $(\mathcal{E}^\cdot(M), d^\cdot)$ is the C^∞ de Rham complex of M. The de Rham cohomology of the differentiable manifold M is, by definition, the homology of the complex $(\mathcal{E}^\cdot(M), d^\cdot)$, i.e.,

$$H^i_{\mathrm{dR}}(M) = h^i(\mathcal{E}^\cdot(M), d^\cdot)$$

for any integer i.

Let $\mathcal{R}^i(M) \subseteq \mathcal{E}^i(M)$ be the vector subspace of complex-valued real algebraic differential i-forms on M. Then, $(\mathcal{R}^\cdot(M), d^\cdot)$ is a complex, the *real algebraic de Rham complex* of M. The homology of this complex does not seem to be well-behaved. Indeed, the homology groups $h^i(\mathcal{R}^\cdot(M), d^\cdot)$ are, in general, infinite-dimensional real vector spaces. This can already be seen in the case of M being the affine real line $\mathbb{R}$: the system

$$\{\frac{dx}{p}\,|\,p = x^2 + bx + c, \text{ where } b, c \in \mathbb{R} \text{ and } b^2 - 4c < 0\}$$

is a linearly independent system in the real vector space $h^1(\mathcal{R}^\cdot(\mathbb{R}), d^\cdot)$. Therefore, instead of the homology of the real algebraic de Rham complex we rather study its image in the homology of the C^∞ de Rham complex. More precisely, let $a^i\colon \mathcal{R}^i(M) \to \mathcal{E}^i(M)$ be the inclusion of the vector space of algebraic differential forms on M into the vector space of C^∞ differential forms on M. Then, a^i induces a map on homology

$$\alpha^i\colon h^i(\mathcal{R}^\cdot(M), d^\cdot) \longrightarrow H^i_{\mathrm{dR}}(M)$$

for any integer i. We define the *i-th real algebraic de Rham cohomology group* $H^i_{\mathrm{dR}}(M)_{\mathrm{alg}}$ of the real algebraic variety M by

$$H^i_{\mathrm{dR}}(M)_{\mathrm{alg}} = \mathrm{im}(\alpha^i).$$

We say that a de Rham cohomology class in $H^i_{\mathrm{dR}}(M)$ of M is *realizable by a real algebraic differential form* if it is contained in $H^i_{\mathrm{dR}}(M)_{\mathrm{alg}}$.

Example 2.1. Any de Rham cohomology class in $H^1_{\mathrm{dR}}(\mathbb{P}^1(\mathbb{R}))$ of the real projective line $\mathbb{P}^1(\mathbb{R})$ is realizable by a real algebraic differential form. Indeed, $H^1_{\mathrm{dR}}(\mathbb{P}^1(\mathbb{R}))$ is generated by the cohomology class of the real algebraic differential 1-form $dx/(x^2+1)$ on $\mathbb{P}^1(\mathbb{R})$. To put it differently, one has $H^1_{\mathrm{dR}}(\mathbb{P}^1(\mathbb{R}))_{\mathrm{alg}} = H^1_{\mathrm{dR}}(\mathbb{P}^1(\mathbb{R}))$.

3 Restriction of scalars of complex algebraic varieties

For the reader's convenience, we recall some notions related to restriction of scalars of a complex algebraic variety.

Denote by Σ the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Its nontrivial element is denoted by σ.

Let X be a complex algebraic variety. Let $s\colon X \to \mathrm{Spec}(\mathbb{C})$ be its structure morphism. The *complex conjugate variety* X^σ of X is the scheme X endowed with the structure morphism

$$\mathrm{Spec}(\sigma) \circ s\colon X \longrightarrow \mathrm{Spec}(\mathbb{C}).$$

It is clear that X^σ is again a complex algebraic variety. Note also that the complex conjugate variety $(X^\sigma)^\sigma$ of X^σ is *equal* to the complex algebraic variety X.

Example 3.1. Let X be an affine complex algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by polynomials $F_1, \ldots, F_m$. Let, for a polynomial F with complex coefficients, F^σ denote the polynomial obtained from F by letting σ act on its coefficients. Then, one easily checks that the complex conjugate variety X^σ of X is isomorphic to the affine complex algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by the polynomials $F_1^\sigma, \ldots, F_m^\sigma$.

Let X and Y be complex algebraic varieties. Let $f\colon X \to Y$ be a morphism of schemes. Then, the morphism f considered as a morphism of schemes from X^σ into Y^σ will be denoted by f^σ. The morphism f^σ is called the *complex conjugate morphism* of f. Of course, if f is a morphism of complex algebraic varieties, then f^σ is one as well.

Example 3.2. Let X and Y be affine complex algebraic subvarieties of $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{A}^m_{\mathbb{C}}$, respectively. Let $f\colon X \to Y$ be a morphism of complex algebraic varieties. Let $F_1, \ldots, F_m \in \mathbb{C}[X_1, \ldots, X_n]$ be coordinate functions of f. Then, one easily checks that $F_1^\sigma, \ldots, F_m^\sigma$ are coordinate functions of f^σ.

The identity map on X, considered as a morphism of schemes from X into X^σ, will be denoted by $\iota = \iota_X$. Then, the complex conjugate morphism $\iota^\sigma\colon X^\sigma \to X$ of ι is equal to the identity map on X considered as a morphism of schemes from X^σ into X.

Example 3.3. Let X be an affine complex algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by polynomials $F_1, \ldots, F_m$. Identify, in the natural way, X^σ with the affine complex algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by the polynomials $F_1^\sigma, \ldots, F_m^\sigma$. The map on the set of complex points $\iota(\mathbb{C})\colon X(\mathbb{C}) \to X^\sigma(\mathbb{C})$ induced by $\iota = \iota_X$ is nothing but the restriction to $X(\mathbb{C})$ of complex conjugation on $\mathbb{C}^n$.

Now we are ready to recall the definition of restriction of scalars with respect to the field extension $\mathbb{C}/\mathbb{R}$ of a complex algebraic variety.

Let X be a complex algebraic variety. Let Y be the complex algebraic variety $X \times X^\sigma$. There is a natural action on the scheme Y of the Galois group Σ of the field extension $\mathbb{C}/\mathbb{R}$. Indeed, let pr_1 and pr_2 be the projections of Y onto the first and second factor, respectively. Let $\tau\colon Y \to Y$ be the morphism of schemes such that $\mathrm{pr}_1 \circ \tau = \iota^\sigma \circ \mathrm{pr}_2$ and $\mathrm{pr}_2 \circ \tau = \iota \circ \mathrm{pr}_1$. Then, the nontrivial element $\sigma \in \Sigma$ acts on Y as the morphism τ. That this defines an action of Σ on Y follows from the fact that $\tau^2 = \mathrm{id}$.

Example 3.4. Let X be an affine complex algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by polynomials $F_1, \ldots, F_m$. Identify, as before, X^σ with the affine complex algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by the polynomials $F_1^\sigma, \ldots, F_m^\sigma$. The map on the sets of complex points $\tau(\mathbb{C})\colon Y(\mathbb{C}) \to Y(\mathbb{C})$ induced by τ is equal to the restriction to $Y(\mathbb{C})$ of the map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ defined by $(z, w) \mapsto (\sigma(w), \sigma(z))$.

Assume that X is a quasiprojective. Then X^σ is quasiprojective as well. Hence, Y is quasiprojective. Then, the quotient Y/Σ of Y in the category of locally ringed spaces is a scheme. We will denote that scheme by $\tilde{X}$. In fact, $\tilde{X}$ is a scheme over $\mathbb{R}$ and $\tilde{X}_\mathbb{C} = Y$. Therefore, $\tilde{X}$ is an algebraic variety over $\mathbb{R}$. It is called *restriction of scalars with respect to the field extension* $\mathbb{C}/\mathbb{R}$ *of* X, or also the *Weil restriction of* X [6].

Let $\Delta\colon X \to Y$ be the diagonal morphism (ι, ι^σ). Let $\pi\colon Y \to \tilde{X}$ be the quotient morphism. Then, the restriction of scalars $\tilde{X}$ comes along with the morphism of schemes

$$\varphi = \pi \circ \Delta\colon X \longrightarrow \tilde{X}.$$

This morphism satisfies the following property. For every complex point $p\colon \mathrm{Spec}(\mathbb{C}) \to X$ of X, there is a unique real point $q\colon \mathrm{Spec}(\mathbb{R}) \to \tilde{X}$ of $\tilde{X}$ such that $\varphi \circ p = q_\mathbb{C}$, where $q_\mathbb{C}\colon \mathrm{Spec}(\mathbb{C}) \to \tilde{X}$ is the complex point of $\tilde{X}$ induced by q. Indeed, $q_\mathbb{C} = (p, p^\sigma)$ is a complex point of $\tilde{X}$ which comes from a real point since $\tau \circ (p, p^\sigma) = (p, p^\sigma)$.

Define a map

$$\psi\colon \tilde{X}(\mathbb{R}) \longrightarrow X(\mathbb{C})$$

by $\psi(q) = \mathrm{pr}_1 \circ q_\mathbb{C}$ for any $q \in \tilde{X}(\mathbb{R})$. Then the above property satisfied by φ implies that ψ is a bijection. In fact, the map ψ is an isomorphism from the real algebraic variety $\tilde{X}(\mathbb{R})$ onto the underlying real algebraic structure on $X(\mathbb{C})$ (see [5] for the definition of the underlying real algebraic structure). This holds trivially in case $X = \mathbb{A}^n_\mathbb{C}$. The general case follows immediately from this trivial case.

Example 3.5. Let X be a nonsingular projective complex algebraic curve and let g be its genus. Then, the underlying real algebraic structure on $X(\mathbb{C})$ is an orientable compact connected affine real algebraic surface of genus g. If X is rational, $X(\mathbb{C})$ is isomorphic to the 2-sphere S^2 as a real algebraic variety. If X is of genus 1, the underlying real algebraic structure on $X(\mathbb{C})$ is a real algebraic torus. The latter class of real algebraic tori has been classified in the paper [5].

Finally, we can state and prove the following lemma.

Lemma 3.6. *Let X be a nonsingular quasiprojective complex algebraic variety. Let U be an open subset of the restriction of scalars $\tilde{X}$ of X such that $U(\mathbb{R}) = \tilde{X}(\mathbb{R})$. Let $f\colon \tilde{X}(\mathbb{R}) \to U(\mathbb{C})$ be the map defined by $f(p) = p_\mathbb{C}$ for $p \in \tilde{X}(\mathbb{R})$. Then, the induced map on de Rham cohomology groups*

$$f^i\colon H^i_{\mathrm{dR}}(U(\mathbb{C})) \longrightarrow H^i_{\mathrm{dR}}(\tilde{X}(\mathbb{R}))$$

is surjective for any integer i.

Proof. With notation as above, $U_{\mathbb{C}}$ is an open subset of Y. Hence, $U(\mathbb{C}) = U_{\mathbb{C}}(\mathbb{C})$ is a subset of $Y(\mathbb{C})$. Let $p\colon U(\mathbb{C}) \longrightarrow X(\mathbb{C})$ be the restriction of the projection of $Y(\mathbb{C})$ onto the first factor $X(\mathbb{C})$. Then, the diagram

$$\begin{array}{ccc} \tilde{X}(\mathbb{R}) & \xrightarrow{f} & U(\mathbb{C}) \\ & \searrow^{\psi} & \downarrow p \\ & & X(\mathbb{C}) \end{array}$$

commutes. Since ψ is a bijection, $g = \psi^{-1} \circ p$ is a left-inverse to f, i.e., $g \circ f = \mathrm{id}$. It follows that the induced maps on de Rham cohomology groups satisfy $f^i \circ g^i = \mathrm{id}$ for any integer i. In particular, f^i is surjective for any integer i. □

4 Affine neighborhoods of the set of real points of an algebraic variety over $\mathbb{R}$

A well-known fact in real algebraic geometry is that any quasiprojective real algebraic variety is affine (see [1], Théorème 3.4.5 and Proposition 3.2.10). An inspection of the proof of that statement reveals that the following stronger statement holds.

Proposition 4.1. *Let X be a quasiprojective algebraic variety over $\mathbb{R}$. Then, there is an affine open subset U of X such that $U(\mathbb{R}) = X(\mathbb{R})$.* □

5 Real algebraic differential forms on complex algebraic varieties

In order to prove our main result, we need to recall a result of A. Grothendieck.

Let V be a nonsingular affine complex algebraic variety. Let Ω^i_V be the sheaf of algebraic differential i-forms on V. Let $(\Gamma(V, \Omega^{\cdot}_V), d^{\cdot})$ be the global algebraic de Rham complex of V.

Recall from the introduction that $(\mathcal{E}^{\cdot}(V(\mathbb{C})), d^{\cdot})$ is the C^∞ de Rham complex of the differentiable manifold $V(\mathbb{C})$ and that its homology is the de Rham cohomology of the differentiable manifold $V(\mathbb{C})$, i.e.,

$$H^i_{\mathrm{dR}}(V(\mathbb{C})) = h^i(\mathcal{E}^{\cdot}(V(\mathbb{C})), d^{\cdot})$$

for any integer i.

Let $b^{\cdot}$ be the inclusion of the global algebraic de Rham complex of V into the C^∞ de Rham complex of the differentiable manifold $V(\mathbb{C})$. Then, $b^{\cdot}$ induces an isomorphism on homology; i.e., $b^{\cdot}$ induces isomorphisms

$$\beta^i \colon h^i(\Gamma(V, \Omega_V^{\cdot}), d^{\cdot}) \longrightarrow H^i_{\mathrm{dR}}(V(\mathbb{C}))$$

for any integer i [3].

Now we are ready to state and prove our main result.

Theorem 5.1. *Let X be a nonsingular quasiprojective complex algebraic variety. Then, every de Rham cohomology class of the differentiable manifold $X(\mathbb{C})$ can be realized by a real algebraic differential form, i.e.,*

$$H^i_{\mathrm{dR}}(X(\mathbb{C}))_{\mathrm{alg}} = H^i_{\mathrm{dR}}(X(\mathbb{C}))$$

for any integer i.

Proof. Let X be a nonsingular quasiprojective complex algebraic variety. Recall from Section 3 that the underlying real algebraic structure on $X(\mathbb{C})$ is isomorphic to the real algebraic variety $\tilde{X}(\mathbb{R})$. Therefore, it suffices to prove that every de Rham cohomology class of $\tilde{X}(\mathbb{R})$ can be realized by a real algebraic differential form.

Since X is quasiprojective, its restriction of scalars $\tilde{X}$ is quasiprojective. By Proposition 4.1, there is an affine open subset U of $\tilde{X}$ such that $U(\mathbb{R}) = \tilde{X}(\mathbb{R})$.

Consider $U(\mathbb{R})$ as a subset of the scheme U. Then, one has a restriction map

$$r^i \colon \Gamma(U, \Omega^i_U) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathcal{R}^i(\tilde{X}(\mathbb{R}))$$

from the vector space of complex-valued algebraic differential i-forms on U into vector space of real algebraic differential i-forms on $\tilde{X}(\mathbb{R})$.

With notation as before, one has a commutative diagram

$$\begin{array}{ccc} \Gamma(U, \Omega^i_U) \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\;b^i\;} & \mathcal{E}^i(U(\mathbb{C})) \\ \downarrow{\scriptstyle r^i} & & \downarrow{\scriptstyle f^*} \\ \mathcal{R}^i(\tilde{X}(\mathbb{R})) & \xrightarrow[\;a^i\;]{} & \mathcal{E}^i(\tilde{X}(\mathbb{R})) \end{array}$$

All these maps induce maps on homology, so that we get a commutative diagram

$$\begin{array}{ccc} h^i(\Gamma(U, \Omega^{\cdot}_U) \otimes_{\mathbb{R}} \mathbb{C}, d^{\cdot}) & \xrightarrow{\;\beta^i\;} & H^i_{\mathrm{dR}}(U(\mathbb{C})) \\ \downarrow{\scriptstyle \rho^i} & & \downarrow{\scriptstyle f^i} \\ h^i(\mathcal{R}^{\cdot}(\tilde{X}(\mathbb{R})), d^{\cdot}) & \xrightarrow[\;\alpha^i\;]{} & H^i_{\mathrm{dR}}(\tilde{X}(\mathbb{R})) \end{array}$$

By Lemma 3.6, f^i is surjective and by the above result of A. Grothendieck, β^i is an isomorphism. It follows that α^i is surjective. Hence, $H^i_{\mathrm{dR}}(\tilde{X}(\mathbb{R}))_{\mathrm{alg}} = H^i_{\mathrm{dR}}(\tilde{X}(\mathbb{R}))$. □

Remark 5.2. I do not know whether every de Rham cohomology class on a nonsingular affine real algebraic variety can be realized by a real algebraic differential form.

6 A QUESTION

Let X be a nonsingular projective complex algebraic variety. Let μ be any real algebraic Kähler metric on $X(\mathbb{C})$ [7]. (Such metrics abound: the restriction of the Fubini-Study metric to the image of any embedding of X into projective space gives rise to a real algebraic Kähler metric on X.) Let $\mathcal{H}^i(X(\mathbb{C}))$ be the vector space of the harmonic differential i-forms on $X(\mathbb{C})$ relative to μ. Then, one can pose the following question.

Question 6.1. *Is any harmonic differential i-form on $X(\mathbb{C})$ real algebraic, i.e., does the inclusion*

$$\mathcal{H}^i(X(\mathbb{C})) \subseteq \mathcal{R}^i(X(\mathbb{C}))$$

hold?

Theorem 5.1 says that there is no cohomological obstruction to an affirmative answer to the above question. Indeed, according to Theorem 5.1, for any harmonic differential i-form η on $X(\mathbb{C})$ there is a closed real algebraic differential form ω on $X(\mathbb{C})$ such that $[\omega] = [\eta]$ in $H^i_{\mathrm{dR}}(X(\mathbb{C}))$.

There is some evidence for an affirmative answer to Question 6.1.

Evidence 6.2. If a harmonic differential i-form η on $X(\mathbb{C})$ is real algebraic, then its complex conjugate harmonic differential form $\overline{\eta}$ is real algebraic too.

Evidence 6.3. All harmonic differential $(i,0)$-forms and $(0,i)$-forms on $X(\mathbb{C})$ are real algebraic. Indeed, harmonic differential $(i,0)$-forms are complex algebraic and, therefore, real algebraic. The statement for $(0,i)$-forms then follows from Evidence 6.2.

Evidence 6.4. Let n be the dimension of X. If every harmonic differential i-form on $X(\mathbb{C})$ is real algebraic, then every harmonic differential $(2n-i)$-form on $X(\mathbb{C})$ is real algebraic. Indeed, let

$$\star\colon \mathcal{E}^i(X(\mathbb{C})) \longrightarrow \mathcal{E}^{2n-i}(X(\mathbb{C}))$$

be the Hodge $\star$-operator (see [7], p. 158) relative to the metric μ. It is easy to check that $\star$ maps the subspace of real algebraic i-forms into the subspace of real algebraic $(2n-i)$-forms. Now, let η be any harmonic differential $(2n-i)$-form. Then, $\star^{-1}\eta$ is a harmonic differential i-form ([7], Proposition 5.2.3), and therefore, real algebraic. But then, $\eta = \star\star^{-1}\eta$ is real algebraic.

Evidence 6.5. Let n be the dimension of X. All harmonic differential i-forms on $X(\mathbb{C})$ are real algebraic for $i = 0, 1, 2n-1$ and $2n$. This follows from Evidence 6.3 and 6.4.

Evidence 6.6. Let X be any nonsingular projective complex algebraic curve and let μ be any real algebraic Kähler metric on $X(\mathbb{C})$. Then, every harmonic differential form on $X(\mathbb{C})$ is real algebraic. This follows from Evidence 6.5.

Evidence 6.7. Let X be a complex Abelian variety and let μ be the flat Kähler metric on $X(\mathbb{C})$. Then, μ is a real algebraic metric and every harmonic differential form on $X(\mathbb{C})$ relative to μ is real algebraic. Indeed, by Evidence 6.5, all harmonic differential 1-forms on X are real algebraic. Since $\mathcal{H}^i(X(\mathbb{C})) = \bigwedge^i \mathcal{H}^1(X(\mathbb{C}))$, all harmonic differential i-forms are real algebraic for any integer i.

Evidence 6.8. Let X be the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ and let μ be the Fubini-Study metric on $X(\mathbb{C})$. Then, every harmonic differential form on $X(\mathbb{C})$ is real algebraic. Indeed, μ is a real algebraic differential 2-form and

$$\mathcal{H}^i(X(\mathbb{C})) = \begin{cases} \mathbb{C}\cdot\wedge^{\frac{i}{2}}\mu & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

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