On real algebraic vector bundles

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1 Introduction

In real algebraic geometry the notion of strongly algebraic vector bundles plays an important role. In this paper we will prove the following statement which seems to be more of theoretical than of practical importance.

1.1 Theorem. *Let X be an affine real algebraic variety and let $\pi: E \to X$ be a real algebraic vector bundle over X. Then, the vector bundle (E, π) is strongly algebraic if and only if the real algebraic variety E is affine.*

This theorem already appeared in [M-R], however the proof presented there is incorrect. (In particular, the argument in the fifth row from below on page 133 of [M-R] is not correct.)

This paper is organized as follows. In Section 2 we recall the definition of a real algebraic variety. We also give a scheme-theoretic definition of the notion of complexification of a real algebraic variety. It turns out that almost all well-known examples of non-affine real algebraic varieties do not have a complexification. We end Section 2 with an example of a non-affine real algebraic variety that does have a complexification. In Section 3 we turn our attention to real algebraic vector bundles, generalize the notion of strongly algebraic vector bundle, and prove a general theorem of which Theorem 1.1 is a consequence.

Conventions and notation. If X is a scheme over $\mathbb{R}$ then the set of real points $X(\mathbb{R})$ is the subset of X consisting of all closed points having residue field $\mathbb{R}$. The symbol $\cong$ denotes isomorphism.

2 Complexifications of real algebraic varieties

Let us recall the definition of a real algebraic variety [B-C-R]. Let $(X, \mathcal{O}_X)$ be a locally ringed space, where $\mathcal{O}_X$ is a sheaf of $\mathbb{R}$-algebras. Such a space is an *affine real algebraic variety* if X is some subset of $\mathbb{R}^n$, $(n \in \mathbb{N})$, defined by polynomial equations and endowed with the Zariski topology, and $\mathcal{O}_X$ is the sheaf of regular functions on X. More generally,

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such a space is a *real algebraic variety* if X is separated and there exists a finite open covering $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine real algebraic variety (or, more precisely, is isomorphic to an affine real algebraic variety). If X and Y are real algebraic varieties then a *morphism* from X into Y is a morphism $f: X \to Y$ of locally ringed spaces such that

$$f^\sharp: \mathcal{O}_Y \longrightarrow f_\star \mathcal{O}_X$$

is a morphism of sheaves of $\mathbb{R}$-algebras.

We will distinguish between real algebraic varieties and algebraic varieties over $\mathbb{R}$. For us, an *algebraic variety over* $\mathbb{R}$ is a reduced, separated scheme of finite type over $\mathbb{R}$. An algebraic variety Y over $\mathbb{R}$ is said to have *sufficiently many real points* if the set of real points $Y(\mathbb{R})$ of Y is dense in Y.

Let Y be an algebraic variety over $\mathbb{R}$ having sufficiently many real points. Then,

$$(Y(\mathbb{R}), \mathcal{O}_Y|_{Y(\mathbb{R})})$$

is a real algebraic variety. We will denote this real algebraic variety by $Y(\mathbb{R})$, by abuse of language. Clearly, the assignment $Y \mapsto Y(\mathbb{R})$ extends to a functor from the category of algebraic varieties over $\mathbb{R}$ having sufficiently many real points, into the category of real algebraic varieties. If $f: Y \to Y'$ is a morphism of algebraic varieties over $\mathbb{R}$ having sufficiently many real points, then we denote the induced morphism of real algebraic varieties by

$$f(\mathbb{R}): Y(\mathbb{R}) \longrightarrow Y'(\mathbb{R}).$$

2.2 Definition. *Let X be a real algebraic variety. A pair (Y, i) is called a complexification of X if Y is an algebraic variety over $\mathbb{R}$ having sufficiently many real points and*

$$i: X \longrightarrow Y(\mathbb{R})$$

is an isomorphism of real algebraic varieties.

2.3 Example. Every quasi-projective real algebraic variety has a complexification. Indeed, if X is such a variety, then X is affine [B-C-R]. Hence, we may assume that X is a closed real algebraic subvariety of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is the set of real points of $\mathbb{A}^n_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[X_1, \ldots, X_n]$, we can take the Zariski closure Y of X in $\mathbb{A}^n_{\mathbb{R}}$. Then, Y is a complexification of X.

2.4 Remark. Given a real algebraic variety X, there does always exist a reduced scheme Y of finite type over $\mathbb{R}$ with sufficiently many real points, and such that $Y(\mathbb{R})$ is isomorphic to X. But, Y does not need to be separated. Of course, it might happen that Y contains a separated open subset U with $U(\mathbb{R}) = Y(\mathbb{R})$. Then U is a complexification of X. However, there exist real algebraic varieties X having the following property. Every reduced scheme Y, of finite type over $\mathbb{R}$, with sufficiently many real points, and such that $Y(\mathbb{R}) \cong X$, is not separated. In fact, it is not hard to see that the non-affine real algebraic varieties that are constructed in [M-R] have this property. Hence, these real algebraic varieties X do not have a complexification.

2.5 Remark. Complexifications have the following property. If X is a real algebraic variety and (Y, i) is a complexification of X, then, for every algebraic variety Z over $\mathbb{R}$ having sufficiently many real points, and for every morphism $j: X \to Z(\mathbb{R})$, there exists a unique rational map

$$f: Y \dashrightarrow Z$$

such that the set of real points $Y(\mathbb{R})$ of Y is contained in the domain of f and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & Y(\mathbb{R}) \\ & {\scriptstyle j}\searrow & \downarrow {\scriptstyle f(\mathbb{R})} \\ & & Z(\mathbb{R}) \end{array}$$

commutes.

If a real algebraic variety has a complexification then this complexification is not unique, unless the variety is zero-dimensional. The set $\mathcal{C}$ of complexifications of a given real algebraic variety X is a projective system (due to Remark 2.5), and the projective limit

$$\mathfrak{X} = \varprojlim_{Y \in \mathcal{C}} Y$$

exists and is a reduced, separated scheme over $\mathbb{R}$ having sufficiently many real points. Moreover,

$$(\mathfrak{X}(\mathbb{R}), \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{X}(\mathbb{R})})$$

is isomorphic to X (as a locally ringed space with a structure sheaf of $\mathbb{R}$-algebras). Of course, $\mathfrak{X}$ is not of finite type, in general. Observe that if Y is a complexification of X then $\mathfrak{X}$ is just the set of all generalizations of real points of Y, with the induced topology, and the structure sheaf is just the restriction of the structure sheaf of Y.

2.6 Definition. *If X is a real algebraic variety having a complexification, then the scheme $\mathfrak{X}$ defined above is called the associated real scheme.*

2.7 Example. If X is a quasi-projective real algebraic variety, then its associated real scheme $\mathfrak{X}$ is isomorphic to the real scheme $\operatorname{Spec}\Gamma(X, \mathcal{O}_X)$. For, we may assume X to be a closed subvariety of $\mathbb{R}^n$, since X is affine. Let Y be its closure in $\mathbb{A}^n_{\mathbb{R}}$. Then

$$\mathfrak{X} = \varprojlim_{f \in S} Y_f = \operatorname{Spec} S^{-1}\Gamma(Y, \mathcal{O}_Y) = \operatorname{Spec}\Gamma(X, \mathcal{O}_X),$$

where S is the multiplicative set of functions $f \in \Gamma(Y, \mathcal{O}_Y)$ whose zero locus $V(f)$ has no real points, and Y_f denotes the complement of $V(f)$ in Y.

It follows from Remark 2.5 that the assignment $X \longmapsto \mathfrak{X}$ defines a functor from the category of real algebraic varieties having a complexification into the category of schemes over $\mathbb{R}$. Clearly, this functor is fully faithful. This functor maps closed embeddings of real algebraic varieties into closed immersions of schemes. However, it does not map open embeddings of real algebraic varieties into open immersions of schemes. For example, let $U = \mathbb{R}^2 \backslash \{(0,0)\}$. Then U is an open subvariety of $X = \mathbb{R}^2$. By Example 2.7 , the morphism of associated real schemes is induced by the restriction map

$$\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(U, \mathcal{O}_U)$$

which is not of finite type. Hence, the induced morphism of associated real schemes $\mathfrak{U} \longrightarrow \mathfrak{X}$ is not an open immersion.

We have seen in Example 2.3 that every quasi-projective, or equivalently, every affine real algebraic variety has a complexification. We observed in Remark 2.4 that there exist non-affine real algebraic varieties that do not have a complexification. We would like to end this section by giving an example of a non-affine real algebraic variety that does have a complexification (see also [G]).

2.8 Example. Let

$$\widetilde{X} = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$$

be the Hirzebruch surface over $\mathbb{P}^1_{\mathbb{R}}$. Denote the structure morphism $\widetilde{X} \to \mathbb{P}^1_{\mathbb{R}}$ by $\widetilde{\pi}$. Choose non-zero rational sections σ_1 of $\mathcal{O}$ and σ_2 of $\mathcal{O}(1)$. The rational sections $0 \oplus \sigma_2$ and $\sigma_1 \oplus 0$ of $\mathcal{O} \oplus \mathcal{O}(1)$ induce sections s_0 and s_∞ of $\widetilde{\pi}$, respectively. Let X be the cokernel of

$$\mathbb{P}^1_{\mathbb{R}} \underset{s_\infty}{\overset{s_0}{\rightrightarrows}} \widetilde{X},$$

i.e., X is the scheme that arises when we identify the points $s_0(P)$ and $s_\infty(P)$ of $\widetilde{X}$, where P runs through $\mathbb{P}^1_{\mathbb{R}}$. Then, X is a complete algebraic variety over $\mathbb{R}$ having sufficiently many real points. Since s_0 and s_∞ are sections of $\widetilde{\pi}$, the morphism $\widetilde{\pi}$ induces a morphism

$$\pi\colon X \longrightarrow \mathbb{P}^1_{\mathbb{R}}.$$

Clearly, X is fibred over $\mathbb{P}^1_{\mathbb{R}}$ with fibres the nodal cubic curve C given by the equation $y^2 = x^3 + x^2$. It is known that $X \otimes \mathbb{C}$ is not projective and, hence, X is not projective. This follows from the following fact [H, Exercise 7.13, p. 171]. For every invertible sheaf $\mathcal{L}$ on X, the invertible sheaf $i_0^\star \mathcal{L}$ on C has degree zero, where

$$i_0\colon C \cong \pi^{-1}(0) \hookrightarrow X.$$

Let us prove, using this fact, that the real algebraic variety $X(\mathbb{R})$ is not affine. Suppose we have an embedding of $X(\mathbb{R})$ into $\mathbb{R}^n$. Then, according to Remark 2.5 , this embedding extends to a rational map

$$f\colon X \dashrightarrow \mathbb{P}^n_{\mathbb{R}}$$

such that the set of real points $X(\mathbb{R})$ is contained in the domain $\mathrm{dom}(f)$ of f. Let $Z = X \backslash \mathrm{dom}(f)$. Then $\operatorname{codim} Z \geq 2$. For, if η is a point of X of codimension 1 then X is either regular at η or not. In the former case, $\eta \in \mathrm{dom}(f)$, since $\mathbb{P}^n_{\mathbb{R}}$ is projective. In the latter case, η is the generic point of the singular locus X^{sing} of X, since X^{sing} is irreducible and of codimension 1. But $X^{\mathrm{sing}}(\mathbb{R}) \neq \phi$ and $X^{\mathrm{sing}}(\mathbb{R}) \subset \mathrm{dom}(f)$. Hence, $\eta \in \mathrm{dom}(f)$. Therefore, Z contains no point of codimension 1, hence $\mathrm{codim}\, Z \geq 2$.

Clearly, there exists an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{L}|_{X\setminus Z} \cong f^\star \mathcal{O}(1)$. Since $\mathrm{codim}\, Z \geq 2$, there exist global sections

$$s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$$

such that $f^\star x_i = s_i|_{X\setminus Z}$, for $i = 0, \ldots, n$. By the fact that $i_0^\star \mathcal{L}$ has degree zero, we conclude that f is constant on the fibre $\pi^{-1}(0)$. Hence, $f(\mathbb{R})$ is constant on $\pi(\mathbb{R})^{-1}(0) \cong C(\mathbb{R})$. Contradiction. This proves that $X(\mathbb{R})$ is a real algebraic variety which is not affine but does have a complexification, namely X.

3 Strongly algebraic vector bundles

In this section X will be a real algebraic variety having a complexification. We will denote its associated real scheme by $\mathfrak{X}$ and

$$\iota: X \longrightarrow \mathfrak{X}(\mathbb{R})$$

will be the canonical isomorphism.

Recall that a *real algebraic vector bundle* over X consists of the following data: a morphism $\pi: E \to X$ of real algebraic varieties, an open covering $\{U_i\}$ of X and isomorphisms

$$\varphi_i: U_i \times \mathbb{R}^{n_i} \xrightarrow{\cong} \pi^{-1}(U_i), \quad n_i \in \mathbb{N}$$

such that $\varphi_j^{-1}\varphi_i(x, v) = (x, g_{ji}(x) \cdot v)$, where $g_{ji}(x)$ is an invertible linear mapping from $\mathbb{R}^{n_i}$ into $\mathbb{R}^{n_j}$, for every $x \in U_i \cap U_j$. Of course, if $U_i \cap U_j$ is not empty then $n_i = n_j$. If Y is a scheme over $\mathbb{R}$ then a *geometric vector bundle* over Y consists of the following data. A morphism $\pi: F \to Y$ of schemes over $\mathbb{R}$, an open covering $\{U_i\}$ of Y and isomorphisms

$$\varphi_i: U_i \times \mathbb{A}_{\mathbb{R}}^{n_i} \xrightarrow{\cong} \pi^{-1}(U_i)$$

such that $\varphi_j^{-1}\varphi_i(x, v) = (x, g_{ji}(x) \cdot v)$, where g_{ji} is a morphism from $U_i \cap U_j$ into the scheme of $\mathbb{R}$-linear isomorphisms from $\mathbb{A}_{\mathbb{R}}^{n_i}$ into $\mathbb{A}_{\mathbb{R}}^{n_j}$. Again, if $U_i \cap U_j$ is not empty then $n_i = n_j$, and then g_{ji} is a morphism from $U_i \cap U_j$ into $\mathrm{GL}_{n,\mathbb{R}}$, where $n = n_i = n_j$ and

$$\mathrm{GL}_{n,\mathbb{R}} = \mathrm{Spec}\, \mathbb{R}[T_{ij} | i, j = 1, \ldots, n]_{\det(T_{ij})}.$$

A *complexification of a real algebraic vector bundle* $E \to X$ is a geometric vector bundle $F \to Y$ such that F is a complexification of E, Y is a complexification of X and, under these isomorphisms, the real algebraic vector bundle $F(\mathbb{R}) \to Y(\mathbb{R})$ is isomorphic to $E \to X$.

3.9 Definition. *If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules which is locally free of finite rank, then $\mathcal{F}$ is called strongly algebraic if there exists a sheaf $\mathcal{G}$ of $\mathcal{O}_{\mathfrak{X}}$-modules, locally free of finite rank, such that $i^\star \mathcal{G} \cong \mathcal{F}$. A real algebraic vector bundle over X is called strongly algebraic if its sheaf of sections is strongly algebraic.*

3.10 Remark. This definition is a generalization of the definition of strongly algebraic vector bundles in [B-C-R]. Indeed, let X be an affine real algebraic variety and let $\pi: E \to X$ be a real algebraic vector bundle over X. Denote the sheaf of sections of E by $\mathcal{F}$. Since $\mathfrak{X} = \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$, the real algebraic vector bundle $E \to X$ is strongly algebraic if and only if there exists a finitely generated projective $\Gamma(X, \mathcal{O}_X)$-module M such that $\mathcal{F} \cong \mathcal{O}_X \otimes M$.

3.11 Lemma. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules, locally free of finite rank. Then, $\mathcal{F}$ is strongly algebraic if and only if there exist a complexification (Y, j) of X and a sheaf $\mathcal{H}$ of $\mathcal{O}_Y$-modules, locally free of finite rank, such that*

$$j^\star \mathcal{H} \cong \mathcal{F}.$$

In particular, if $E \to X$ is a real algebraic vector bundle over X then, E is strongly algebraic if and only if there exists a complexification of E as a vector bundle.

Proof. Clearly, if there exist a complexification (Y, j) of X and a sheaf $\mathcal{H}$ of $\mathcal{O}_Y$-modules, locally free of finite rank, such that $j^\star \mathcal{H} \cong \mathcal{F}$, then $\mathcal{F}$ is strongly algebraic .

To prove the converse, we may assume X connected. Let (Y, j) be a complexification of X. Let us identify X with the set of real points of Y via j. Moreover, we identify the associated real scheme $\mathfrak{X}$ of X with the set of all generalizations of real points of Y. By definition of a strongly algebraic sheaf, there exists a sheaf $\mathcal{G}$ of $\mathcal{O}_{\mathfrak{X}}$-modules, locally free of finite rank, such that

$$\mathcal{G}|_X \cong \mathcal{F}.$$

In particular, there exist finitely many open subsets U_i of Y such that $\{U_i\}$ covers $\mathfrak{X}$ and

$$\mathcal{G}|_{U_i \cap \mathfrak{X}} \cong (\mathcal{O}_{\mathfrak{X}}|_{U_i \cap \mathfrak{X}})^n.$$

This defines morphisms

$$g_{ij}: U_i \cap U_j \cap \mathfrak{X} \longrightarrow \mathrm{GL}_{n,\mathbb{R}},$$

satisfying the cocycle condition. Each g_{ij} extends uniquely to a rational map

$$h_{ij}: U_i \cap U_j \dashrightarrow \mathrm{GL}_{n,\mathbb{R}}$$

with $U_i \cap U_j \cap \mathfrak{X} \subseteq \operatorname{dom}(h_{ij})$. We may assume that each h_{ij} is defined on the whole of $U_i \cap U_j$. For, if η is a generic point of an irreducible component of $U_i \cap U_j \backslash \operatorname{dom}(h_{ij})$ then $\eta \notin U_i \cap \mathfrak{X}$ or $\eta \notin U_j \cap \mathfrak{X}$. Say $\eta \notin U_i \cap \mathfrak{X}$. Then no specialization of η is in $U_i \cap \mathfrak{X}$. Replace U_i by $U_i \backslash \overline{\{\eta\}}$. Then, $\{U_i\}$ still covers $\mathfrak{X}$. After a finite number of steps we have established that each h_{ij} is defined on $U_i \cap U_j$. Replacing Y by $\bigcup U_i$, we may assume that $\{U_i\}$ covers Y. Since the g_{ij} satisfy the cocycle condition and since $\mathfrak{X}$ is dense in Y, the h_{ij} satisfy the cocycle condition also. Hence, the h_{ij} define a sheaf $\mathcal{H}$ of $\mathcal{O}_Y$-modules which is locally free of finite rank. Clearly, $\mathcal{H}|_{\mathfrak{X}} \cong \mathcal{G}$ and hence, $\mathcal{H}|_X \cong \mathcal{F}$. □

3.12 Theorem. *Let X be a real algebraic variety having a complexification and let $\pi: E \to X$ be a real algebraic vector bundle over X. Then, the following conditions are equivalent.*

(i) *The real algebraic vector bundle $E \to X$ is strongly algebraic.*
(ii) *The real algebraic vector bundle $E \to X$ has a complexification.*
(iii) *The real algebraic variety E has a complexification.*

Proof. By Lemma 3.11 , (i) and (ii) are equivalent. By definition, if $E \to X$ has a complexification as a vector bundle, then E has a complexification as a real algebraic variety. Hence, (ii) implies (iii).

Let us now prove that (iii) implies (i). Suppose the real algebraic variety E has a complexification F. We may assume that the morphism of real algebraic varieties $\pi\colon E \to X$ extends to a morphism

$$\rho\colon F \longrightarrow Y,$$

of algebraic varieties over $\mathbb{R}$, where Y is a complexification of X. Consider the associated real scheme $\mathfrak{X}$ of X as a subset of Y and let

$$\mathfrak{F} = \mathfrak{X} \times_Y F.$$

Then, $\mathfrak{F}$ is a reduced, separated scheme over $\mathbb{R}$ and $\rho'\colon \mathfrak{F} \to \mathfrak{X}$ is of finite type. Moreover, $\mathfrak{F}(\mathbb{R})$ is isomorphic to E. Since the set of points of $\mathfrak{F}$ at which ρ' is flat is open [M, Theorem 53] and contains the set of real points of $\mathfrak{F}$, we may assume ρ' to be faithfully flat. Furthermore, every fibre of ρ' is reduced. Since $\pi\colon E \to X$ is a real algebraic vector bundle over X we get a rational group law on $\mathfrak{F}$ relative to $\mathfrak{X}$. It follows from Corollaire 3.13(ii) of [A] that there exists a reduced, faithfully flat group scheme G over $\mathfrak{X}$, of finite type over $\mathfrak{X}$, such that G is isomorphic to $\mathfrak{F}$ as a rational group law relative to $\mathfrak{X}$. Let $\varphi\colon G \to \mathfrak{X}$ be the structural morphism. If $\{U_i\}$ is an open covering of $\mathfrak{X}$ such that

$$\pi^{-1}(U_i(\mathbb{R})) \cong U_i(\mathbb{R}) \times \mathbb{R}^{n_i}$$

then $\varphi^{-1}(U_i)$ and $U_i \times \mathbb{A}_{\mathbb{R}}^{n_i}$ are group schemes over U_i which are isomorphic as rational group laws relative to U_i. Hence,

$$\varphi^{-1}(U_i) \cong U_i \times \mathbb{A}_{\mathbb{R}}^{n_i}.$$

Now, it is clear that G is a geometric vector bundle over $\mathfrak{X}$. Its sheaf of sections $\mathcal{G}$ is a locally free sheaf of finite rank whose restriction to X is the sheaf of sections of E. Hence, the real algebraic vector bundle E is strongly algebraic. This finishes the proof of the theorem. □

Proof of Theorem 1.1 . Let X be an affine real algebraic variety and let $E \to X$ be a real algebraic vector bundle over X. If the real algebraic vector bundle E is strongly algebraic, then, according to Theorem 3.12 , there exists a complexification $F \to Y$ of the real algebraic vector bundle $E \to X$. Since X is affine, we may assume Y to be affine. Then, F is affine since the map $F \to Y$ is affine. Therefore, E is affine. Conversely, if E is an affine real algebraic variety, then, according to Example 2.3 , E has a complexificaton. By Theorem 3.12 , the real algebraic vector bundle $E \to X$ is strongly algebraic. □

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