

J. Huisman

Non-special divisors on real algebraic curves and embeddings into real projective spaces

Received: October 17, 2001

Published online: February 20, 2003 – © Springer-Verlag 2003

Abstract. We show that there is a large class of non-special divisors of relatively small degree on a given real algebraic curve. If the real algebraic curve has many real components, such a divisor gives rise to an embedding (birational embedding, resp.) of the real algebraic curve into the real projective space \mathbb{P}^r for $r \geq 3$ ($r = 2$, resp.). We study these embeddings in quite some detail.

Mathematics Subject Classification (2000). 14C20, 14H50, 14P99

Key words. real algebraic curve – non-special divisor – M -curve – plane curve – double point – multiple point – inflection point – space curve – ramification

1. Introduction

In an earlier paper we showed that there is a large class of effective non-special divisors of relatively small degree on real algebraic curves that have many real components, so-called M -curves [7]. In this paper we generalize that result. The current generalization seems to give the largest uniform class of non-special divisors of relatively small degree on real algebraic curves. It has several applications, especially to M -curves. Here, we will study the embeddings of M -curves into the real projective spaces to which they give rise. We will show that any M -curve embeds into an odd-dimensional real projective space in such a way that the image curve is really unramified. In particular, the image curve does not have real inflection points. As for the embeddings of M -curves into even-dimensional real projective spaces, we will determine the exact number of real inflection points on the image curve. The interesting thing is that this inflectionary behavior, in turn, characterizes M -curves.

In an attempt to whet the reader's appetite, we give an example of the results we obtained (cf. Figure 1.1):

Theorem 1.1. *Let $C \subseteq \mathbb{P}^2$ be a, not necessarily non-singular, geometrically irreducible real algebraic curve of degree $d \geq 2$. Suppose that the set of real points $C(\mathbb{R})$ of C consists of $d - 1$ global real analytic branches and suppose that at least*

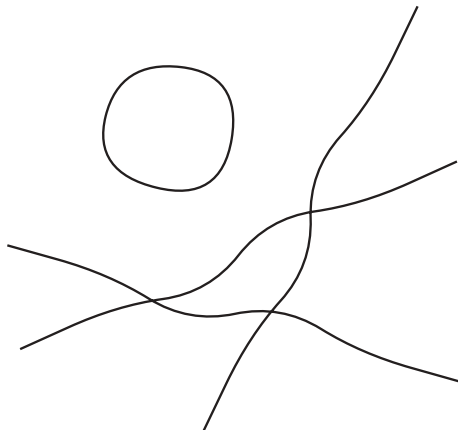


Fig. 1.1. A real algebraic curve satisfying the conditions of Theorem 1.1 with $d = 5$

$d - 2$ of them represent the non-trivial homology class in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Then the following statements hold:

1. The normalization of C is an M -curve, i.e., its genus g is equal to $d - 2$.
2. Any of the $d - 1$ real analytic branches of $C(\mathbb{R})$ is a compact connected non-singular real analytic curve in $\mathbb{P}^2(\mathbb{R})$. Exactly one of them, say X , is homologous to zero in $\mathbb{P}^2(\mathbb{R})$.
3. The curve C has only split real ordinary multiple points as singularities and the total multiplicity of the singular locus is equal to $\frac{1}{2}g(g - 1)$.
4. Each of the real analytic branches of $C(\mathbb{R})$ different from X has exactly 3 inflection points, all ordinary. The real analytic branch X has no inflection points. In particular, the curve C has exactly $3(d - 2) = 3g$ ordinary real inflection points.
5. The curve C does not have real multitangent lines.

If $d = 2$, the real algebraic curve C is necessarily a non-singular conic having real points and all statements, of the theorem hold trivially. For $d = 3$ the real algebraic curve C is necessarily a non-singular real cubic curve such that $C(\mathbb{R})$ has 2 connected components. Statement 4 is the well-known fact that such a curve has exactly 3 ordinary real inflection points. More generally, i.e. for arbitrary $d \geq 2$, the generalized Klein Equation for real plane curves imply that C has exactly $3(d - 2)$ real inflection points [11, 12]. However, it does not imply anything on their distribution over the branches of $C(\mathbb{R})$. Therefore, to my knowledge, the result is new for $d \geq 4$. To my opinion, the interest of the result, and its generalizations to higher dimensions, is that the topological conditions imposed on the plane real algebraic curve imply very precise geometrical properties of the curve. We will see more examples of this in the paper.

The paper is organized as follows. In Section 2 we derive our main result on non-special divisors on real algebraic curves. In Section 3 we study the embeddings of M -curves into the odd-dimensional real projective spaces to which they give

rise. Section 4 deals with the even-dimensional case. In Section 5 we specialize to the case of plane birational embeddings.

Convention and notation. A real algebraic curve is a reduced non-singular proper scheme over \mathbb{R} of dimension 1; it is supposed to be geometrically irreducible and non-singular, unless stated otherwise. The projective n -space over \mathbb{R} is denoted by \mathbb{P}^n instead of $\mathbb{P}_{\mathbb{R}}^n$.

2. Non-special divisors on real algebraic curves

Let C be a real algebraic curve and let X be a connected component of the set of real points $C(\mathbb{R})$ of C . Let $\text{res}_X: \text{Div}(C) \rightarrow \text{Div}(C)$ be the restriction-to- X morphism. This morphism is defined by letting $\text{res}_X(P) = P$ if $P \in X$ and $\text{res}_X(P) = 0$ if $P \notin X$, for any closed point P of C . Now, for any divisor D on C , we define the *degree of D on X* to be the natural number $\text{deg}_X(D) = \text{deg}(\text{res}_X(D))$.

Recall the following statement [3, Corollary 4.2.2], of which we give a topological proof:

Proposition 2.1. *Let C be a real algebraic curve. If ω is a non-zero rational differential form on C then $\text{deg}_X(\text{div}(\omega))$ is even for each connected component X of $C(\mathbb{R})$.*

Proof. The restriction of ω to X defines a non-zero meromorphic real differential form on the real analytic curve X . Since X is isomorphic to the real analytic curve $\mathbb{P}^1(\mathbb{R})$, the statement follows from the next lemma. \square

Lemma 2.2. *Let ω be a non-zero meromorphic real differential form on the real analytic curve $\mathbb{P}^1(\mathbb{R})$. Then, $\text{deg}(\text{div}(\omega))$ is even.*

Proof. Let η be the differential form $dx/(x^2 + 1)$ on $\mathbb{P}^1(\mathbb{R})$. Clearly, η has no zeros and no poles on $\mathbb{P}^1(\mathbb{R})$. Let f be the unique meromorphic real analytic function on $\mathbb{P}^1(\mathbb{R})$ such that $\omega = f \cdot \eta$. Then, $\text{div}(\omega) = \text{div}(f)$. If f is constant, $f \neq 0$ and $\text{div}(f) = 0$. Hence, $\text{deg}(\text{div}(\omega))$ is even if f is constant. Suppose, therefore, that f is non-constant and consider f as a real analytic map from $\mathbb{P}^1(\mathbb{R})$ into itself. Since f is non-constant, $\text{div}(f) = f^*0 - f^*\infty$. But, it is easily seen that $\text{deg}(f^*0)$ and $\text{deg}(f^*\infty)$ are both congruent to the topological degree mod 2 of f (see [8] for the definition and properties of the topological degree mod 2). Hence, $\text{deg}(\text{div}(f)) = \text{deg}(f^*0) - \text{deg}(f^*\infty)$ is even. This shows that $\text{deg}(\text{div}(\omega))$ is also even if f is non-constant. \square

One should compare Proposition 2.1 with the following well-known statement [4, Lemma 4.1]:

Proposition 2.3. *Let C be a real algebraic curve. If f is a non-zero rational function on C then $\text{deg}_X(\text{div}(f))$ is even for each connected component X of C . \square*

For future reference we mention the following consequence of Proposition 2.3:

Corollary 2.4. *Let C be a real algebraic curve. Let D and E be linearly equivalent divisors on C . Then, for all connected components X of $C(\mathbb{R})$, one has*

$$\deg_X(D) \equiv \deg_X(E) \pmod{2}. \quad \square$$

Recall that a divisor D on an algebraic curve C is said to be *non-special* if $h^0(D) = \deg(D) - g + 1$, where g is the genus of C . By Riemann–Roch, D is non-special if and only if $h^0(\Omega(-D)) = 0$. Here, $\Omega(-D)$ is the sheaf on C whose non-zero sections over a non-empty open subset U constitute the $\mathcal{O}(U)$ -module of non-zero rational differential forms ω on C satisfying $\operatorname{div}(\omega) \geq D$ on U . It follows that D is non-special if D' is non-special and $D \geq D'$.

It follows from Riemann–Roch that divisors of degree at least $2g - 1$ are non-special. Using Proposition 2.1 one gets a large class of non-special divisors of relatively small degree, i.e., of degree less than $2g - 1$, on any real algebraic curve:

Theorem 2.5. *Let C be a real algebraic curve and let g be its genus. Let D be a divisor on C and let d be its degree. Let k be the number of connected components X of $C(\mathbb{R})$ such that $\deg_X(D)$ is odd. If $d + k \geq 2g - 1$ then D is non-special.*

Proof. We show that $h^0(\Omega(-D)) = 0$. Let ω be a global section of $\Omega(-D)$. Suppose that ω is non-zero. Then, one can consider ω as a rational section of Ω such that $\operatorname{div}(\omega) \geq D$. By Proposition 2.1, $\deg_X(\operatorname{div}(\omega))$ is even for each connected component X of $C(\mathbb{R})$. In particular, $\deg_X(\operatorname{div}(\omega))$ is at least $\deg_X(D) + 1$ for each of the k connected components X of $C(\mathbb{R})$ for which $\deg_X(D)$ is odd. It follows that $\deg(\operatorname{div}(\omega)) \geq d + k \geq 2g - 1$. Contradiction since $\deg(\operatorname{div}(\omega)) = 2g - 2$. \square

Note that Theorem 2.5 generalizes Theorem 2.4 of [7] in two ways; it holds for any real algebraic curve, and its condition on the divisor is considerably *weaker*.

One easily gets Harnack’s Inequality for real algebraic curves [5] as a consequence of Theorem 2.5:

Corollary 2.6 (Harnack’s inequality). *Let C be a real algebraic curve. Let g be the genus of C and let s be the number of connected components of $C(\mathbb{R})$. Then, $s \leq g + 1$.*

Proof. Choose real points P_1, \dots, P_s on C such that P_i and P_j are on different connected components of $C(\mathbb{R})$, for $i \neq j$. Choose non-real closed points Q_1, \dots, Q_g on C and let D be the divisor $\sum Q_i - \sum P_i$. Then, the degree d of D is equal to $2g - s$ and the number k of connected components X such that $\deg_X(D)$ is odd is equal to s . One has $d + k = 2g \geq 2g - 1$. By Theorem 2.5, D is non-special. It follows that $h^0(D) = d - g + 1 = 2g - s - g + 1 = g + 1 - s$. Since $h^0(D) \geq 0$, one has $s \leq g + 1$. \square

One also has the following consequence of Theorem 2.5:

Corollary 2.7. *Let C be a real algebraic curve and let g be its genus. Let D be a divisor on C and let d be its degree. Let k be the number of connected components X of $C(\mathbb{R})$ such that $\deg_X(D)$ is odd:*

1. If $d + k \geq 2g + 1$ then the linear system $|D|$ is base point-free.
2. If $d + k \geq 2g + 1$ then $|D|$ separates points and tangent vectors on any connected component X of $C(\mathbb{R})$.
3. If $d + k \geq 2g + 1$ and $\deg_X(D)$ is even for at least 1 connected component X of $C(\mathbb{R})$ then there is a real point P of C such that $|D|$ separates P from all other points and $|D|$ separates tangent vectors at P .
4. If $d + k \geq 2g + 3$ then $|D|$ separates points and tangent vectors.

Proof. Suppose that $d + k \geq 2g + 1$. Let P be any closed point of C . Let D' be the divisor $D - P$ and let d' be its degree. We show that D' is non-special. It will then follow that $h^0(D - P) = h^0(D) - \deg(P)$, i.e., that $|D|$ is base point-free.

Let k' be the number of connected components X of $C(\mathbb{R})$ such that $\deg_X(D')$ is odd. If P is a real point of C then $d' = d - 1$ and $k' \geq k - 1$. Hence, $d' + k' \geq d + k - 2 \geq 2g - 1$. By Theorem 2.5, D' is non-special if P is real. If P is non-real then $d' = d - 2$ and $k' = k$. Hence, $d' + k' = d + k - 2 \geq 2g - 1$. Again by Theorem 2.5, D' is non-special if P is non-real. Therefore, in both cases, D' is non-special. This shows Statement 1.

Suppose that $d + k \geq 2g + 1$ and let X be a connected component of $C(\mathbb{R})$. In order to show that $|D|$ separates points and tangent vectors on X , it suffices to show, as before, that $D' = D - P - Q$ is non-special for all $P, Q \in X$. Now, D' is of degree $d' = d - 2$ since P and Q are both real. Since P and Q are on the same connected component of $C(\mathbb{R})$, the number k' of connected components of $C(\mathbb{R})$ at which the degree of D' is odd, is equal to k . Now, $d' + k' = d - 2 + k \geq 2g - 1$. By Theorem 2.5, D' is non-special. This shows Statement 2.

Suppose that $d + k \geq 2g + 1$ and let X be a connected component of $C(\mathbb{R})$ such that $\deg_X(D)$ is even. Choose $P \in X$. By Statement 2, $|D|$ separates tangent vectors at P and $|D|$ separates P from any other point Q of X . Therefore, in order to show Statement 3, it suffices to show that $D' = D - P - Q$ is non-special for all closed points Q of C not contained in X . Again let d' be the degree of D' and let k' be the number of connected components of $C(\mathbb{R})$ at which the degree of D' is odd. There are two cases to consider: the case Q real and the case Q non-real. If Q is real then $d' = d - 2$ and $k' \geq k + 1 - 1$ since $P \in X$ and $\deg_X(D)$ is even. Now, $d' + k' \geq d + k - 2 \geq 2g - 1$ and $D - P - Q$ is non-special. If Q is non-real then $d' = d - 3$ and $k' = k + 1$. Hence, $d' + k' = d + k - 2 \geq 2g - 1$ and $D - P - Q$ is non-special. This shows Statement 3.

Suppose that $d + k \geq 2g + 3$. In order to show that $|D|$ separates points and tangent vectors, it suffices to show that $D - P - Q$ is non-special for all closed points P and Q of C . Again let d' be the degree of D' and let k' be the number of connected components of $C(\mathbb{R})$ on which D' has odd degree. There are three cases to consider. The case P and Q non-real, the case P real and Q non-real, and the case P and Q real. In the first case, $d' = d - 4$ and $k' = k$. Hence, $d' + k' = d - 4 + k \geq 2g - 1$ and $D - P - Q$ is non-special. In the second case, $d' = d - 3$ and $k' \geq k - 1$. Again, $d' + k' \geq d - 3 + k - 1 \geq 2g - 1$ and $D - P - Q$ is non-special. In the third case, $d' = d - 2$ and $k' \geq k - 2$ so that $d' + k' \geq d - 2 + k - 2 \geq 2g - 2$ and $D - P - Q$ is non-special. Therefore, $D - P - Q$ is non-special for all closed points P and Q of C . This shows Statement 4. \square

Theorem 2.5 and its Corollary 2.7 are the most interesting when the real algebraic curve C has many real components and the divisor D has odd degree on many of these components.

Let C be a real algebraic curve and let g be its genus. By Harnack's inequality, the number of connected components s of $C(\mathbb{R})$ is at most $g + 1$. The real algebraic curve C is called an M -curve if $s = g + 1$. Klein showed the existence of M -curves of any genus. In fact, there are many M -curves of a given genus: the moduli space of M -curves of genus g is a connected semi-analytic variety of dimension $3g - 3$, if $g > 1$ [9,6].

It is well known that a general linear system of dimension $r \geq 3$ and degree $d = g + r$ on a real algebraic curve of genus g is base point-free and gives rise to an embedding of the curve in the projective space \mathbb{P}^r . The following consequence of Corollary 2.7.4 sharpens this fact for an explicit, i.e. non-generic, large class of linear systems on M -curves:

Corollary 2.8. *Let C be an M -curve and let g be its genus. Let $r \geq 1$ be an integer. Let D be a divisor on C of degree $d = g + r$ such that $\deg_X(D)$ is odd for at least g connected components X of $C(\mathbb{R})$. Then, the linear system $|D|$ has dimension r and is base point-free. Let $f: C \rightarrow \mathbb{P}^r$ be its associated morphism. Then:*

1. if $r = 1$, the restriction of f to any connected component of $C(\mathbb{R})$ is a real analytic isomorphism onto $\mathbb{P}^1(\mathbb{R})$;
2. if $r = 2$, the morphism f is a birational embedding of C into \mathbb{P}^2 ;
3. if $r \geq 3$, the morphism f is an embedding of C into \mathbb{P}^r . □

Let C , g , r and D be as in the statement of the preceding corollary. Observe that if r is even then $\deg_X(D)$ is odd for exactly g connected components X of $C(\mathbb{R})$. If r is odd then $\deg_X(D)$ is odd for all $g + 1$ connected components X of $C(\mathbb{R})$. We will study separately the associated embeddings of C into \mathbb{P}^r for r even and for r odd in the next two sections. The last section is devoted to the case $r = 2$.

In the next two sections we will need the following converse to Corollary 2.8.1, whose easy proof is left to the reader (or see [7] for details).

Proposition 2.9. *Let C be a real algebraic curve and let g be its genus. Suppose that there is a morphism $f: C \rightarrow \mathbb{P}^1$ satisfying the following 3 conditions:*

1. The morphism f is of degree $g + 1$.
2. The restriction of f to any connected component of $C(\mathbb{R})$ is a homeomorphism onto $\mathbb{P}^1(\mathbb{R})$.
3. Any closed point $P \in C$ such that $f(P) \in \mathbb{P}^1(\mathbb{R})$ is real.

Then C is an M -curve. □

3. Morphism of M -curves into odd-dimensional real projective spaces

The embedding of an M -curve C into \mathbb{P}^r as in Corollary 2.8 turns out to be particularly regular when r is odd, i.e., when $\deg_X(D)$ is odd for all connected

components X of $C(\mathbb{R})$. In order to make this precise, we need to introduce the following terminology and notation.

Let C be an algebraic curve over a field K and let D be a divisor on C . Define D_{red} to be the *reduced divisor* of D . More precisely,

$$D_{\text{red}} = \sum_{P \in \text{Supp}(D)} P,$$

where $\text{Supp}(D)$ is the support of D . Define the *weight* of D on C to be the natural number $w(D) = \deg(D - D_{\text{red}})$.

Let $r \geq 1$ be an integer. Recall that a morphism $f: C \rightarrow \mathbb{P}^r$ over K is *non-degenerate* if the image $f(C)$ is not contained in a hyperplane of \mathbb{P}^r . Let $f: C \rightarrow \mathbb{P}^r$ be a non-degenerate morphism. A hyperplane H in \mathbb{P}^r is said to be *unramified over K* – or simply *unramified* if no confusion is possible – with respect to f if the weight $w(f^*H)$ of f^*H is at most $r - 1$. The morphism f is said to be *unramified* if all hyperplanes in \mathbb{P}^r are unramified with respect to f . An algebraic curve $C \subseteq \mathbb{P}^r$ is *unramified* if the inclusion morphism of C into \mathbb{P}^r is unramified.

Let $f: C \rightarrow \mathbb{P}^r$ be a non-degenerate morphism. Let P be a K -rational point of C . Then, P is an *inflection point* of C with respect to f if the osculating hyperplane H to $f(C)$ at $f(P)$ has order of contact at least $r + 1$ at P . An inflection point is an *ordinary inflection point* if the order of contact of the osculating hyperplane is equal to $r + 1$.

It is clear that, if P is an inflection point of C , its osculating hyperplane H is not unramified. In particular, an unramified curve $C \subseteq \mathbb{P}^r$ has no K -rational inflection points.

The standard example of an unramified morphism from an algebraic curve into projective space is the *rational normal map* $f: \mathbb{P}^1 \rightarrow \mathbb{P}^r$ affinely defined by $f(t) = (t, t^2, \dots, t^r)$. The morphism f is an embedding of \mathbb{P}^1 into \mathbb{P}^r . Its image is the *rational normal curve*. In particular, the rational normal curve does not have inflection points.

Now, if K is algebraically closed, the rational normal curves are, essentially, the only unramified curves. Indeed, let $f: C \rightarrow \mathbb{P}^r$ be a really unramified morphism. Then, C does not contain any inflection points with respect to f . By the Plücker formulas, f is a rational normal map, after a change of coordinates of \mathbb{P}^r (see [1, Exercise I.C] for more details). Therefore, the notion of unramified curves is less interesting over algebraically closed fields.

The main result of this section is that, over \mathbb{R} , there are unramified curves of any genus and in any odd dimension. In fact, we have already seen that for any M -curve C of any genus there is a really unramified morphism $f: C \rightarrow \mathbb{P}^1$. Indeed, let $r = 1$ and let D be a divisor as in Corollary 2.8. Then, the associated morphism $f: C \rightarrow \mathbb{P}^1$ is of degree $g + 1$ and, according to Corollary 2.8.1, maps any connected component of $C(\mathbb{R})$ isomorphically onto $\mathbb{P}^1(\mathbb{R})$. Since C is an M -curve, f is unramified over all real points of \mathbb{P}^1 . This means exactly that f is really unramified, i.e. unramified over \mathbb{R} .

Theorem 3.1. *Let C be an M -curve and let g be its genus. Let $r \geq 3$ be an odd integer. Let D be a divisor on C of degree $d = g + r$ such that $\deg_X(D)$ is odd for*

all connected components X of $C(\mathbb{R})$. Let $f: C \rightarrow \mathbb{P}^r$ be the morphism of C into \mathbb{P}^r associated to D . Then, f is an embedding and f is unramified. In particular, C has no real inflection points with respect to f .

Proof. By Corollary 2.7, f is an embedding of C into \mathbb{P}^r . Since f is associated to D , f is non-degenerate. We have to show that f is unramified.

Let H be any hyperplane of \mathbb{P}^r . Since $\deg_X(D)$ is odd for all connected components X of $C(\mathbb{R})$, the same holds for the divisor f^*H on C by Corollary 2.4. Hence

$$w(f^*H) = \deg((f^*H) - (f^*H)_{\text{red}}) \leq d - (g + 1) = r - 1.$$

This shows that f is unramified. \square

Observe that there are many divisors D satisfying the hypothesis of the theorem. Indeed, since $d - (g + 1) = r - 1$ is even, one can, for example, choose D to satisfy $\deg_X(D) = 1$ for all connected components X of $C(\mathbb{R})$.

Note that, because of the hypothesis in Theorem 3.1 of $\deg_X(D)$ being odd for all X , the image of each connected component X of $C(\mathbb{R})$ in $\mathbb{P}^r(\mathbb{R})$ realizes the non-trivial homology class in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. The final result of this section may then be considered as a converse to Theorem 3.1.

Theorem 3.2. *Let $d \geq r \geq 3$ be integers such that r is odd. Let $C \subseteq \mathbb{P}^r$ be a non-degenerate real algebraic curve of degree d such that $C(\mathbb{R})$ has $d - r + 1$ connected components, each representing the non-trivial homology class in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Then, C is an M -curve.*

Proof. Since $r - 1$ is even, let t be an integer such that $2t = r - 1$. Choose t non-real closed points P_1, \dots, P_t on C in a general position. The smallest linear subspace V of \mathbb{P}^r containing P_1, \dots, P_t is of codimension 2. Consider the restriction to C of the projection of \mathbb{P}^r onto \mathbb{P}^1 with center V . This restriction is a morphism $f: C \rightarrow \mathbb{P}^1$ of degree $d - r + 1$. We want to show that f satisfies the conditions of Proposition 2.9 in order to conclude that C is an M -curve.

Since each connected component of $C(\mathbb{R})$ represents the non-trivial homology class of $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$, each hyperplane in \mathbb{P}^r intersects each of the $d - r + 1$ connected components of $C(\mathbb{R})$. Since each hyperplane containing V contains the non-real closed points P_1, \dots, P_t of C , and $d - r + 1 + 2t = d$, each hyperplane containing V intersects each connected component of $C(\mathbb{R})$ in exactly one point. Hence, the restriction of f to any connected component of $C(\mathbb{R})$ is a homeomorphism onto $\mathbb{P}^1(\mathbb{R})$. Moreover, if P is a closed point of C such that $f(P)$ is real, then P is real. By Proposition 2.9, C is an M -curve. \square

It is interesting to observe that the hypothesis on the curve $C \subseteq \mathbb{P}^r$ of degree d in the preceding theorem imply that the genus of C is equal to $d - r$. This is a statement far from being true for any curve $C \subseteq \mathbb{P}^r$ of degree d .

Corollary 3.3. *Let $d \geq r \geq 3$ be integers such that r is odd. Let $C \subseteq \mathbb{P}^r$ be a non-degenerate real algebraic curve of degree d such that $C(\mathbb{R})$ has $d - r + 1$ connected components, each representing the non-trivial homology class in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Then, C is unramified. In particular, C has no real inflection points.*

Proof. By Theorem 3.2, C is an M -curve. In particular, its genus g is equal to $d - r$. Let $H \subseteq \mathbb{P}^r$ be a hyperplane and let D be the divisor $H \cdot C$ on C . Let X be a connected component of $C(\mathbb{R})$. Since X represents the non-trivial homology class in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$, $\deg_X(D)$ is odd. Hence, D is a divisor of degree $d = g + r$ such that $\deg_X(D)$ is odd for all connected components X of $C(\mathbb{R})$. By Theorem 2.5, the linear system $|D|$ is of dimension $d - g = r$. Therefore, the inclusion map from C into \mathbb{P}^r is the morphism associated to the divisor D . By Theorem 3.1, C is unramified. \square

The preceding results and, also, the results of the next section seem to justify the following conjecture:

Conjecture 3.4. Let $r \geq 3$ be an odd integer. Let $C \subseteq \mathbb{P}^r$ be a non-degenerate real algebraic curve. If C is unramified then C is an M -curve and, moreover, each connected component of $C(\mathbb{R})$ represents the non-trivial homology class of $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$.

4. Morphism of M -curves into even-dimensional real projective spaces

In this section we show that the embeddings (birational embeddings, resp.) of M -curves of positive genus into even projective spaces, as in Corollary 2.8 are not unramified. In fact, we will explicitly determine the number of real inflection points of the curve and show that all inflection points are ordinary.

We start with the following lemma:

Lemma 4.1. *Let C be an M -curve of genus $g \geq 1$. Let $r \geq 2$ be even and let D be a divisor of degree $d = g + r + 1$ such that $\deg_X(D)$ is odd for all connected components X of $C(\mathbb{R})$. Let $f: C \rightarrow \mathbb{P}^{r+1}$ be the associated embedding and identify C with its image in \mathbb{P}^{r+1} . Let X and X' be distinct connected components of $C(\mathbb{R})$. Let $\iota: X \rightarrow X'$ be defined by letting $\iota(P)$ be the unique intersection point of X' and the osculating hyperplane H_P to C at P . Then, each fiber of ι contains exactly $r + 1$ points.*

Proof. Let us first justify the implicitly mentioned statements. By Corollary 2.7, f is an embedding. The osculating hyperplane H_P has an order of contact to C at least $r + 1$ at P . By Corollary 2.4, H_P intersects all g connected components Y of $C(\mathbb{R})$ different from X . Since $(r + 1) + g = d$, the hyperplane H_P intersects each connected component Y , $Y \neq X$, in exactly 1 point. Hence, ι is a well-defined map from X into X' .

Now we show that each fiber of ι contains exactly $r + 1$ points. In fact, we show that ι is real analytic, unramified and of degree $r + 1$ for some choice of orientations of X and X' . It is clear that this will imply that the cardinality of each fiber of ι is equal to $r + 1$.

It is easily seen that ι is real analytic. In order to show that ι is unramified, observe that the above arguments show that the divisor $H_P \cdot C$ has entirely real support. It follows that one can extend ι to an injective complex analytic map from

an open neighborhood of X in $C(\mathbb{C})$ into an open neighborhood of X' in $C(\mathbb{C})$. Since this extension is injective, ι is unramified.

Now, we show that ι is of degree $r + 1$ for some choice of orientations. Let \mathcal{C} be a finitely presented proper flat real analytic family of, not necessarily non-singular or irreducible, real algebraic curves over the interval $[0, 1]$ having the following properties:

1. The real algebraic curve \mathcal{C}_t is non-singular and geometrically irreducible, and is an M -curve of genus g , for $t \neq 0$.
2. The real algebraic curve $C' = \mathcal{C}_0$ has two non-singular geometrically irreducible components C'_0 and C'_1 intersecting each other in a non-real ordinary double point of C' . The real algebraic curve C'_0 is isomorphic to \mathbb{P}^1 and the real algebraic curve C'_1 is an M -curve of genus $g - 1$.
3. The real algebraic curve \mathcal{C}_1 is isomorphic to C .
4. Identifying C with \mathcal{C}_1 , the subsets X and $C'_0(\mathbb{R})$ of $\mathcal{C}(\mathbb{R})$ belong to the same connected component of $\mathcal{C}(\mathbb{R})$.

We want to extend the divisor D on C over the whole family \mathcal{C} into a relative Weil divisor \mathcal{D} . We do this as follows. We may assume that $\deg_X(D) = r + 1$. Since $\deg(D) = g + r$ and $\deg_Y(D)$ is odd for each connected component Y of $C(\mathbb{R})$, $Y \neq X$, the degree of D on Y is equal to 1. It follows that $\text{Supp}(D) \subseteq C(\mathbb{R})$. Now, choose sections $\sigma_i: [0, 1] \rightarrow \mathcal{C}$, for $i = 1, \dots, d$ such that $D = \sum \sigma_i(1)$. Put $\mathcal{D} = \sum \sigma_i([0, 1])$. Then \mathcal{D} is a relative Weil divisor on \mathcal{C} such that $\mathcal{D}_1 = D$. By construction, the divisor $D' = \mathcal{D}_0$ on C' has degree $r + 1$ on the component C'_0 and degree g on C'_1 . It is easily seen that the divisor \mathcal{D} induces a morphism $F: \mathcal{C} \rightarrow \mathbb{P}^r \times [0, 1]$ of families of real algebraic varieties over the interval $[0, 1]$. Clearly, $F_1 = f$. Put $f' = F_0$. Since the degree of D' on C'_0 is equal to $r + 1$, the restriction of f' to C'_0 is a rational normal map of C'_0 into \mathbb{P}^{r+1} . Since the degree of D' on C'_1 is equal to g , the induced linear system on C'_1 is 1-dimensional. Hence, f' maps C'_1 onto a projective line L in \mathbb{P}^{r+1} . Clearly, L intersects $f'(C'_0)$ in a non-real point. Now, one has a map $\iota': C'_0(\mathbb{R}) \rightarrow L(\mathbb{R})$ defined as ι . Since ι and ι' vary in a connected family of maps, it suffices to show that ι' is of degree $r + 1$ for some choice of orientations on $C'_0(\mathbb{R})$ and $L(\mathbb{R})$. But this is an elementary matter to check. \square

Theorem 4.2. *Let C be an M -curve and let g be its genus. Let $r \geq 2$ be an even integer. Let D be a divisor of degree $d = g + r$ such that $\deg_X(D)$ is odd for exactly g connected components X of $C(\mathbb{R})$. Let $f: C \rightarrow \mathbb{P}^r$ be the morphism associated to D . Then, f is an embedding (birational embedding, resp.) if $r > 2$ (if $r = 2$, resp.). Moreover, the following assertions hold:*

1. *Let X be a connected component of $C(\mathbb{R})$ such that $\deg_X(D)$ is odd. Then, X contains exactly $r + 1$ inflection points of C with respect to f , and they are all ordinary.*
2. *Let X' be the connected component of $C(\mathbb{R})$ such that $\deg_{X'}(D)$ is even. Then, X' does not contain any inflection point of C with respect to f .*

Proof. Corollary 2.8 states that f is an embedding (birational embedding, resp.) if $r > 2$ (if $r = 2$, resp.).

Choose a point $Q \in X'$. Let $D' = D + Q$. Then D' is a divisor of degree $g + r + 1$ having odd degree at all connected components of $C(\mathbb{R})$. Let $f' : C \rightarrow \mathbb{P}^{r+1}$ be the morphism associated to D' . Observe that the composition of f' with the projection from \mathbb{P}^{r+1} onto \mathbb{P}^r with center Q coincides with the morphism f . According to the preceding lemma, there are exactly $r + 1$ points $P \in X$ such that Q belongs to the osculating hyperplane H'_P to $f'(C)$ at $f(P)$ in \mathbb{P}^{r+1} . Projecting down to \mathbb{P}^r , these points correspond exactly to the points $P \in X$ such that osculating hyperplane H_P to $f(C)$ at $f(P)$ in \mathbb{P}^r has order of contact at least $r + 1$ at P , i.e., the real inflection points of C with respect to f . This proves the first assertion of Statement 1.

Now, let $P \in X$ be such that the osculating hyperplane $H_P \subseteq \mathbb{P}^r$ has order of contact to $f(C)$ at least $r + 1$ at $f(P)$. Since there are $g - 1$ other connected components of $C(\mathbb{R})$ such that $\deg_X(D)$ is odd, H_P intersects at least $g - 1$ other connected components of $C(\mathbb{R})$. Since $r + 1 + g - 1 = d$, the order of contact of H_P to $f(C)$ is equal to $r + 1$ at P . This finishes the proof of Statement 1.

In order to prove Statement 2, let $P \in X'$ and let H_P be the osculating hyperplane to $f(C)$ at $f(P)$. Since $\deg_Y(D)$ is odd for all g connected components Y of $C(\mathbb{R})$ different from X' , the degree of f^*H_P on X' is at most $d - g = r$. It follows that P is not an inflection point of C with respect to f . \square

Note that there are many divisors D satisfying the hypothesis of the theorem. Indeed, since $d - g = r$ is even, one can take $\deg_X(D) = 1$ for g connected components X of $C(\mathbb{R})$ and $\deg_X(D) = 0$ for 1 connected component X of $C(\mathbb{R})$.

Corollary 4.3. *Let C be an M -curve and let g be its genus. Let $r \geq 2$ be an even integer. Let D be a divisor of degree $d = g + r$ such that $\deg_X(D)$ is odd for exactly g connected components X of $C(\mathbb{R})$. Let $f : C \rightarrow \mathbb{P}^r$ be the morphism associated to D . Then C has exactly $g(r + 1)$ real inflection points with respect to f . Moreover, they are all ordinary. In particular, $f : C \rightarrow \mathbb{P}^r$ is ramified if $g \geq 1$. \square*

With notation as in the preceding corollary, Corollary 4.3 implies that there are at least $g(r + 1)$ hyperplanes H in $\mathbb{P}^r(\mathbb{R})$, for r even, such that the weight $w(f^*H)$ is at least r . It is easy to see that their weight is necessarily equal to r . It would then be interesting to determine exactly the number of hyperplanes of weight r .

Observe also that the real analytic curves $f(X)$ represent the non-trivial homology class in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ if $\deg_X(D)$ is odd. The real analytic curve $f(X)$ is homologically trivial in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ if $\deg_X(D)$ is even.

We then again have a converse to Theorem 4.2:

Theorem 4.4. *Let $d \geq r \geq 2$ be integers such that r is even. Let C be a non-degenerate real algebraic curve such that $C(\mathbb{R})$ has $d - r + 1$ connected components. Let $f : C \rightarrow \mathbb{P}^r$ be a birational embedding such that $f(C)$ is of degree d . Suppose that for exactly $d - r$ connected components X of $C(\mathbb{R})$, the image curve $f(X)$ represents the non-trivial homology class in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Then, C is an M -curve.*

Proof. Since $r - 1$ is odd, there is an integer t such that $2t + 1 = r - 1$. Choose closed points P_0, P_1, \dots, P_t on C in a general position such that P_1, \dots, P_t are non-real and P_0 is real and is on the connected component of $C(\mathbb{R})$ which is

homologically trivial in $\mathbb{P}^r(\mathbb{R})$. The smallest linear subspace V of \mathbb{P}^r containing P_0, \dots, P_t is of codimension 2. Let $f: C \rightarrow \mathbb{P}^1$ be the restriction to C of the projection from \mathbb{P}^r onto \mathbb{P}^1 with center V . Then, as in the proof of Theorem 3.2, one shows that f satisfies the conditions of Proposition 2.9. It follows that C is an M -curve. \square

Observe again that the hypotheses on the curve $C \subseteq \mathbb{P}^r$ of degree d in the preceding theorem imply that the genus of C is equal to $d - r$.

Corollary 4.5. *Let $d \geq r \geq 2$ be integers such that r is even. Let C be a non-degenerate real algebraic curve of genus g such that $C(\mathbb{R})$ has $d - r + 1$ connected components. Let $f: C \rightarrow \mathbb{P}^r$ be a birational embedding such that $f(C)$ is of degree d . Suppose that for exactly $d - r$ connected components X of $C(\mathbb{R})$, the image $f(X)$ represents the non-trivial homology class in $H_1(\mathbb{P}^r(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. Then, C has exactly $g(r + 1)$ real inflection points with respect to f and all of them are ordinary inflection points. In particular, f is ramified if $g \geq 1$.*

Proof. By Theorem 4.4, C is an M -curve. Hence, g is equal to $d - r$. Let $H \subseteq \mathbb{P}^r$ be a hyperplane and let D be the divisor f^*H on C . By the hypotheses, the degree of D is equal to d and the degree of D is odd on exactly g connected components of $C(\mathbb{R})$. By Theorem 2.5, the linear system $|D|$ is of dimension $d - g = r$. Therefore, the morphism f is the morphism associated to D . It follows from Corollary 4.3 that C has exactly $g(r + 1)$ real inflection points with respect to f . \square

In fact, the proof shows more. Indeed, since the morphism f is the morphism induced by the divisor D , the morphism f is, in fact, an embedding of C into \mathbb{P}^r if $r > 2$.

All results that we have established up to now seem to justify yet another conjecture:

Conjecture 4.6. Let $r \geq 2$ be an even integer. Let C be a real algebraic curve and let $f: C \rightarrow \mathbb{P}^r$ be a non-degenerate birational embedding. If f is unramified then either C is a rational curve and f is a rational normal map, or C is a twisted form of a rational curve and f is a twisted form of a rational normal map.

5. Morphisms of M -curves into \mathbb{P}^2

In this section we study more closely the birational embeddings of an M -curve C into the projective plane \mathbb{P}^2 that are associated to divisors having odd degree on all but one connected component of $C(\mathbb{R})$. The following statement summarizes previous results on morphisms of M -curves into \mathbb{P}^r for $r = 2$ and adds statements that are specific to the case $r = 2$.

Theorem 5.1. *Let C be an M -curve and let g be its genus. Let D be a divisor on C of degree $g + 2$ such that $\deg_X(D)$ is odd for exactly g connected components X of $C(\mathbb{R})$. Let X_0 be the connected component at which D has even degree and let X_1, \dots, X_g be the other connected components of $C(\mathbb{R})$. Let $f: C \rightarrow \mathbb{P}^2$ be the morphism associated to D . Then:*

1. The morphism f from C onto its image $f(C)$ is birational.
2. The image curve $f(C) \subseteq \mathbb{P}^2$ is of degree $g + 2$.
3. The restriction of f to any connected component of $C(\mathbb{R})$ is a real analytic embedding into $\mathbb{P}^2(\mathbb{R})$.
4. The real analytic curve $f(X_0) \subseteq \mathbb{P}^2(\mathbb{R})$ realizes the zero homology class in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. The real analytic curve $f(X_i) \subseteq \mathbb{P}^2(\mathbb{R})$ realizes the non-zero homology class in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ if $i \in \{1, \dots, g\}$.
5. For $i \neq j$ and $1 \leq i, j \leq g$, the real analytic curves $f(X_i)$ and $f(X_j)$ intersect in one point only. Moreover, this intersection is transversal.
6. The singularities of Statement 5 are the only singularities of $f(C)$. In particular, $f(C)$ has only split real ordinary multiple points as singularities. The total multiplicity of the singular locus of $f(C)$ is equal to $\frac{1}{2}g(g - 1)$.
7. Each real analytic curve $f(X_i)$, $i = 1, \dots, g$, has exactly 3 inflection points, all ordinary. The real analytic curve $f(X_0)$ contains no inflection points. In particular, the real algebraic curve C has exactly $3g$ real ordinary inflection points with respect to f .
8. Let $L \subseteq \mathbb{P}^2$ be a projective line. If the weight $w(f^*L)$ is greater than 1 then there is a real inflection point P of C such that L is tangent to $f(C)$ at $f(P)$. In particular, the curve $f(C)$ does not have real multitangent lines.

Proof. Statement 1 is part of Corollary 2.8. Statement 2 follows from the fact that D is of degree $g + 2$. Statement 3 follows from Corollary 2.7.2. Statement 4 follows from the fact that $\deg_{X_0}(D)$ is even and $\deg_{X_i}(D)$, $i \neq 0$, is odd. Statement 7 follows from Theorem 4.2 when one takes $r = 2$.

In order to show Statement 8, suppose that L is a line in \mathbb{P}^2 such that $w(L) > 1$. It follows from Statement 4 that $\deg_{X_i}(f^*L)$ is odd, for $i = 1, \dots, g$. Hence, the reduced divisor f^*L has degree at least g . Suppose that $\deg(f^*L) > g$. Then, $w(f^*L) < (g + 2) - g = 2$. This is a contradiction. Hence, $(f^*L)_{\text{red}}$ is exactly of degree g . It follows that the support of f^*L is contained in $C(\mathbb{R})$ and that $\text{Supp}(f^*L) \cap X_i$ consists of one point only, for $i \neq 0$. Since $\deg(f^*L) = g + 2$ and $\deg_{X_i}(f^*L)$ is odd, for $i \neq 0$, L is a tangent line to a real inflection point of C .

In order to show Statements 5 and 6, let μ be the total multiplicity of the singular locus of $f(C)$. It follows from Statement 4 that $f(X_i) \cap f(X_j) \neq \emptyset$ if $i, j \in \{1, \dots, g\}$. Since f is birational, $f(X_i) \neq f(X_j)$ if $i \neq j$. Therefore, all points in $f(X_i) \cap f(X_j)$ are singular points of $f(C)$, where $i \neq j$ and $1 \leq i, j \leq g$. It follows that $\mu \geq \frac{1}{2}g(g - 1)$. Since the image curve $f(C)$ is of degree $g + 2$, the normalization of $f(C)$ is of genus $\frac{1}{2}(g + 2 - 1)(g + 2 - 2) - \mu$. But, C is the normalization of $f(C)$. Therefore,

$$\begin{aligned} g &= \frac{1}{2}(g + 2 - 1)(g + 2 - 2) - \mu \\ &\leq \frac{1}{2}(g + 2 - 1)(g + 2 - 2) - \frac{1}{2}g(g - 1) = g. \end{aligned}$$

It follows that $\mu = \frac{1}{2}g(g - 1)$, i.e., $f(X_i)$ and $f(X_j)$ only intersect in one point $Q_{(i,j)}$, for $i \neq j$ and $1 \leq i, j \leq g$, this intersection is transversal, and the points $Q_{(i,j)}$ are the only singular points of $f(C)$. This shows statements 5 and 6. \square

If the genus of C is equal to 1, then the morphism f is an embedding of C into \mathbb{P}^2 as a non-singular real cubic curve having 2 real components. Statement 5

of Theorem 5.1 then corresponds to the well-known fact that the complement in $\mathbb{P}^2(\mathbb{R})$ of one of the connected components of $C(\mathbb{R})$ is connected whereas the complement of the other connected component is not connected.

Statement 7 of Theorem 5.1 is also well known for curves of genus 1: there is a structure of a real Lie group on $C(\mathbb{R})$ for which the inflection points of $C(\mathbb{R})$ correspond to the points of the 3-torsion subgroup [10]. The connected component X_1 of $C(\mathbb{R})$ is the neutral component of $C(\mathbb{R})$ and the group of connected components of $C(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence, the 3-torsion subgroup of $C(\mathbb{R})$ is equal to the 3-torsion subgroup of X_1 . Since X_1 is homeomorphic to S^1 , X_1 is isomorphic to the circle group. In particular, the 3-torsion subgroup of X_1 has exactly 3 elements. Therefore, X_1 contains exactly 3 inflection points and X_0 contains no inflection points.

One can try to generalize this proof to the case of an M -curve C of any genus g . It is not hard to define an additive law on $f(X_i)$, for given $i \neq 0$, by a similar geometric procedure as for the case $g = 1$. Unfortunately, this law satisfies all properties of a commutative group law except associativity, so that the proof breaks down, if $g > 1$.

Using Brusotti's Theorem on independent smoothing of ordinary double points [2], and the connectedness of the moduli space of M -curves of genus g [9], one has the following consequence:

Corollary 5.2. *Let g be a natural integer. Any M -curve of genus g is isomorphic to the normalization of a deformation of the union of g distinct lines in \mathbb{P}^2 and a conic in \mathbb{P}^2 having real points and intersecting each of the lines in a non-real point.* \square

An easy consequence of Theorem 5.1 is the following:

Corollary 5.3. *Let C be an M -curve and let g be its genus. Then C embeds into a real algebraic surface obtained from \mathbb{P}^2 by blowing up at most $\frac{1}{2}g(g-1)$ real points.* \square

Proof of Theorem 1.1. Let $f: \tilde{C} \rightarrow C$ be the normalization of C . Let g be the genus of \tilde{C} and let s be the number of connected components $C(\mathbb{R})$. Since $\tilde{C}(\mathbb{R})$ has $d-1$ global real analytic branches, $s = d-1$. Since at least $d-2$ real analytic branches of $\tilde{C}(\mathbb{R})$ represent the non-trivial homology class in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$, these branches intersect mutually. Hence, the multiplicity of the singular locus of \tilde{C} is at least $\frac{1}{2}(d-2)(d-3)$. Then,

$$g \geq \frac{1}{2}(d-1)(d-2) - \frac{1}{2}(d-2)(d-3) = d-2.$$

Hence, $s = d-1 \geq g+1$. But, by Harnack's inequality, $s \leq g+1$. Therefore, $s = g+1$, i.e., $g = d-2$. It follows that there are exactly $d-2$ branches of $\tilde{C}(\mathbb{R})$ representing the non-trivial homology class in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$. The statement now follows from Theorems 4.4 and 5.1. \square

References

1. Arbarello, E., Cornalba, M., Griffiths, P.A., Harris, J.: Geometry of algebraic curves. Vol. I, Grundlehren Math. Wiss. **267**, Springer Verlag 1985
2. Brusotti, L.: Sulla "piccola variazione" di una curva piana algebrica reali. Rend. Rom. Ac. Lincei. **30**, 375–379 (1921)
3. Ciliberto, C., Pedrini, C.: Real abelian varieties and real algebraic curves. Lectures in real geometry, F. Broglia, ed., 167–256 (1994)
4. Gross, B.H., Harris, J.: Real algebraic curves. Ann. Sci. Éc. Norm. Supér., IV. Sér. **14**, 157–182 (1981)
5. Harnack, A.: Über die Vieltheiligkeit der ebenen algebraischen Curven. Math. Ann. **10**, 189–198 (1876)
6. Huisman, J.: Real quotient singularities and non-singular real algebraic curves in the boundary of the moduli space. Compos. Math. **118**, 43–60 (1999)
7. Huisman, J.: On the geometry of algebraic curves having many real components. Rev. Mat. Complut. **14**, 83–92 (2001)
8. Milnor, J.W.: Topology from the differentiable viewpoint. Univ. Press Virginia 1965
9. Seppälä, M., Silhol, R.: Moduli spaces for real algebraic curves and real abelian varieties. Math. Z. **201**, 151–165 (1989)
10. Silverman, J.H.: The arithmetic of elliptic curves. Grad. Texts in Math. **106**, Springer Verlag 1986
11. Viro, O.Ya.: Some integral calculus based on Euler characteristic. Topology and geometry – Rohlin Seminar, LNM **1346**, 127–138, Springer Verlag 1988
12. Wall, C.T.C.: Duality of real projective plane curves: Klein's equation. Topology **35**, 355–362 (1996)