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ON THE NUMBER OF REAL HYPERSURFACES HYPERTANGENT TO A GIVEN REAL SPACE CURVE

J. HUISMAN

ABSTRACT. Let C be a smooth geometrically integral real algebraic curve in projective n-space $\mathbb{P}^n$. Let c be its degree and let g be its genus. Let d, s and m be nonzero natural integers. Let ν be the number of real hypersurfaces of degree d that are tangent to at least s real branches of C with order of tangency at least m. We show that ν is finite if $s = g$, $gm = cd$ and the restriction map $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(C, \mathcal{O}(d))$ is an isomorphism. Moreover, we determine explicitly the value of ν in that case.

1. Introduction

In real enumerative geometry, one often considers the number of real solutions of a complex enumerative problem defined over the reals [9, 7, 10]. In this paper, we study a purely real enumerative problem, i.e., the enumerative problem has no meaning over the complex numbers, or, when it is given a meaning over the complex numbers, it will have infinitely many complex solutions.

The enumerative problem we study is as follows. Let C be a smooth geometrically integral real algebraic curve in projective n-space $\mathbb{P}^n$, where $n \geq 2$. Let d, s and m be nonzero natural integers. Let ν be the number of real hypersurfaces of degree d that are tangent to at least s real branches of C with order of tangency at least m. We want to find conditions on d, s and m that imply that ν is finite, and possibly nonzero. Let c be the degree of C and let g be the genus of C. We show that ν is finite, and possibly nonzero, if $s = g$, $gm = cd$ and the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow H^0(C, \mathcal{O}(d))$$

is an isomorphism (Theorem 3.1). Moreover, we determine explicitly the value of ν in that case. As an example, let $C \subseteq \mathbb{P}^2$ be a smooth geometrically integral real quartic curve. Let ν be the number of real cubics tangent to at

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least 3 real branches of C with order of tangency at least 4. Then ν is finite, and $\nu = 64$ if C has exactly 3 real branches, and $\nu = 256$ if C has exactly 4 real branches (Example 4.4).

2. Divisors on real algebraic curves

We need to recall some facts about real algebraic curves.

Let C be a smooth proper geometrically integral real algebraic curve. A connected component of the set of real points $C(\mathbb{R})$ of C is called a *real branch* of C. Since C is smooth and proper, a real branch of C is necessarily homeomorphic to the unit circle. Let $\mathcal{B}$ be the set of real branches of C. Since C is proper, the set $\mathcal{B}$ is finite. By Harnack's Inequality [4], the cardinality of $\mathcal{B}$ is at most $g+1$, where g is the genus of C. Moreover, Harnack's Inequality is sharp, i.e., for any $g \in \mathbb{N}$ there are smooth proper geometrically integral real algebraic curves C of genus g having $g+1$ real branches.

Let D be a divisor on C. For a real branch B of C, let $\deg_B(D)$ denote the degree of D on B. Define an element

$$\delta(D) \in \operatorname{Hom}(\mathcal{B}, \mathbb{Z}/2\mathbb{Z})$$

by letting $\delta(D)(B) \equiv \deg_B(D) \pmod 2$ for all $B \in \mathcal{B}$. If E is a divisor on C which is linearly equivalent to D, then $\delta(E) = \delta(D)$. This follows from the elementary fact that the divisor of a nonzero rational function on C has even degree on any real branch. Denote again by δ the induced morphism

$$\delta\colon \operatorname{Pic}(C) \longrightarrow \operatorname{Hom}(\mathcal{B}, \mathbb{Z}/2\mathbb{Z})$$

from the Picard group $\operatorname{Pic}(C)$ into the group $\operatorname{Hom}(\mathcal{B}, \mathbb{Z}/2\mathbb{Z})$.

The group $\operatorname{Pic}(C)$ comes along with a natural topology. For $d \in \mathbb{Z}$, the subset $\operatorname{Pic}^d(C)$ of all divisor classes on C of degree d is open and closed in $\operatorname{Pic}(C)$. Two divisor classes $\mathbf{d}$ and $\mathbf{e}$ of degree d belong to the same connected component of $\operatorname{Pic}^d(C)$ if and only if $\delta(\mathbf{d}) = \delta(\mathbf{e})$ [1, §4.1]. For $\delta \in \operatorname{Hom}(\mathcal{B}, \mathbb{Z}/2\mathbb{Z})$, define

$$\operatorname{Pic}^{d,\delta}(C) = \{\mathbf{d} \in \operatorname{Pic}(C) \mid \deg(\mathbf{d}) = d \text{ and } \delta(\mathbf{d}) = \delta\}.$$

Then $\operatorname{Pic}^{d,\delta}(C)$ is nonempty if and only if $d \equiv \sum \delta(B) \pmod 2$, and in that case $\operatorname{Pic}^{d,\delta}(C)$ is a connected component of $\operatorname{Pic}^d(C)$. The neutral component of $\operatorname{Pic}(C)$ is $\operatorname{Pic}^{0,0}(C)$. It is a connected compact commutative real Lie group of dimension g. Each connected component of $\operatorname{Pic}(C)$ is a principal homogeneous space under the action of $\operatorname{Pic}^{0,0}(C)$.

Let $m \in \mathbb{Z}$, $m \neq 0$. Denote also by m the multiplication-by-m map on $\operatorname{Pic}(C)$. The kernel of the restriction of m to $\operatorname{Pic}^{0,0}(C)$ is obviously isomorphic to the group $(\mathbb{Z}/m\mathbb{Z})^g$. Moreover,

$$m \cdot \operatorname{Pic}^{d,\delta}(C) = \operatorname{Pic}^{md,m\delta}(C).$$

Recall from [5] the following statement:

THEOREM 2.1. *Let C be a smooth proper geometrically integral real algebraic curve. Let g be its genus. Let D be a divisor on C. Let d be the degree of D and let k be the number of real branches B of C such that $\deg_B(D)$ is odd. If $d+k>2g-2$ then D is nonspecial.* □

COROLLARY 2.2. *Let C be a smooth proper geometrically integral real algebraic curve. Let g be its genus. Suppose that C has at least g real branches. Let*

$$X=\bigcup_{\substack{\mathcal{B}'\subset\mathcal{B}\\ \#\mathcal{B}'=g}}\prod_{B\in\mathcal{B}'}B,$$

where the product is taken in some chosen order. Let

$$\varphi\colon X\longrightarrow \operatorname{Pic}^g(C)$$

be the map defined by letting $\varphi(P_1,\dots,P_g)$ be the divisor class of $P_1+\cdots+P_g$. Then φ is injective. Moreover, the image of φ is consists of all $\mathbf{e}\in\operatorname{Pic}^g(C)$ such that $\deg_B(\mathbf{e})\not\equiv 0 \pmod 2$ for exactly g real branches B of C.

Proof. Suppose that $\varphi(P_1,\dots,P_g)=\varphi(Q_1,\dots,Q_g)$. Let D be the divisor $P_1+\cdots+P_g$ and let E be the divisor $Q_1+\cdots+Q_g$. By hypothesis, E is linearly equivalent to D, i.e., $E\in|D|$. By Theorem 2.1, D is nonspecial. In particular, the dimension of the linear system $|D|$ is equal to $\deg(D)-g=0$. But D and E belong to $|D|$. Hence $D=E$. It follows that $(P_1,\dots,P_g)=(Q_1,\dots,Q_g)$. This shows that φ is injective.

Let $P\in X$. It is clear that $\deg_B(\varphi(P))\not\equiv 0 \pmod 2$ for exactly g real branches B of C. Conversely, suppose that $\mathbf{e}\in\operatorname{Pic}^g(C)$ is such that $\deg_B(\mathbf{e})\not\equiv 0 \pmod 2$ for exactly g real branches B of C. Since $\deg(\mathbf{e})=g$, there is an effective divisor E on C such that its class is equal to $\mathbf{e}$, by Riemann-Roch. Then $\deg(E)=g$ and $\deg_B(E)\neq 0$ for at least g real branches B of C. Since E is effective, there are real points $P_1,\dots,P_g$ of C, each on a different real branch of C, such that $E\geq P_1+\cdots+P_g$. But then $E=P_1+\cdots+P_g$. Let $P=(P_1,\dots,P_g)$. Then $P\in X$ and $\varphi(P)=\mathbf{e}$. □

3. Real space curves

Let $n\geq 2$ and let $C\subseteq\mathbb{P}^n$ be a smooth geometrically integral real algebraic curve. We say that C is *nondegenerate* if C is not contained in a real hyperplane of $\mathbb{P}^n$. We assume, in what follows, that C is nondegenerate.

Let X be a real branch of C. Let $[X]$ be the homology class of X in the first homology group $H_1(\mathbb{P}^n(\mathbb{R}),\mathbb{Z}/2\mathbb{Z})$. One says that X is a *pseudo-line* of C if $[X]\neq 0$. Otherwise, X is an *oval* of C. Equivalently, X is a pseudo-line of C if and only if each hyperplane H in $\mathbb{P}^n(\mathbb{R})$ intersects X in an odd number of points, when counted with multiplicities.

The main result of the paper is the following statement.

Theorem 3.1. *Let $n \geq 2$ be an integer. Let C be a nondegenerate smooth geometrically integral real algebraic curve in $\mathbb{P}^n$. Let c be its degree and let g be its genus. Suppose that C has at least g real branches. Let d be a nonzero natural integer such that the restriction map*

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow H^0(C, \mathcal{O}(d))$$

is an isomorphism. Suppose that there is a nonzero natural integer m such that $gm = cd$. Let ν be the number of real hypersurfaces D in $\mathbb{P}^n$ of degree d such that D is tangent to at least g real branches of C with order of contact at least m. Then ν is finite. Moreover, $\nu \neq 0$ if and only if

(1) *m and d are odd, and C has exactly g pseudo-lines, or*
(2) *m is even and either d is even or all real branches of C are ovals.*

Furthermore, in case (1), $\nu = m^g$, and, in case (2),

$$\nu = \begin{cases} (g+1)\cdot m^g & \text{if } C \text{ has } g+1 \text{ real branches,} \\ m^g & \text{if } C \text{ has } g \text{ real branches.} \end{cases}$$

Proof. We have to determine the number ν of real hypersurfaces D in $\mathbb{P}^n$ of degree d such that the intersection divisor $D \cdot C$ satisfies $D \cdot C \geq m(P_1 + \cdots + P_g)$, for some real points $P_1, \ldots, P_g$ of C, each on a different real branch of C. Since $D \cdot C$ is of degree $dc = mg$, the condition $D \cdot C \geq m(P_1 + \cdots + P_g)$ is equivalent to the condition $D \cdot C = m(P_1 + \cdots + P_g)$. Therefore we have to determine the number ν of real hypersurfaces D in $\mathbb{P}^n$ of degree d such that $D \cdot C = m(P_1 + \cdots + P_g)$, for some real points $P_1, \ldots, P_g$ of C, each on a different real branch of C. By the hypothesis on d, the number ν is equal to the number of divisors E on C of the form $P_1 + \cdots + P_g$ (where, again, each P_i is a real point on a different real branch of C) such that $m \cdot E$ belongs to the linear system $|dH|$ on C. Let $\mathbf{h}$ be the divisor class of the hyperplane section H on C. By Corollary 2.2, the number ν is also equal to the number of $\mathbf{e} \in \mathrm{Pic}^g(C)$ such that $m \cdot \mathbf{e} = d \cdot \mathbf{h}$ and $\deg_B(\mathbf{e}) \not\equiv 0 \pmod 2$ for exactly g real branches B of C. In particular, since $m \neq 0$, one has that ν is finite.

Now, suppose that $\nu \neq 0$. Then there is an $\mathbf{e} \in \mathrm{Pic}^g(C)$ such that $m \cdot \mathbf{e} = d \cdot \mathbf{h}$ and such that $\deg_B(\mathbf{e}) \neq 0$ for exactly g real branches B of C.

Suppose that m is odd. Then $m \cdot \mathbf{e} = d \cdot \mathbf{h}$ implies that $d \deg_B(\mathbf{h}) \not\equiv 0 \pmod 2$ for exactly g real branches B of C. In particular, d is odd and C has exactly g pseudo-lines, i.e., we are in case (1). Since m is odd, the connected component of $\mathrm{Pic}(C)$ containing $\mathbf{e}$ is the only connected component of $\mathrm{Pic}(C)$ whose image by the multiplication-by-m map is equal to the connected component of $\mathrm{Pic}(C)$ that contains $d \cdot \mathbf{h}$. Hence ν is equal to m^g.

Suppose that m is even. Then $m \cdot \mathbf{e} = d \cdot \mathbf{h}$ implies that $d \deg_B(\mathbf{h}) \equiv 0 \pmod 2$ for all real branches B of C. In particular, either d is even, or all real branches of C are ovals, i.e., we are in case (2). If C has exactly g real branches then the connected component $\mathrm{Pic}^{g,\delta}(C)$ of $\mathrm{Pic}(C)$ containing $\mathbf{e}$ is the only connected component of $\mathrm{Pic}(C)$ whose image by the multiplication-by-m map is equal to the connected component of $\mathrm{Pic}(C)$ that contains $d \cdot \mathbf{h}$, and such that $\delta(B) \neq 0$ for exactly g real branches B of C. Hence $\nu = m^g$ if C has exactly g real branches. If C has exactly $g+1$ real branches then there are exactly $g+1$ connected components $\mathrm{Pic}^{g,\delta}(C)$ of $\mathrm{Pic}(C)$ whose image by the multiplication-by-m map is equal to the connected component of $\mathrm{Pic}(C)$ that contains $d \cdot \mathbf{h}$, and such that $\delta(B) \neq 0$ for exactly g real branches B of C. Hence $\nu = (g+1) \cdot m^g$ if C has exactly $g+1$ real branches.

We have shown, in particular, that m, d and C satisfy condition (1) or (2) if $\nu \neq 0$. It is clear that, conversely, if m, d and C satisfy condition (1) or (2) then $\nu \neq 0$. □

Remark 3.2. As we have seen in the proof above, if D is a real hypersurface in $\mathbb{P}^n$ of degree d that is tangent to at least g real branches of C with order of contact at least m then D is tangent to exactly g real branches of C with order of contact exactly equal to m. Moreover, D intersects each of these g real branches in exactly one point. Furthermore, all intersection points of D and C are real.

Remark 3.3. If $m = 1$ in Theorem 3.1, then, according to the preceding remark, all real hypersurfaces of degree d that intersect at least g real branches of C, intersect each of these real branches transversely. Hence, if there is one such hypersurface, then there should be infinitely many. Therefore there are no real hypersurfaces D of degree d that intersect at least g real branches of C, i.e., $\nu = 0$.

Remark 3.4. Observe that the curve C of the statement of Theorem 3.1 is necessarily nonrational. A nonrational nondegenerate smooth curve in $\mathbb{P}^n$ has degree strictly greater than n. Hence $c > n$ in Theorem 3.1.

Remark 3.5. Let $n \geq 2$ and let $C \subseteq \mathbb{P}^n$ be a nondegenerate smooth geometrically integral real algebraic curve. Let c be its degree and let g be its genus. According to [6, Corollary 5.2], C has at least $n+1$ ovals if $g > c - n$. Therefore case (1) of Theorem 3.1 only occurs when $g \leq c - n$.

According to Remark 3.3, the most interesting applications of Theorem 3.1 are to be expected when $m \geq 2$. Here is a reformulation of Theorem 3.1 in that case.

Corollary 3.6. *Let $n \geq 2$ be an integer. Let C be a nondegenerate smooth geometrically integral real algebraic curve in $\mathbb{P}^n$. Let c be its degree and let g be its genus. Suppose that C has at least g real branches. Let d and m be nonzero natural integers, $m \geq 2$, such that*

$$\binom{n+d}{d} = cd - g + 1 \quad \textit{and} \quad gm = cd.$$

Suppose that

(1) *C is not contained in a real hypersurface of degree d, or*
(2) *the restriction map $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(C, \mathcal{O}(d))$ is surjective.*

Let ν be the number of real hypersurfaces D in $\mathbb{P}^n$ of degree d such that D is tangent to at least g real branches of C with order of contact at least m. Then ν is finite. Moreover, $\nu \neq 0$ if and only if

(1) *m and d are odd, and C has exactly g pseudo-lines, or*
(2) *m is even and either d is even or all real branches of C are ovals.*

Furthermore, in case (1), *$\nu = m^g$, and, in case* (2),

$$\nu = \begin{cases} (g+1)\cdot m^g & \text{if } C \text{ has } g+1 \text{ real branches,} \\ m^g & \text{if } C \text{ has } g \text{ real branches.} \end{cases}$$

Proof. By hypothesis, the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow H^0(C, \mathcal{O}(d))$$

is either injective or surjective. One has

$$\dim H^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d}.$$

Moreover, since $cd = gm \geq 2g > 2g-2$, the invertible sheaf $\mathcal{O}(d)$ on C is nonspecial. In particular,

$$\dim H^0(C, \mathcal{O}(d)) = cd - g + 1,$$

by Riemann-Roch. It follows from the hypothesis that $\dim H^0(\mathbb{P}^n, \mathcal{O}(d)) = \dim H^0(C, \mathcal{O}(d))$. Hence the above restriction map is an isomorphism. Therefore all conditions of Theorem 3.1 are satisfied. □

Example 3.7. Let g be a nonzero natural integer. Let C be a smooth proper geometrically integral real algebraic curve of genus g having at least g real branches. Let c be a nonzero multiple of g, $c > g$, such that there is a nonspecial very ample divisor D on C of degree c. Let n be the dimension of the linear system $|D|$. Identify C with the image of the induced embedding of C into $\mathbb{P}^n$. Since D is nonspecial, $n = c - g$. Put $d = 1$ and $m = c/g$. Then the conditions of Corollary 3.6 are satisfied. Let ν be the number of real hyperplanes in $\mathbb{P}^n$ that are tangent to at least g real branches of C with order of contact at least m. Then ν is finite. Moreover, $\nu \neq 0$ if and only if

(1) m is odd, and C has exactly g pseudo-lines, or
(2) m is even and all real branches of C are ovals.

Furthermore, in case (1), $\nu = m^g$, and, in case (2),

$$\nu = \begin{cases} (g+1)\cdot m^g & \text{if } C \text{ has } g+1 \text{ real branches,} \\ m^g & \text{if } C \text{ has } g \text{ real branches.} \end{cases}$$

EXAMPLE 3.8. Let $C \subseteq \mathbb{P}^3$ be a nondegenerate smooth geometrically integral real algebraic curve of degree $c = 5$ and of genus $g = 1$. Since C is of odd degree, $C(\mathbb{R}) \neq \emptyset$. Hence C has at least one real branch. Put $d = 2$ and $m = 10$. Since C is nonrational and $d \geq c-3$, the curve C is not contained in a real quadric surface [2]. Therefore we can apply Corollary 3.6 in order to conclude that, if C has exactly one real branch, there are exactly 10 real quadrics in $\mathbb{P}^3$ that are tangent to C with order of contact at least 10. If C has exactly 2 real branches then there are 20 such real quadrics.

4. Real plane curves

If one specializes Corollary 3.6 to the case of real plane curves, one gets the following statement.

COROLLARY 4.1. *Let C be a nondegenerate smooth geometrically integral real algebraic curve in $\mathbb{P}^2$. Let c be its degree. The genus g of C is equal to $\frac{1}{2}(c-1)(c-2)$. Suppose that C has at least g real branches. Let d and m be nonzero natural integers, $d < c$, such that*

$$\tfrac{1}{2}(d+2)(d+1) = cd - g + 1 \quad \text{and} \quad mg = cd.$$

Let ν be the number of real curves D in $\mathbb{P}^2$ of degree d such that D is tangent to at least g real branches of C with order of contact at least m. Then ν is finite. Moreover, $\nu \neq 0$ if and only if

(1) *C is a real cubic, i.e., $c = 3$, $g = 1$, $d = 1$ and $m = 3$, or*
(2) *m is even and, either d or c is even.*

Furthermore, in case (1), $\nu = 3$, and, in case (2),

$$\nu = \begin{cases} (g+1) \cdot m^g & \text{if } C \text{ has } g+1 \text{ real branches,} \\ m^g & \text{if } C \text{ has } g \text{ real branches.} \end{cases}$$

Proof. It is clear that the conditions of Corollary 3.6 are satisfied with $n = 2$. Therefore Corollary 3.6 applies. Note that a smooth real algebraic curve in $\mathbb{P}^2$ has at most 1 pseudo-line. This shows that case (1) of Theorem 3.1 only occurs when C is a real cubic, and then $c = 3$, $g = 1$, $d = 1$ and $m = 3$. By the same argument, case (2) of Theorem 3.1 only occurs when C is of even degree. □

REMARK 4.2. The integer m in Corollary 4.1 necessarily satisfies $m \geq 3$. Indeed, suppose that $m \leq 2$. Since

$$cd = gm = \tfrac{1}{2}m \cdot (c-1)(c-2),$$

c divides $(c-1)(c-2)$. Hence c divides $c-2$ and therefore $c = 2$. This contradicts Remark 3.4, i.e., $m \geq 3$.

Example 4.3. Let C be a smooth real cubic in $\mathbb{P}^2$. Then $c = 3$ and $g = 1$. The only values for (d, m) that satisfy the conditions of Corollary 4.1 are $(1, 3)$ and $(2, 6)$. If $(d, m) = (1, 3)$ then Corollary 4.1 is the well known fact that a real cubic has exactly 3 real inflection points [8]. If $(d, m) = (2, 6)$ then Corollary 4.1 states that there are either 6 or 12 real quadrics tangent to a given real cubic with order of tangency equal to 6. This can also be shown directly.

Example 4.4. Let C be a smooth real quartic in $\mathbb{P}^2$. Assume that C has at least 3 real branches. Such real plane curves abound. Indeed, let C be a nonhyperelliptic smooth proper geometrically integral real algebraic curve of genus 3 having at least 3 real branches. Then the image of the canonical embedding, again denoted by C, is a real quartic having the above properties.

Put $c = 4$ and $g = 3$. The only values of d and m that satisfy the conditions of Corollary 4.1 are $d = 3$ and $m = 4$. Then, by Corollary 4.1, the number of real cubics tangent to at least 3 real branches of C with order of contact at least 4, is equal to 64 if C has 3 real branches, and 256 if C has 4 real branches.

Example 4.5. Let C be a smooth real sextic in $\mathbb{P}^2$. Assume that C has at least 10 real branches. Such curves abound [3]. Put $c = 6$ and $g = 10$. The only values of d and m that satisfy the conditions of Corollary 4.1 are $d = 5$ and $m = 3$. Then, by Corollary 4.1, no real quintic in $\mathbb{P}^2$ is tangent to at least 10 real branches of C with order of contact at least 3. This can also be shown directly.

Proposition 4.6. *The following values for (c, d, m) are the only ones satisfying the conditions of Corollary 4.1:*

$$(3, 1, 3),\ (3, 2, 6),\ (4, 3, 4),\ (6, 5, 3).$$

Proof. We have already seen in the preceding examples that these values of (c, d, m) satisfy the conditions of Corollary 4.1.

Conversely, suppose that (c, d, m) satisfies the conditions of Corollary 4.1. We distinguish the cases c even and c odd.

Suppose that c is even. Then $cd = m(c-1) \cdot \frac{1}{2}(c-2)$. Since $c-1$ and c are coprime, $c-1$ divides d. Since $d < c$, $d = c-1$ and $c(c-1) = \frac{1}{2}m(c-1)(c-2)$. Since $c \neq 1$, one gets $c = \frac{1}{2}m(c-2)$, i.e., $2c = m(c-2)$. In particular, $c-2$ divides $2c$. Since $c-2$ also divides $2c-4$, one has that $c-2$ divides 4, i.e., $c-2 = 1, 2$, or 4. Then $c = 3$, 4, or 6. We have already above in the preceding examples that then the value of (c, d, m) is necessarily one of the list above.

Suppose that c is odd. Then $cd = m \cdot \frac{1}{2}(c-1) \cdot (c-2)$. Since $c-2$ and c are coprime, $c-2$ divides d. Since $d < c$, $d = c-2$, or $d = c-1$ and $c = 3$. Since we have already treated the case $c = 3$ above, we may assume that $d = c-2$. Then $c(c-2) = \frac{1}{2}m(c-1)(c-2)$. Since $c \neq 2$

(Remark 3.4), one gets $c = \frac{1}{2}m(c-1)$, i.e., $2c = m(c-1)$. In particular, $c-1$ divides $2c$. Since $c-1$ also divides $2c-2$, one has that $c-1$ divides 2, i.e., $c-1=1$, or 2. Then $c=2$, or 3. Since $c \neq 2$, $c=3$, which case has already been dealt with above. □

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Institut de Recherche Mathématique de Rennes, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France

E-mail address: huisman@univ-rennes1.fr