MODULES OVER TWISTED GROUP RINGS AND VECTOR BUNDLES OVER THE ANISOTROPIC REAL CONIC

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Abstract. We prove, in an elementary way, that a locally free sheaf of finite rank over the anisotropic real conic is the direct sum of indecomposable locally free sheaves of rank 1 or 2. Our proof is purely algebraic, and is based on a classification of graded $\mathbb{C}[X,Y]$-modules endowed with a certain action of the cyclic group $\mathbb{Z}/4\mathbb{Z}$.

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1. Introduction

Let Q be the anisotropic real conic, i.e., Q is the real curve defined by the equation $U^2+V^2+W^2=0$ in the real projective plane $\mathbb{P}^2$. It is known that a locally free sheaf of finite rank over Q admits a unique decomposition as a direct sum of some explicit indecomposable locally free sheaves of rank 1 or 2. The literature contains at least two proofs of this fact ([4, §2], [1]), but none of them can be considered to be really elementary.

In this article, we present an elementary and purely algebraic proof, that makes use of the Theorem of Grothendieck on vector bundles over the complexification $Q_{\mathbb{C}}$ of Q, as do the other proofs mentioned above. Our proof has the interesting feature that actions of a cyclic group of 4 elements intervene, which is rather unusual in real algebraic geometry where the cyclic group of order 2 reigns for more than a century.

Let A be the homogeneous coordinate ring of Q. We classify graded A-modules that are locally free of finite rank, and obtain the above statement as a consequence. As usual in real algebraic geometry, the classification of such A-modules is obtained from the classification of locally free graded $\mathbb{C}\otimes_{\mathbb{R}}A$-modules of finite rank that come along with an action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. In order to classify the latter modules, we observe the following.

The complexification $Q_{\mathbb{C}}$ of Q is isomorphic to the complex projective line $\mathbb{P}^1$. The action of complex conjugation on $Q_{\mathbb{C}}$ corresponds to the involution $[x:y]\mapsto[-\overline{y}:\overline{x}]$ on the set of complex points $\mathbb{P}^1(\mathbb{C})$ of $\mathbb{P}^1$. On the homogeneous coordinate ring $\mathbb{C}[X,Y]$ of $\mathbb{P}^1$ this is the automorphism $P\mapsto\overline{P}(-Y,X)$. Now, this automorphism is of order 4, and not of order 2. Therefore, we get an action of the cyclic group $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{C}[X,Y]$. We obtain the classification of the above mentioned $\mathbb{C}\otimes_{\mathbb{R}}A$-modules, as a consequence of the classification of locally free graded $\mathbb{C}[X,Y]$-modules of finite rank that are equipped with an action of $\mathbb{Z}/4\mathbb{Z}$. The fact that the group that acts is $\mathbb{Z}/4\mathbb{Z}$, and not $\mathbb{Z}/2\mathbb{Z}$, explains somehow the existence of those mysterious indecomposable locally free sheaves of rank 2 on the real curve Q.

Convention. All actions, modules and algebras will be left actions, left modules and left algebras, respectively.

2. Modules over the twisted group ring $\mathbb{C}(\mathbb{Z}/4\mathbb{Z})$

Let G be a cyclic group of order 4, and let σ be a generator. Define an action of G on the $\mathbb{R}$-algebra $\mathbb{C}$ of complex numbers by letting $\sigma\alpha$ be the complex conjugate $\overline{\alpha}$ of α, for all $\alpha \in \mathbb{C}$. Note that the action of G on $\mathbb{C}$ is not faithful.

We will study complex vector spaces M that are endowed with an $\mathbb{R}$-linear action of G such that $\sigma(\alpha x) = (\sigma\alpha)(\sigma x)$, for all $\alpha \in \mathbb{C}$ and for all $x \in M$. Equivalently, we will study modules over the twisted group ring $\mathbb{C}(G)$.

Recall that the twisted group ring $\mathbb{C}(G)$ of $\mathbb{C}$ is the $\mathbb{C}$-vector space with basis the set G. There is a unique $\mathbb{C}$-algebra structure on $\mathbb{C}(G)$ if one defines the product $\cdot$ on $\mathbb{C}(G)$ by $\sigma^i \cdot \sigma^j = \sigma^{i+j}$ and $\sigma \cdot \alpha = (\sigma\alpha) \cdot \sigma$, for all $i, j \in \{0, 1, 2, 3\}$ and $\alpha \in \mathbb{C}$. It is clear that a complex vector space, which is endowed with a G action as above, is naturally a $\mathbb{C}(G)$-module, and conversely. Also, a morphism between two such vector spaces corresponds to a morphism of $\mathbb{C}(G)$-modules, and conversely.

The action of G on $\mathbb{C}$ defines the structure of a $\mathbb{C}(G)$-module on $\mathbb{C}$. When we speak of $\mathbb{C}$ as a $\mathbb{C}(G)$ module, we mean that particular structure of a $\mathbb{C}(G)$-module. A less trivial example of a $\mathbb{C}(G)$-module is the $\mathbb{C}(G)$-module induced by the action of G on $\mathbb{C}^2$ defined by $\sigma(x, y) = (-\overline{y}, \overline{x})$. We denote that particular $\mathbb{C}(G)$-module by V.

Let M be a $\mathbb{C}(G)$-module. Define $f \colon M \to M$ by $f(x) = \sigma^2 x$ for all $x \in M$. It is easy to see that f is a $\mathbb{C}$-linear endomorphism of M with the property that $f^2 = \mathrm{id}$. We denote by M_1 and M_{-1} the eigenspaces of f for the eigenvalues 1 and -1, respectively. Since f commutes with the action of σ on M, the sub-vector spaces M_1 and M_{-1} of M are, in fact, sub-$\mathbb{C}(G)$-modules of M. Therefore, every $\mathbb{C}(G)$-module M decomposes as a direct sum of its $\mathbb{C}(G)$-submodules M_1 and M_{-1}.

Let M be a $\mathbb{C}(G)$-module. The sub-$\mathbb{C}(G)$-modules M_1 and M_{-1} are natural. Indeed, if $g \colon M \to N$ is a morphism of $\mathbb{C}(G)$-modules, then g commutes with the action of σ^2 on M and N, and therefore, $g(M_1) \subseteq N_1$ and $g(M_{-1}) \subseteq N_{-1}$.

Proposition 1. *Let M be a $\mathbb{C}(G)$-module of finite type. Then there are natural integers m, n such that M_1 is isomorphic to the $\mathbb{C}(G)$-module $\mathbb{C}^m$, and M_{-1} is isomorphic to the $\mathbb{C}(G)$-module V^n. In particular, M is isomorphic to the $\mathbb{C}(G)$-module $\mathbb{C}^m \oplus V^n$. Moreover, the integers m and n are uniquely determined by M.*

Proof. Let us first prove uniqueness of m and n. Suppose that M is isomorphic to $\mathbb{C}^m \oplus V^n$, for some natural integers m and n. Then, by naturality, $(\mathbb{C}^m \oplus V^n)_1$ is isomorphic to M_1. But, $(\mathbb{C}^m \oplus V^n)_1 = \mathbb{C}^m$. Therefore, m is the complex dimension of M_1, and is, indeed, uniquely determined by M. Similarly, $2n$ is equal to the complex dimension of M_{-1}, and is also uniquely determined by M. This proves uniqueness.

Next, we prove existence of the natural integers m and n. It suffices to prove that there are natural integers m and n such that $M_1 \cong \mathbb{C}^m$ and $M_{-1} \cong V^n$.

Let g be the $\mathbb{R}$-linear endomorphism of M defined by $g(m) = \sigma m$, for all $m \in M$. Since g commutes with the endomorphism f defined above, one has $g(M_1) \subseteq M_1$ and $g(M_{-1}) \subseteq M_{-1}$. Since $g^2 = f$ and since the restriction of f to M_1 is the identity, M_1 is the direct sum of the real eigenspaces $M_{1,1}$ and $M_{1,-1}$ of the restriction of g to M_1. Since $iM_{1,1} = M_{1,-1}$, the complexification $\mathbb{C} \otimes_{\mathbb{R}} M_{1,1}$ of $M_{1,1}$ is $\mathbb{C}$-linearly isomorphic to M_1. Even more is true, the complexification of $M_{1,1}$ is a $\mathbb{C}(G)$-module in a natural way, and the $\mathbb{C}$-linear isomorphism is, in fact, $\mathbb{C}(G)$-linear. Since M is of finite type as a $\mathbb{C}(G)$-module, M is finite dimensional as a complex vector space. It follows that $M_{1,1}$ is finite-dimensional as a real vector space. Hence, there is a natural integer m such that $M_{1,1}$ is isomorphic to $\mathbb{R}^m$ as real vector spaces. Then, $\mathbb{C}^m$ is $\mathbb{C}(G)$-linearly isomorphic to $\mathbb{C} \otimes M_{1,1}$. It follows that M_1 is isomorphic to $\mathbb{C}^m$, as a $\mathbb{C}(G)$-module.

Finally, we show that M_{-1} is isomorphic to V^n, for some natural integer n. Since $g^2 = f$ and since the restriction of f to M_{-1} is $-\mathrm{id}$, the finite-dimensional vector space M_{-1} is easily seen to possess a basis $x_1, \ldots, x_{2n}$, for some natural integer n, that has the property that $g(x_{2i-1}) = x_{2i}$ and $g(x_{2i}) = -x_{2i-1}$, for $i = 1, \ldots, n$. It follows that M_{-1} is $\mathbb{C}(G)$-linearly isomorphic to V^n. □

Proposition 2. *Let* $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ *be a short exact sequence of* $\mathbb{C}(G)$*-modules of finite type. Then this sequence is split.*

Proof. By naturality, one has two induced short exact sequences of $\mathbb{C}(G)$-modules of finite type

$$0 \longrightarrow L_1 \xrightarrow{\varphi_1} M_1 \xrightarrow{\psi_1} N_1 \longrightarrow 0$$

and

$$0 \longrightarrow L_{-1} \xrightarrow{\varphi_{-1}} M_{-1} \xrightarrow{\psi_{-1}} N_{-1} \longrightarrow 0$$

It suffices to prove that they are both split. For the first exact sequence, this is trivial since it is isomorphic to the complexification of the short exact sequence of real vector spaces

$$0 \longrightarrow L_{1,1} \xrightarrow{\varphi_{1,1}} M_{1,1} \xrightarrow{\psi_{1,1}} N_{1,1} \longrightarrow 0$$

with notation as in the proof of Proposition 1. For the second exact sequence, observe that, by Proposition 1, there is a complex basis $x_1, \ldots, x_{2n}$ of N_{-1} such that $\sigma x_{2i-1} = x_{2i}$ and $\sigma x_{2i} = -x_{2i-1}$, for $i = 1, \ldots, n$. Choose $y_1, y_3, \ldots, y_{2n-1} \in M_{-1}$ such that $\psi_{-1}(y_{2i-1}) = x_{2i-1}$, for $i = 1, \ldots, n$. Put $y_{2i} = \sigma y_{2i-1}$, for all $i = 1, \ldots, n$. Since ψ_{-1} is $\mathbb{C}(G)$-linear, one has $\psi_{-1}(y_i) = x_i$, for all i. The complex subvector space K of M_{-1} generated by $y_1, \ldots, y_{2n}$ is a sub-$\mathbb{C}(G)$-module of M_{-1}, and the restriction of ψ_{-1} to K is a $\mathbb{C}(G)$-linear isomorphism onto N_{-1}. It follows that the second exact sequence is also split. □

3. Modules over the graded twisted group ring $\mathbb{C}[X,Y](\mathbb{Z}/4\mathbb{Z})$

Let $B = \mathbb{C}[X,Y]$ be the graded $\mathbb{C}$-algebra of polynomials in X, Y with coefficients in $\mathbb{C}$. If P is a complex polynomial, $\overline{P}$ denotes the complex conjugate polynomial. We define a left action of G on the graded $\mathbb{R}$-algebra B by

$$P^\sigma = \overline{P}(-Y, X)$$

for all $P \in B$. Let $B(G)$ be the twisted group ring, i.e., $B(G)$ is the free B-module on the set G. It is a B-algebra, and one has $\sigma^i \cdot \sigma^j = \sigma^{i+j}$ and $\sigma \cdot P = P^\sigma \cdot \sigma$ in $B(G)$, for all $i, j \in \{0, 1, 2, 3\}$ and $P \in B$. The B-algebra $B(G)$ is naturally graded. The twisted group ring $B(G)$ contains $\mathbb{C}(G)$ as a subring.

Let M be a graded $B(G)$-module. If d is an integer, M_d denotes the complex subvector space of M of all homogeneous elements of M of degree d. It is clear that M_d is a $\mathbb{C}(G)$-module. For example, B itself is a graded $B(G)$-module, and B_d is the $\mathbb{C}(G)$-module of all homogeneous polynomials in $\mathbb{C}[X,Y]$ of degree d.

If M is a graded $B(G)$-module and n is a natural integer, then $M[n]$ denotes the twisted graded $B(G)$-module defined by $M[n]_i = M_{n+i}$ for all $i \in \mathbb{Z}$. In the sequel, the twisted $B(G)$-modules $B[n]$ will play an important rôle.

Let M be a $\mathbb{C}(G)$-module. The B-module $B \otimes_{\mathbb{C}} M$ becomes a $B(G)$-module if one defines the action of G on $B \otimes_{\mathbb{C}} M$ by $\sigma(P \otimes m) = P^\sigma \otimes (\sigma m)$. This $B(G)$-module is naturally graded. One has $(B \otimes_{\mathbb{C}} M)[n] = B[n] \otimes_{\mathbb{C}} M$, for all integers n.

We need to recall some facts from algebraic geometry (see [3] for details). Let $\mathbb{P}^1$ be the complex projective line $\mathrm{Proj}(B)$, and let $\mathcal{O}$ be the structure sheaf on $\mathbb{P}^1$.

If M is a graded B-module, then one has an associated quasi-coherent sheaf $\widetilde{M}$ of $\mathcal{O}$-modules over $\mathbb{P}^1$. Conversely, if $\mathcal{E}$ is a quasi-coherent sheaf of $\mathcal{O}$-modules on $\mathbb{P}^1$, then

$$\Gamma_\star(\mathcal{E}) = \bigoplus_{d\in\mathbb{Z}} H^0(\mathbb{P}^1, \mathcal{O}(d) \otimes_{\mathcal{O}} \mathcal{E}).$$

is a graded B-module. One has a natural isomorphism between the sheaves $\widetilde{\Gamma_\star(\mathcal{E})}$ and $\mathcal{E}$ [3, Prop II.5.15]. The natural morphism of graded B-algebras from M into $\Gamma_\star(\widetilde{M})$ is not necessarily an isomorphism.

Let M be a graded B-module. we will say that M is *locally free of finite rank* if there is a locally free sheaf $\mathcal{E}$ of $\mathcal{O}$-modules of finite rank on $\mathbb{P}^1$ such that M is isomorphic to $\Gamma_\star(\mathcal{E})$. It follows from what has been said above that, if M is a locally free graded B-module of finite rank, then $\widetilde{M}$ is a locally free sheaf of finite rank over $\mathbb{P}^1$. For such a module M, the natural morphism from M into $\Gamma_\star(\widetilde{M})$ is an isomorphism. Therefore, the functor $M \mapsto \widetilde{M}$ is an equivalence from the category of locally free graded B-modules of finite rank, onto the category of locally free sheaves of finite rank over $\mathbb{P}^1$. The inverse functor is the functor $\mathcal{E} \mapsto \Gamma_\star(\mathcal{E})$.

Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules of finite rank over $\mathbb{P}^1$. A Theorem of Grothendieck states that there are integers $d_1, \ldots, d_n$ such that $\mathcal{E}$ is isomorphic to the direct sum of the invertible sheaves $\mathcal{O}(d_i)$ on $\mathbb{P}^1$ [2]. Moreover, the integers $d_1, \ldots, d_n$ are uniquely determined by $\mathcal{E}$, up to permutation. In particular, the graded B-module $\Gamma_\star(\mathcal{E})$ is isomorphic to the direct sum of the graded B-modules $B[d_i]$.

By the Theorem of Grothendieck, a graded B-module M is locally free of finite rank if and only if there are integers $d_1, \ldots, d_n$ such that M is isomorphic to the direct sum of the graded B-modules $B[d_i]$. Moreover, the integers $d_1, \ldots, d_n$ are uniquely determined by M, up to permutation.

Let M be a graded B-module that is locally free of finite rank. We will say that M is *gradually pure* if all the integers d_i above are equal. Equivalently, M is *gradually pure* if M is isomorphic to the graded B-module $B[d]^n$, for some $n \in \mathbb{N}$ and some $d \in \mathbb{Z}$. In that case, d is called the *degree* of the gradually pure B-module M. The theorem of Grothendieck above states that M decomposes, in a unique way, as a direct sum of gradually pure locally free graded B-modules of finite rank and of distinct degrees.

Proposition 3. *Let M be a graded $B(G)$-module which is locally free of finite rank as a graded B-module. Then there are integers $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ such that*

$$M \cong \left(\bigoplus_{i=1}^{m} B[a_i]\right) \oplus \left(\bigoplus_{i=1}^{n} (B[b_i] \otimes_{\mathbb{C}} V)\right),$$

where V is the $\mathbb{C}(G)$-module defined above. The integers $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ are uniquely determined by M, up to permutation.

Proof. We start be showing that M admits a decomposition as a direct sum of graded sub-$B(G)$-modules K_i, for $i = 1, \ldots, n$, such that each K_i is locally free of finite rank as a graded B-module and gradually pure. We may assume that $M \neq \{0\}$. In order to construct K_1, let d_1 be the smallest integer such that $M_{d_1} \neq \{0\}$. Such an integer exists since M is locally free of finite rank as a graded B-module. The complex subvector space M_{d_1} of M is, in fact, a sub-$\mathbb{C}(G)$-module of M. Let K_1 be the sub-B-module of M generated by M_{d_1}. It is clear that K_1 is a graded sub-$B(G)$-module of M. It follows from the Theorem of Grothendieck above that K_1 is a gradually pure locally free graded B-module of finite rank. Moreover, $K_1 \neq \{0\}$.

If $K_1 = M$ we are done. If not, then there is a smallest integer d_2 such that $(K_1)_{d_2} \neq M_{d_2}$. Of course M_{d_2} is a sub-$\mathbb{C}(G)$-module of M, and contains $(K_1)_{d_2}$

as a sub-$\mathbb{C}(G)$-module. By Proposition 2, there is a sub-$\mathbb{C}(G)$-module $(K_2)_{d_2}$ of M_{d_2} that is a direct summand of $(K_1)_{d_2}$. Let K_2 be the graded sub-B-module of M generated by $(K_2)_{d_2}$. Then, K_2 is a sub-$B(G)$-module of M, and, by the Theorem of Grothendieck, K_2 is a gradually pure locally free graded B-module of finite rank. Moreover, $K_1 \cap K_2 = \{0\}$.

We iterate and obtain, by induction, a sequence $K_1, \dots, K_n$ of graded sub-$B(G)$-modules of M, such that each K_i is a gradually pure locally free graded B-module of finite rank with $(K_1 + \cdots + K_{i-1}) \cap K_i = \{0\}$. Moreover, the degrees of the gradually pure submodules K_i are distinct. In particular, we have a direct sum decomposition

$$M = \bigoplus_{i=1}^{n} K_i$$

of M as a $B(G)$-module.

It suffices, therefore, to prove the statement in case M is gradually pure, and for such modules it suffices to treat the case of gradually pure modules of degree 0. Let M be a gradually pure locally free graded $B(G)$-module of finite rank of degree 0. By the Theorem of Grothendieck, M_0 is a $\mathbb{C}(G)$-module of finite type. By Proposition 1, M_0 is isomorphic to $\mathbb{C}^m \oplus V^n$. Since M is $B(G)$-isomorphic to $B \otimes_{\mathbb{C}} M_0$, it follows that M is isomorphic to $B \otimes (\mathbb{C}^m \oplus V^n) \cong B^m \oplus (B^n \otimes_{\mathbb{C}} V)$, as graded $B(G)$-modules. □

4. Vector bundles over the anisotropic conic

Let $Q \subseteq \mathbb{P}^2$ be the anisotropic real conic defined by the equation $U^2 + V^2 + W^2 = 0$. The homogeneous coordinate ring of Q is the graded $\mathbb{R}$-algebra $A = \mathbb{R}[U,V,W]/(U^2 + V^2 + W^2)$. Define an $\mathbb{R}$-algebra morphism

$$\varphi \colon A \longrightarrow B$$

by $\varphi(U) = X^2 + Y^2$, $\varphi(V) = i(X^2 - Y^2)$ and $\varphi(W) = 2iXY$. Observe that φ is not a morphism of graded $\mathbb{R}$-algebras. Nevertheless, φ is homogeneous of degree 2, i.e., one has $\varphi(A_d) \subseteq B_{2d}$. Since $\varphi(U), \varphi(V), \varphi(W)$ belong to the graded sub-$\mathbb{R}$-algebra B^G of fixed points of B for the action of G, one has $\varphi(A) \subseteq B^G$. Therefore, φ is a homogeneous morphism of $\mathbb{R}$-algebras from A into B^G of degree 2.

Proposition 4. *The morphism $\varphi \colon A \to B^G$ is a homogeneous isomorphism of degree 2 of graded $\mathbb{R}$-algebras.*

Proof. It is clear that $\varphi \colon A \to B$ is injective. In order to show that φ is surjective, we show that the real vector spaces A_d and B^G_{2d} are of the same dimension, and that $B_{2d+1} = \{0\}$, for all natural integers d.

Let us first prove that $B^G_{2d+1} = \{0\}$. Let $P \in B^G_{2d+1}$. Then

$$P = \sigma^2 P = \sigma(P^\sigma(-Y, X)) = (P^\sigma)^\sigma(-X, -Y) = P(-X, -Y) = -P,$$

since $\deg(P)$ is odd. Therefore, $P = 0$. This proves that $B^G_{2d+1} = \{0\}$.

Next, we show that A_d and B^G_{2d} have the same dimension. The real vector space A_d is equal to the quotient of $\mathbb{R}[U,V,W]_d$ by $(U^2+V^2+W^2)_d$. The dimension of $\mathbb{R}[U,V,W]_d$ is equal to $\frac{1}{2}(d+1)(d+2)$. Therefore, the dimension of $(U^2 + V^2 + W^2)_d$ is equal to $\frac{1}{2}(d-1)d$. It follows that the dimension of A_d is equal to

$$\tfrac{1}{2}(d+1)(d+2) - \tfrac{1}{2}(d-1)d = \tfrac{1}{2}(4d+2) = 2d+1.$$

Next, we determine the dimension of B^G_{2d}. The complex dimension of B_{2d} is equal to $2d+1$. Now, the calculation of $\sigma^2 P$ above shows that $\sigma^2 P = P$ if $\deg(P)$ is even. Therefore, the $\mathbb{C}(G)$-vector space B^G_{2d} is isomorphic to $\mathbb{C}^{2d+1}$ by Proposition 1. It follows that B^G_{2d} is isomorphic to $\mathbb{R}^{2d+1}$. Hence, the dimension of B^G_{2d} is equal to $2d+1$ too. □

Corollary 5. *The morphism φ induces an isomorphism of real schemes*

$$\mathit{Proj}(\varphi)\colon\ \mathit{Proj}(B^G) \longrightarrow \mathit{Proj}(A) = Q. \quad \square$$

Let $\iota\colon \mathbb{C}\otimes_{\mathbb{R}} B^G \longrightarrow B$ be the morphism of graded $\mathbb{C}(G)$-algebras that is induced by the inclusion of B^G into B. The actions of G on the graded algebras $\mathbb{C}\otimes_{\mathbb{R}} B^G$ and B induce actions of G on the schemes $\mathrm{Proj}(\mathbb{C}\otimes_{\mathbb{R}} B^G)$ and $\mathrm{Proj}(B)$. One has the following statement.

Proposition 6. *The morphism ι induces an isomorphism of complex schemes*

$$\mathit{Proj}(\iota)\colon \mathbb{P}^1 = \mathit{Proj}(B) \longrightarrow \mathit{Proj}(\mathbb{C}\otimes_{\mathbb{R}} B^G).$$

Moreover, this morphism is equivariant with respect to the actions of G.

Proof. If d is en even integer, the $\mathbb{C}$-linear map ι_d is an isomorphism. It easily follows that $\mathrm{Proj}(\iota)$ is an isomorphism. The isomorphism $\mathrm{Proj}(\iota)$ is G-equivariant, since ι is G-equivariant. □

As a consequence of Proposition 6, one gets an equivalence $\mathrm{Proj}(\iota)^\star$ between the category of locally free G-sheaves of finite rank on $\mathbb{P}^1$, and the category of such sheaves on $\mathrm{Proj}(\mathbb{C}\otimes_{\mathbb{R}} B^G)$. A G-sheaf on a scheme on which G acts is an action of G on a sheaf of modules, lying over the action of G on the scheme. For the scheme $\mathbb{P}^1$, such a sheaf is the G-sheaf associated to a graded $B(G)$-module which is locally free of finite rank as a graded B-module. In fact, the functor $M\mapsto \widetilde{M}$ is an equivalence from the category of graded $B(G)$-modules which are locally free of finite rank as graded B-modules, into the category of locally free G-sheaves of finite rank over $\mathbb{P}^1$.

Note that the algebra $\mathbb{C}\otimes_{\mathbb{R}} B^G$ is evenly graded. A graded module over such an algebra is also supposed to be evenly graded. It follows, in particular, that only twists of graded $\mathbb{C}\otimes_{\mathbb{R}} B^G$-module by an even integer are defined. Apart from that, the usual definitions concerning $\mathbb{Z}$-graded modules apply verbatim. In particular, we will say that a graded $\mathbb{C}\otimes_{\mathbb{R}} B^G$-algebra is locally free of finite rank, if it is isomorphic to the graded module

$$\Gamma_{\mathrm{even}}(\mathcal{E}) = \bigoplus_{d\in 2\mathbb{Z}} H^0(\mathrm{Proj}(\mathbb{C}\otimes_{\mathbb{R}} B^G), \mathcal{O}(d)\otimes_{\mathcal{O}} \mathcal{E})$$

associated to a locally free sheaf of finite rank over $\mathrm{Proj}(\mathbb{C}\otimes_{\mathbb{R}} B^G)$. Here $\mathcal{O}$ denotes the structure sheaf on $\mathrm{Proj}(\mathbb{C}\otimes_{\mathbb{R}} B^G)$. Again, the functor $M\mapsto\widetilde{M}$ from the category of graded $\mathbb{C}\otimes_{\mathbb{R}} B^G$-modules which are locally free of finite rank, into the category of locally free sheaves of finite rank on $\mathrm{Proj}(\mathbb{C}\otimes_{\mathbb{R}} B^G)$ is an equivalence. A similar statement holds for the corresponding G-equivariant categories.

The equivalence $\mathrm{Proj}(\iota)^\star$ above induces an equivalence between the category of graded $B(G)$-modules that are locally free of finite rank as graded B-modules, and the category of graded $(\mathbb{C}\otimes_{\mathbb{R}} B^G)(G)$-modules that are locally free of finite rank as graded $\mathbb{C}\otimes_{\mathbb{R}} B^G$-modules. Explicitly, let M be a locally free graded $B(G)$-module of finite rank. Then, the induced graded $(\mathbb{C}\otimes_{\mathbb{R}} B^G)(G)$-module is

$$M_{\mathrm{even}} = \bigoplus_{d\in 2\mathbb{Z}} M_d,$$

which is a $(\mathbb{C}\otimes_{\mathbb{R}} B^G)(G)$-module if we identify $\mathbb{C}\otimes_{\mathbb{R}} B^G$ with a subring of B via ι, and is locally free of finite rank as a $\mathbb{C}\otimes_{\mathbb{R}} B^G$-algebra by the Theorem of

Grothendieck. As an example, one has the following important locally free graded $(\mathbb{C}\otimes_{\mathbb{R}} B^G)(G)$-modules of finite rank:

$$B_{\text{even}},\quad B[1]_{\text{even}},\quad (B\otimes_{\mathbb{C}} V)_{\text{even}}\quad\text{and}\quad (B[1]\otimes_{\mathbb{C}} V)_{\text{even}}.$$

Proposition 3 immediately implies the following statement.

Proposition 7. *Let M be a graded $(\mathbb{C}\otimes_{\mathbb{R}} B^G)(G)$-module which is locally free of finite rank as a graded $\mathbb{C}\otimes_{\mathbb{R}} B^G$-module. Then there are even integers $a_1,\dots,a_m$, $b_1,\dots,b_n$, $c_1,\dots,c_p$ and $d_1,\dots,d_q$ such that*

$$M\cong\left(\bigoplus_{i=1}^{m} B_{\text{even}}[a_i]\right)\oplus\left(\bigoplus_{i=1}^{n}(B[1]_{\text{even}})[b_i]\right)\oplus$$
$$\oplus\left(\bigoplus_{i=1}^{p}(B\otimes_{\mathbb{C}} V)_{\text{even}}[c_i]\right)\oplus\left(\bigoplus_{i=1}^{p}(B[1]\otimes_{\mathbb{C}} V)_{\text{even}}[d_i]\right),$$

where V is the $\mathbb{C}(G)$-module defined in Section 2. The even integers $a_1,\dots,a_m$, $b_1,\dots,b_n$, $c_1,\dots,c_p$ and $d_1,\dots,d_q$ are uniquely determined by M, up to permutation. □

A graded B^G-module M is said to be locally free of finite rank if there is a locally free sheaf $\mathcal{E}$ of finite rank over $\mathrm{Proj}(B^G)$ such that the associated graded B^G-module $\Gamma_{\text{even}}(\mathcal{E})$ is isomorphic to M. Complexification induces an equivalence from the category of locally free graded B^G-modules of finite rank onto a full subcategory of the category of graded $(\mathbb{C}\otimes_{\mathbb{R}} B^G)(G)$-modules that are locally free of finite rank as $\mathbb{C}\otimes_{\mathbb{R}} B^G$-modules. This full subcategory consists exactly of the $(\mathbb{C}\otimes_{\mathbb{R}} B^G)(G)$-modules M on which the action of σ^2 is trivial. Indeed, the action of σ^2 on the complexification of a locally free sheaf of finite rank on $\mathrm{Proj}(B^G)$ is trivial. Moreover, if a locally free sheaf of finite rank over $\mathrm{Proj}(\mathbb{C}\otimes_{\mathbb{R}}B^G)$ is a G-sheaf for which the action of σ^2 is trivial, then it is isomorphic to the complexification of a unique locally free sheaf of finite rank over $\mathrm{Proj}(B^G)$.

Now, the action of σ^2 on B_{even} and $B[1]\otimes_{\mathbb{C}} V$ is trivial, whereas the action of σ^2 on $B[1]_{\text{even}}$ and $B\otimes_{\mathbb{C}} V$ is not. Therefore, from Proposition 7 one deduces the following statement.

Proposition 8. *Let M be a graded B^G-module which is locally free of finite rank. Then there are even integers $a_1,\dots,a_m$ and $b_1,\dots,b_n$ such that*

$$M\cong\left(\bigoplus_{i=1}^{m} B^G_{\text{even}}[a_i]\right)\oplus\left(\bigoplus_{i=1}^{n}(B[1]\otimes_{\mathbb{C}} V)^G_{\text{even}}[b_i]\right),$$

where V is the $\mathbb{C}(G)$-module defined in Section 2. The even integers $a_1,\dots,a_m$ and $b_1,\dots,b_n$ are uniquely determined by M, up to permutation. □

Note that the graded B^G-module

$$(B[1]\otimes_{\mathbb{C}} V)^G_{\text{even}}=\{P(X,Y)\otimes(1,0)+\overline{P}(-Y,X)\otimes(0,1)\,|\,P\in B_{\text{odd}}\},$$

is isomorphic to the graded B^G-module $B_{\text{odd}}[1]$. Its complexification $\mathbb{C}\otimes_{\mathbb{R}} B_{\text{odd}}[1]$ is a locally free graded $\mathbb{C}\otimes_{\mathbb{R}} B^G$-module of rank 2, since it corresponds, via the isomorphism $\mathrm{Proj}(\iota)$, to the locally free graded B-module $\mathbb{C}\otimes_{\mathbb{R}} B[1]$, which is obviously of rank 2. Moreover the determinant module of the latter is $B(G)$-isomorphic to $B[2]$. Therefore, the graded B^G-module $(B[1]\otimes_{\mathbb{C}} V)^G_{\text{even}}$ is of rank 2 and its determinant module is isomorphic to $B^G_{\text{even}}[2]$.

The locally free sheaf of finite rank on $\mathrm{Proj}(B^G)$ associated to B^G_{even} is, of course, the structure sheaf $\mathcal{O}$ of $\mathrm{Proj}(B^G)$. Let $\mathcal{V}$ be the locally free sheaf of finite rank on $\mathrm{Proj}(B^G)$ associated to $(B[1]\otimes_{\mathbb{C}} V)^G_{\text{even}}$. By the preceding observations, $\mathcal{V}$ is a locally free sheaf of rank 2 on $\mathrm{Proj}(B^G)$ and its determinant line bundle is isomorphic to $\mathcal{O}[2]$. Moreover, $\mathcal{V}$ is indecomposable on $\mathrm{Proj}(B^G)$, i.e., $\mathcal{V}$ is not isomorphic to the direct sum of locally free sheaves of rank 1, since the B^G-module $(B[1]\otimes\mathbb{C}V)^G_{\mathrm{even}}$ is indecomposable by Proposition 8. Denote again by $\mathcal{V}$ the corresponding sheaf on the anisotropic conic Q. Then, the following statement is the result we wanted to prove. It follows from the preceding observations.

Theorem 9. *Let $\mathcal{E}$ be a locally free sheaf of finite rank over the anisotropic conic Q. Then, there are integers $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ such that*

$$\mathcal{E} \cong \left(\bigoplus_{i=1}^{m} \mathcal{O}(a_i) \right) \oplus \left(\bigoplus_{i=1}^{n} (\mathcal{O}(b_i) \otimes_{\mathcal{O}} \mathcal{V}) \right),$$

where $\mathcal{O}$ is the structure sheaf on Q, $\mathcal{O}(d)$ is the restriction to Q of the sheaf with the same notation on $\mathbb{P}^2$, and $\mathcal{V}$ is the locally free sheaf on Q defined above. The integers $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ are uniquely determined by $\mathcal{E}$, up to permutation. Moreover, $\mathcal{V}$ is an indecomposable locally free sheaf of rank 2 on Q whose determinant is isomorphic to the invertible sheaf $\mathcal{O}(1)$ of degree 2 on Q. □

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