A geometric description of the neutral component of the Jacobian of a real plane curve having many pseudo-lines

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Abstract

A pseudo-line of a real plane curve is a real branch that is not homologically trivial in $\mathbb{P}^2(\mathbb{R})$. A real plane curve C of degree d is said to have many pseudo-lines if it has exactly $d-2$ pseudo-lines and if the genus of its normalization $\tilde{C}$ is equal to $d-2$. Let C be such a curve. We give a planar description of the neutral component of the set of real points of the Jacobian of the normalization $\tilde{C}$ of C.

MSC 1991: 14H40, 14P99

Keywords: real algebraic plane curve, pseudo-line, Jacobian

1 Introduction

Let C be a real plane curve. The set of real points $C(\mathbb{R})$ of C is a real analytic subset of $\mathbb{P}^2(\mathbb{R})$. A *real branch* of C is an irreducible real analytic subset of $C(\mathbb{R})$ of dimension 1. A real branch B of C is said to be a *pseudo-line* of C if its homology class in $H_1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is nonzero. We are interested in real plane curves having many pseudo-lines: we say that a geometrically integral real plane curve of degree d has *many pseudo-lines* if it has exactly $d-2$ pseudo-lines and if the genus of its normalization is equal to $d-2$ (see Figure 1). We refer the reader to [8] for the reasons behind this precise definition. Real plane curves having many pseudo-lines abound: all nonsingular real quadrics and all nonsingular real cubics are examples of such curves. In fact, there are "many" real plane curves having many pseudo-lines in any degree ≥ 2 [6, Theorem 5.1].

Let C be a real plane curve of degree d having many pseudo-lines. Let $\tilde{C}$ be its normalization. The set of real points $\mathrm{Jac}(\mathbb{R})$ of the Jacobian of $\tilde{C}$ is a—not necessarily connected—compact commutative real Lie group. The

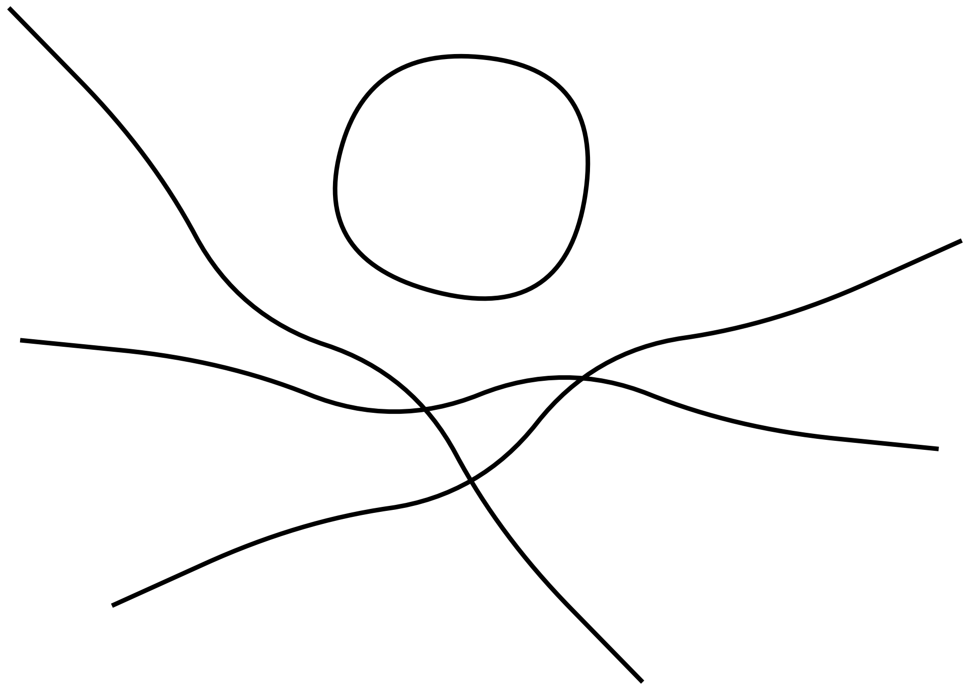

Figure 1: An example of a real plane having many pseudo-lines of degree 5.

object of the paper is to give a geometric description of the neutral component of the set of real points of $\mathrm{Jac}(\mathbb{R})$. More precisely, $\mathrm{Jac}(\mathbb{R})$ may be identified with the subgroup $\mathrm{Pic}^0(\tilde{C})$ of the Picard group $\mathrm{Pic}(\tilde{C})$ of $\tilde{C}$. It is a compact commutative real Lie group of dimension g, where $g = d - 2$ is the genus of $\tilde{C}$ (see [2, §4.1] for more details). Its neutral component is isomorphic to the real Lie group $(S^1)^g$. We show in this paper that there is a planar description of the neutral component of $\mathrm{Pic}^0(\tilde{C})$. This planar description is a direct generalization of the classical planar description of the group law on the neutral component of the set of real points of a real elliptic curve [10]. Such a generalization has already been constructed [7], but used an embedding of $\tilde{C}$ into $\mathbb{P}^{2g}$. The actual description is a true planar description, i.e., it is based on intersecting the real plane curve C with certain real plane curves of degree g. This also generalizes the construction of [3] that applied to hyperelliptic curves only. The curious reader may immediately refer to Section 4, where the planar description is given.

The paper is organized as follows. In Section 2 we recall some properties of real plane curves having many pseudo-lines. These curves are, in general, singular. Their singularities are all real and ordinary. Therefore, the normalization of a real plane curve having many pseudo-lines embeds into the projective plane blown-up in a finite number of real points. In Section 3 we show that the neutral real component of the Jacobian of the normalization of a real plane curve having many pseudo-lines admits a geometric description coming from this embedding. We deduce from this, in Section 4, the planar description of this neutral component.

Acknowledgement. The second author thanks A. Hirschowitz for having pointed out the paper [9] to him.

2 Real plane curves with many pseudo-lines

In this section we recall some elementary properties of real plane curves having many pseudo-lines [8] that will be useful in the sequel.

Proposition 2.1. *Let $d \geq 2$ be an integer and let C be a real plane curve of degree d having many pseudo-lines.*

1. *The curve C has either $d-2$ or $d-1$ real branches.*
2. *Two distinct pseudo-lines of C intersect in one point only. These singularities are the only singularities of C. They are all real ordinary multiple points. In particular, each real branch of C is a smooth real analytic curve in $\mathbb{P}^2(\mathbb{R})$.*

Proof. 1. Let $\nu\colon \tilde{C} \to C$ be the normalization of C and let g be the genus of $\tilde{C}$. By definition of a real plane curve having many pseudo-lines, $g = d-2$. By Harnack's Inequality [4], the number s of real branches of $\tilde{C}$, or what amounts to the same, of C itself, satisfies $s \leq g+1 = d-1$. On the other hand, since C has many pseudo-lines, $s \geq d-2$. Therefore, $s = d-2$ or $d-1$.

2. Observe that, for topological reasons, any two distinct pseudo-lines of C intersect each other. Hence, by the genus formula [1], the genus g of $\tilde{C}$ satisfies:

$$d-2 = g \leq \tfrac{1}{2}(d-1)(d-2) - \tfrac{1}{2}(d-2)(d-3) = d-2.$$

It follows that the above inequality is an equality. Therefore, any two distinct pseudo-lines of C intersect in one point only—the intersection being transverse—and C has no other singularities. (By definition, the intersection of two distinct real branches of C is transverse in a point P if and only if both branches are smooth at P and both tangent lines are distinct.) $\square$

3 The geometric description in terms of the normalization

Let C be a real plane curve having many pseudo-lines. Let g be the genus of the normalization of C. Let $X_1, \ldots, X_r$ be the singular points of C. By Proposition 2.1, each X_i is a real ordinary multiple point of C. Let b_i be the multiplicity of C at X_i. By the genus formula [1], one has

$$g = \tfrac{1}{2}g(g+1) - \sum_{i=1}^{r} \tfrac{1}{2}b_i(b_i-1).$$

Or, equivalently,

$$\sum_{i=1}^{r} \tfrac{1}{2}b_i(b_i-1) = \tfrac{1}{2}g(g-1).$$

This formula will be used throughout this section without further reference.

Let S be the surface obtained from $\mathbb{P}^2$ by blowing-up the points $X_1, \ldots, X_r$. Denote by $\tilde{C}$ the strict transform of C in S. Since each singular point of C is ordinary, $\tilde{C}$ is the normalization of C.

For $i = 1, \ldots, r$, denote by E_i the special divisor on S lying over X_i. Denote by $\tilde{H}$ the pull-back on S of the hyperplane section H of $\mathbb{P}^2$. Define the divisor D on S by

$$D = g\tilde{H} - \sum_{i=1}^{r}(b_i-1)E_i.$$

It is well-known—but only of interest to us in the next section—that the linear system $|D|$ on S can be identified with the linear system of curves on $\mathbb{P}^2$ of degree g having multiplicity at least $b_i - 1$ at the point X_i, for $i = 1, \ldots, r$ (cf. [5, Remark V.4.8.1]). It is also well known (cf. [9, p. 193]) that

$$\dim|D| \geq \tfrac{1}{2}g(g+3) - \sum_{i=1}^{r} \tfrac{1}{2}b_i(b_i-1).$$

Hence,

$$\dim|D| \geq \tfrac{1}{2}g(g+3) - \tfrac{1}{2}g(g-1) = 2g.$$

In fact, we will prove that this inequality is an equality:

Proposition 3.1. $\dim|D| = 2g$.

Proof. Consider the intersection product $D \cdot \tilde{C}$ of D and $\tilde{C}$. It is a divisor on $\tilde{C}$ of degree

$$g(g+2) - \sum_{i=1}^{r}(b_i-1)b_i = g(g+2) - g(g-1) = 3g.$$

Since $3g > 2g-2$, the divisor $D\cdot\tilde{C}$ is nonspecial. Hence, the linear system $|D\cdot\tilde{C}|$ on $\tilde{C}$ defined by the divisor $D\cdot\tilde{C}$ satisfies

$$\dim|D\cdot\tilde{C}| = 3g-g = 2g.$$

Now, no element of $|D|$ contains $\tilde{C}$ as a component. Therefore, there is a natural map from $|D|$ into $|D\cdot\tilde{C}|$ mapping a divisor F onto $F\cdot\tilde{C}$. Since this map is a linear map of projective spaces, it is necessarily injective. It follows that

$$\dim|D| \leq \dim|D\cdot\tilde{C}| = 2g.$$

Therefore, $\dim|D| = 2g$. □

We have actually showed the following statement:

Proposition 3.2. *The natural map from $|D|$ into $|D\cdot\tilde{C}|$ is a bijection.* □

Let $B_1,\dots,B_g$ be the pseudo-lines of C. According to Proposition 2.1, C has either g or $g+1$ real branches. If C has $g+1$ real branches, let B_0 be the real branch different from $B_1,\dots,B_g$; if not, let B_0 be the empty subset of $C(\mathbb{R})$. By abuse of language, we will call B_0 a branch of C even if $B_0=\emptyset$. No confusion will occur if we also denote by B_i the real branch of $\tilde{C}$ that corresponds to the real branch B_i of C, for $i=0,\dots,g$. Put

$$B = \prod_{i=1}^{g} B_i.$$

The map from B into $\mathrm{Pic}^g(\tilde{C})$ that sends a point $P=(P_1,\dots,P_g)$ of B onto the class of the divisor $P_1+\cdots+P_g$, is a real analytic isomorphism onto a connected component of $\mathrm{Pic}^g(\tilde{C})$ [7, Theorem 3.1]. We identify B with this connected component of $\mathrm{Pic}^g(\tilde{C})$ through that map. Sometimes, an element P of B will also be identified with the divisor $P_1+\cdots+P_g$ instead of the class of this divisor.

Let us recall the following well-known fact on the connected components of the Picard group $\mathrm{Pic}(\tilde{C})$ of $\tilde{C}$ [2, §4.1]. Let F be a divisor on $\tilde{C}$. Define $\deg_i(F)$ to be the degree of F on the real branch B_i. Let F, F' be divisors on $\tilde{C}$. The classes of F and F' in $\mathrm{Pic}(\tilde{C})$ belong to the same connected component of if and only if

1. $\deg(F)=\deg(F')$, and
2. $\deg_i(F)\equiv\deg_i(F') \pmod 2$, for $i=0,\dots,g$.

Proposition 3.3. *There are exactly 3^g elements $O\in B$ such that the divisor $3O$ on $\tilde{C}$ belongs to the linear system $|D\cdot\tilde{C}|$.*

Proof. The divisor $D \cdot \tilde{C}$ on $\tilde{C}$ has degree 0 (mod 2) on B_0 and degree $g - (g-1) \equiv 1 \pmod 2$ on B_i, for $i = 1, \ldots, g$. Therefore, its class in $\mathrm{Pic}^{3g}(\tilde{C})$ belongs to the connected component $3B$ of $\mathrm{Pic}^{3g}(\tilde{C})$. It follows that there is a point $O = (O_1, \ldots, O_g)$ in B such that $3O$ is linearly equivalent to $D \cdot \tilde{C}$. The set of all elements $O' \in B$ such that $3O'$ is linearly equivalent to $D \cdot \tilde{C}$ is equal to

$$(O + \mathrm{Pic}(\tilde{C})_3) \cap B,$$

where $\mathrm{Pic}(\tilde{C})_3$ denotes the 3-torsion subgroup of $\mathrm{Pic}(\tilde{C})$. Of course, the latter 3-torsion subgroup is equal to the 3-torsion subgroup of $\mathrm{Pic}^0(\tilde{C})$. Since the neutral component of $\mathrm{Pic}^0(\tilde{C})$ is of 2-primary index in $\mathrm{Pic}^0(\tilde{C})$ [2, Theorem 4.1.7(ii)], the 3-torsion subgroup of $\mathrm{Pic}^0(\tilde{C})$ is equal to the 3-torsion subgroup of the neutral component of $\mathrm{Pic}^0(\tilde{C})$. In particular, $\mathrm{Pic}(\tilde{C})_3$ is entirely contained in the neutral component of $\mathrm{Pic}^0(\tilde{C})$ and has cardinality 3^g. It follows that $O + \mathrm{Pic}(\tilde{C})_3$ is contained in B and is of cardinality 3^g. □

Recall the following statement (cf. [6, Theorem 2.3] or [7, Theorem 2.3]):

Theorem 3.4. *Let F be a divisor on $\tilde{C}$ and let d be its degree. Let k be the number of integers $i \in \{0, \ldots, g\}$ such that the degree $\deg_i(F)$ of F on B_i is odd. If $d + k > 2g - 2$ then F is nonspecial.* □

Using this fact, we will prove:

Proposition 3.5. *For all $P, Q \in B$ there is a unique divisor $F \in |D \cdot \tilde{C}|$ such that $F \geq P + Q$. Moreover, there is a unique $R \in B$ such that $F = P + Q + R$.*

Proof. Consider the divisor $(D \cdot \tilde{C}) - P - Q$ on $\tilde{C}$. It satisfies the hypothesis of Theorem 3.4. Therefore, it is nonspecial. Then, by Riemann-Roch,

$$\dim |(D \cdot \tilde{C}) - P - Q| = \deg((D \cdot \tilde{C}) - P - Q) - g = (3g - g - g) - g = 0,$$

i.e., there is a unique $F \in |D \cdot \tilde{C}|$ such that $F \geq P + Q$. Now, since F is linearly equivalent to $D \cdot \tilde{C}$,

$$\deg_i(F) \equiv \deg_i(D \cdot \tilde{C}) \equiv 1 \pmod 2$$

for $i = 1, \ldots, g$. Therefore, there is an element $R \in B$ such that $F \geq P + Q + R$. Since $\deg(F) = 3g = \deg(P + Q + R)$, one has $F = P + Q + R$. It also follows that $R \in B$ is unique. □

Now, we will construct a unary law $\ominus$ and a binary law $\oplus$ on B. First, by Proposition 3.3, one may choose $O \in B$ such that $3O \in |D \cdot \tilde{C}|$. Equivalently by Proposition 3.2, and, more geometrically, $O \in B$ is such that the divisor $3O$ on $\tilde{C}$ is cut out by an element of the linear system $|D|$ on the surface S. The element $O \in B$ is going to be the neutral element for the binary law $\oplus$.

In order to define $\ominus$, choose $P \in B$. By Proposition 3.5, there is a unique divisor $F \in |D \cdot \tilde{C}|$ such that $F \geq O + P$. Moreover, by the same proposition, there is a unique element of B, denoted by $\ominus P$, such that $F = O + P + \ominus P$. This defines the unary law $\ominus$ on B. Equivalently by Proposition 3.2, and, more geometrically, there is a unique divisor F on the surface S belonging to the linear system $|D|$ such that $F \cdot \tilde{C} \geq O + P$. Then, $\ominus P$ is the unique element of B such that $F \cdot \tilde{C} = O + P + \ominus P$.

Finally, we define the binary law $\oplus$ on B. Choose $P, Q \in B$. By Proposition 3.5, there is a unique divisor $F \in |D \cdot \tilde{C}|$ such that $F \geq P + Q$. Moreover, by the same proposition, there is a unique element R of B such that $F = P + Q + R$. One defines the binary law $\oplus$ on B by letting $P \oplus Q = \ominus R$. Again equivalently by Proposition 3.2, and, more geometrically, there is a unique divisor F on the surface S belonging to the linear system $|D|$ such that $F \cdot \tilde{C} \geq P + Q$. Then, $P \oplus Q$ is the unique element of B such that $F \cdot \tilde{C} = P + Q + \ominus(P \oplus Q)$.

Theorem 3.6. *The map τ from B into the neutral component of the Jacobian $\mathrm{Pic}^0(\tilde{C})$ of $\tilde{C}$ defined by $\tau(P) = P - O$, is a real analytic isomorphism. Moreover, it satisfies*

$$\tau(O) = 0, \quad \tau(\ominus P) = -\tau(P) \quad \textit{and} \quad \tau(P \oplus Q) = \tau(P) + \tau(Q),$$

for all $P, Q \in B$. In particular, $(B, O, \oplus, \ominus)$ is a real Lie group isomorphic to the neutral component of the set of real points of the Jacobian of $\tilde{C}$.

Proof. The map τ is known to be a real analytic isomorphism onto the neutral component of $\mathrm{Pic}^0(\tilde{C})$ [7, Theorem 3.1]. Of course, $\tau(O) = 0$. Choose $P, Q \in B$. Let $R \in B$ be the unique element such that the divisor $P + Q + R$ on $\tilde{C}$ is linearly equivalent to $D \cdot \tilde{C}$. Then

$$0 = P + Q + R - 3O = \tau(P) + \tau(Q) + \tau(R)$$

in $\mathrm{Pic}^0(\tilde{C})$. Taking $Q = O$, one gets $\tau(\ominus P) = -\tau(P)$ for all $P \in B$. Then, taking $Q \in B$ arbitrary, one gets $\tau(P \oplus Q) = \tau(P) + \tau(Q)$. □

4 The geometric description in terms of the plane curve

Let C be a real plane curve having many pseudo-lines. Let g be the genus of the normalization $\tilde{C}$ of C. Then C has g distinct pseudo-lines $B_1, \ldots, B_g$.

Put

$$B = \prod_{i=1}^{g} B_i.$$

Let $X_1, \dots, X_r$ be the singular points of C. By Proposition 2.1, each X_i is a real point of C belonging to some pseudo-line. For $i = 1, \dots, r$, let b_i be the multiplicity of C at X_i. By Proposition 2.1, b_i is equal to the number of real pseudo-lines of C passing through X_i. By Proposition 2.1, each pseudo-lines B_i is a nonsingular real analytic curve. Define a divisor D_i on B_i by

$$D_i = \sum_{j\colon X_j \in B_i} (b_j - 1) X_j.$$

Loosely speaking, the following statement says that there are exactly 3^g real plane curves F of degree g passing through the points X_j with multiplicity at least $b_j - 1$, and passing through exactly one other point O_i of B_i at which F has order of contact equal to 3.

Theorem 4.1. *There are exactly 3^g real plane curves F of degree g such that, for $i = 1, \dots, g$, the divisor $F \cdot B_i$ on B_i is of the form $3O_i + D_i$ for some $O_i \in B_i$.*

Proof. The statement is a direct consequence of Propositions 3.2 and 3.3. □

By Theorem 4.1, one may choose $O = (O_1, \dots, O_g) \in B$ such that there is a plane curve F of degree g such that $F \cdot B_i = 3O_i + D_i$, for $i = 1, \dots, g$. We are going to define a unary law $\ominus$ and a binary law $\oplus$ on B. The element O is going to be the neutral element for $\oplus$.

First, let us define $\ominus$. Choose $P \in B$. By what we have seen in Section 3, there is a unique real plane curve F of degree g passing through the points X_j with multiplicity at least $b_j - 1$ and passing through the points O_i and P_i. Moreover, there are then unique points $Q_i \in B_i$ such that F passes also through Q_i. More precisely,

Theorem 4.2. *There is a unique real plane curve F of degree g such that, for $i = 1, \dots, g$,*

$$F \cdot B_i \geq O_i + P_i + D_i.$$

Moreover, there is, for $i = 1, \dots, g$, a unique point $Q_i \in B_i$ such that

$$F \cdot B_i = O_i + P_i + Q_i + D_i. \quad \square$$

One defines $\ominus P$ to be the element $Q = (Q_1, \dots, Q_g) \in B$.

In order to define the binary law $\oplus$, choose $P, Q \in B$. There is a unique real plane curve F of degree g passing through the points X_j with multiplicity at least $b_j - 1$ and passing through the points P_i and Q_i. Moreover, there are then unique points $R_i \in B_i$ such that F passes also through R_i. More precisely,

Theorem 4.3. *There is a unique real plane curve F of degree g such that, for $i = 1, \dots, g$,*

$$F \cdot B_i \geq P_i + Q_i + D_i.$$

Moreover, there is, for $i = 1, \dots, g$, a unique point $R_i \in B_i$ such that

$$F \cdot B_i = P_i + Q_i + R_i + D_i. \quad \square$$

One defines $P \oplus Q$ to be the element $\ominus R$, where $R = (R_1, \dots, R_g) \in B$.

The following statement is then a direct consequence of Theorem 3.6:

Theorem 4.4. *The map τ from B into the Jacobian $\mathrm{Pic}^0(\tilde{C})$ of $\tilde{C}$ defined by $\tau(P) = \sum(P_i - O_i)$ is a real analytic isomorphism onto the neutral component of $\mathrm{Pic}^0(\tilde{C})$ and satisfies*

$$\tau(O) = 0, \quad \tau(\ominus P) = -\tau(P) \quad \textit{and} \quad \tau(P \oplus Q) = \tau(P) + \tau(Q),$$

for all $P, Q \in B$. In particular, $(B, O, \oplus, \ominus)$ is a real Lie group isomorphic to the neutral component of the Jacobian of $\tilde{C}$. $\square$

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