Imaginary automorphisms on real hyperelliptic curves

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Abstract

A real hyperelliptic curve X is said to be Gaussian if there is an automorphism $\alpha : X_{\mathbb{C}} \to X_{\mathbb{C}}$ such that $\overline{\alpha} = [-1]_{\mathbb{C}} \circ \alpha$, where $[-1]$ denotes the hyperelliptic involution on X. Gaussian curves arise naturally in several contexts, for example when one studies real Jacobians. In the present paper we study the properties of Gaussian curves and we describe their moduli spaces.

MSC 2000: 14H15, 14H37, 14P99, 30F50

Keywords: hyperelliptic curve, automorphism, moduli space

1 Introduction

Complex hyperelliptic curves are uniquely determined, up to isomorphism, by their branch locus. For real curves this is false, in general. For example, let p be a strictly positive reduced real polynomial. Then the real hyperelliptic curves X and X^- defined by the affine plane equations $y^2 = p(x)$ and $y^2 = -p(x)$ have the same branch locus but they are not isomorphic, since X has real points, while X^- has not.

So a natural question arises: "when is a real hyperelliptic curve X uniquely determined by its branch locus?" A first answer is that this happens if and only if there is an automorphism α of the complexification $X_{\mathbb{C}}$ of X such that $\overline{\alpha} = [-1]_{\mathbb{C}} \circ \alpha$, where $[-1]$ denotes the hyperelliptic involution on X (see Theorem 2.3). In this case we say that α is an imaginary automorphism for X, and that X is a Gaussian curve.

But why is it interesting to study Gaussian curves? They arise naturally when one studies the problem of characterizing the Jacobians of real curves

[*]Financially supported by the European Research Training Networks EAGER (EC contract HPRN-CT-2000-00099) and RAAG (EC contract HPRN-CT-2001-00271).

among all principally polarized real abelian varieties (see [1] or [8] for details about real abelian varieties). Since the complexification of a real Jacobian is a complex Jacobian (but the converse is not true, in general), one usually restricts one's attention to the moduli space $\mathcal{J}$ of principally polarized real abelian varieties whose complexification is a complex Jacobian. In fact, there is a natural involution I acting on $\mathcal{J}$, having the nice property that it exchanges Jacobians of non-hyperelliptic real curves with principally polarized real abelian varieties that are not real Jacobians (see [7, Proposition 3.4] for the precise definition of I). On the contrary, the locus of hyperelliptic real Jacobians is stable for I. However its general point is not fixed by I and the link with our problem is just the following: a real curve is Gaussian if and only if its Jacobian is fixed by I.

In the present paper we study the Gaussian curves whose real locus is not empty. The Gaussian curves whose real locus is empty will be studied in a forthcoming paper [5]. Here, then, we can restrict our attenction to hyperelliptic curves that are double coverings of the real projective line $\mathbb{P}^1$ (see Section 2 for more details about this assumption).

The main results of the present paper concern the classification of Gaussian curves. Topologically, a Gaussian curve X is classified by only two invariants: its genus g and the number k of connected components of its real locus. To classify X geometrically it is useful to look at the real automorphism β of $\mathbb{P}^1$ induced, via the hyperelliptic covering, by the imaginary automorphism α that defines the structure of Gaussian curve on X. Then X is said to be of type I or of type II according to whether β has real fixed points or not. In Theorem 4.3 we classify Gaussian curves of type I: first we determine for what pairs (g, k) the moduli space of Gaussian curves of type I having genus g and k real components is not empty. Next, for such pairs, we compute the dimension and the number of irreducible components of this space. Similarly in Theorems 5.3 and 5.5 we classify Gaussian curves of type II.

This paper is organized as follows. In Section 2 we develop the theory that we need to study Gaussian curves. In Sections 3 we introduce the moduli spaces of Gaussian curves. Section 4 and 5 are devoted to Gaussian curves of type I and of type II, respectively. Finally, in Section 6 we apply our results to the case of real curves of genus 2.

Convention. A curve over a field is supposed to be smooth, proper and geometrically integral [3].

A hyperelliptic curve is usually supposed to be of genus at least 2. Although we are primarily interested in curves of genus at least 2, here it will be convenient to remove this assumption. This will make some notations easier, as we need to refer generically to curves of arbitrary genus, in some constructions.

More precisely, a *real hyperelliptic curve* is a pair (X, f), where X is a real algebraic curve and $f\colon X \to Y$ is a morphism of degree 2 from X into a real algebraic curve Y of genus 0 (when the genus of X is greater than 1, the hyperelliptic covering f is uniquely determined by X).

We say that (X, f) is *isotropic* if Y is isomorphic to the real projective line $\mathbb{P}^1$. We say that (X, f) is *anisotropic* if Y is isomorphic to the "empty circle" (that is the real curve of genus 0 having no real points).

In the present paper we only consider isotropic curves: the anisotropic ones will be studied, via real line arrangements, in a forthcoming paper [5]. So, from now on, we will always assume $Y = \mathbb{P}^1$ and we will omit to specify "isotropic" for a real hyperelliptic curve.

A *morphism* of hyperelliptic real curves from (X, f) into (X', f') is a pair of morphisms $\gamma\colon X \to X'$, $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $f' \circ \gamma = \beta \circ f$. The following statement (see [2, Lemma 2.3]) gives a necessary and sufficient condition for real hyperelliptic curves to be isomorphic.

Lemma 2.1. *Let (X, f) and (X', f') be real hyperelliptic curves having branch loci B and B'. Then the following conditions are equivalent.*

1. *There is an isomorphism $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that:*
   - *(a) $\beta(B) = B'$, and*
   - *(b) $\beta(f(X(\mathbb{R}))) = f'(X'(\mathbb{R}))$.*
2. *The real hyperelliptic curves (X, f) and (X', f') are isomorphic.*

Given a real hyperelliptic curve (X, f), one can construct another real hyperelliptic curve (X^-, f^-), having the same branch locus as (X, f) and which is not isomorphic to (X, f), in general. The following is the construction of (X^-, f^-). Let us consider the Galois group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, acting naturally on the complexification $X_\mathbb{C} = X \times_\mathbb{R} \mathbb{C}$. It gives rise to a morphism of groups φ from G into the group $\mathrm{Aut}_\mathbb{R}(X_\mathbb{C})$ of $\mathbb{R}$-automorphism of $X_\mathbb{C}$. One can twist the φ-action of G on $X_\mathbb{C}$ by defining a morphism $\psi\colon G \to \mathrm{Aut}_\mathbb{R}(X_\mathbb{C})$ by $\psi(\sigma) = [-1]_\mathbb{C} \circ \varphi(\sigma)$ (where σ is the nontrivial element of G and $[-1]$ is the hyperelliptic involution on (X, f), that is the unique nontrivial automorphism of X over $\mathbb{P}^1$). The morphism ψ defines another action of G on $X_\mathbb{C}$.

The quotient of $X_{\mathbb{C}}$ by the ψ-action of G is a real algebraic curve X^-, and $f_{\mathbb{C}}$ induces a ramified double covering $f^-\colon X^- \to \mathbb{P}^1$ which is said to be obtained from f by *twisting the real structure*.

The following statement is well known. We include it for future reference.

Lemma 2.2. *Let (X, f) a real hyperelliptic curve. Let B the branch locus of f. If (X', f') is another real hyperelliptic curve with branch locus B, then (X', f') is either isomorphic to (X, f) or to (X^-, f^-).*

The natural action of G on $X_{\mathbb{C}}$ induces an action of G on the group $\mathrm{Aut}(X_{\mathbb{C}})$ of automorphisms of $X_{\mathbb{C}}$. As usual, for $\alpha \in \mathrm{Aut}(X_{\mathbb{C}})$, we denote by $\overline{\alpha}$ the element $\sigma \cdot \alpha$, where σ is the nontrival element of G. In fact, with the previous notation, $\overline{\alpha} = \varphi(\sigma) \circ \alpha \circ \varphi(\sigma)^{-1}$.

The following theorem gives necessary and sufficient conditions for (X, f) and (X^-, f^-) to be isomorphic.

Theorem 2.3. *Let (X, f) be a real hyperelliptic curve. Let B be the branch locus of f. Let $[-1]$ be the hyperelliptic involution of (X, f). Then the following conditions are equivalent.*

1. *The real hyperelliptic curves (X, f) and (X^-, f^-) are isomorphic.*
2. *There is an automorphism α of $X_{\mathbb{C}}$, of finite order, such that α and $[-1]_{\mathbb{C}}$ commute, and $\overline{\alpha} = [-1]_{\mathbb{C}} \circ \alpha$.*
3. *There is an automorphism β of $\mathbb{P}^1$, of finite order, such that $\beta(B) = B$ and $\beta(f(X(\mathbb{R}))) \neq f(X(\mathbb{R}))$.*

Moreover, if the above conditions are satisfied, the number of automorphisms β of $\mathbb{P}^1$ satisfying condition 3 is finite, and any such automorphism has even order.

Proof. 1⇒2: Suppose that (X, f) and (X^-, f^-) are isomorphic. Then, there are isomorphisms $\gamma\colon X \to X^-$ and $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $f^- \circ \gamma = \beta \circ f$. Since $X_{\mathbb{C}} = X^-_{\mathbb{C}}$, $\alpha = \gamma_{\mathbb{C}}$ is an automorphism of $X_{\mathbb{C}}$. We check that α satisfies the two conditions of 2.

Since

$$f^- \circ (\gamma \circ [-1] \circ \gamma^{-1}) = \beta \circ f \circ [-1] \circ \gamma^{-1} = \beta \circ f \circ \gamma^{-1} = f^-,$$

one has $\gamma \circ [-1] \circ \gamma^{-1} = [-1]^-$, the nontrivial automorphism of X^- over $\mathbb{P}^1$. It follows that $\gamma \circ [-1] = [-1]^- \circ \gamma$, and, taking complexifications:

$$\alpha \circ [-1]_{\mathbb{C}} = [-1]_{\mathbb{C}} \circ \alpha.$$

Therefore, α and $[-1]_{\mathbb{C}}$ commute. Moreover, since γ is a morphism over $\mathbb{R}$, its complexification α satisfies

$$\alpha \circ \varphi(\sigma) = \psi(\sigma) \circ \alpha = [-1]_{\mathbb{C}} \circ \varphi(\sigma) \circ \alpha.$$

It follows that $\overline{\alpha} = [-1]_{\mathbb{C}} \circ \alpha$.

2⇒3: Let α be an automorphism of $X_{\mathbb{C}}$ satisfying condition 2. As in the proof of the implication 2⇒1, the automorphism α induces an automorphism β of $\mathbb{P}^1$.

One clearly has $\beta(B) = B$. In order to show that $\beta(f(X(\mathbb{R}))) \neq f(X(\mathbb{R}))$, suppose that $\beta(f(X(\mathbb{R})))$ is equal to $f(X(\mathbb{R}))$. Then, by Lemma 2.1, β would lift to an automorphism γ of X. Then, $\alpha = \gamma_{\mathbb{C}}$ or $\alpha = [-1]_{\mathbb{C}} \circ \gamma_{\mathbb{C}}$. In both cases, $\overline{\alpha} = \alpha$, which contradicts the hypothesis that $\overline{\alpha} = [-1]_{\mathbb{C}} \circ \alpha$. Therefore, $\beta(f(X(\mathbb{R}))) \neq f(X(\mathbb{R}))$. Moreover, it is clear that the automorphism β induced by α is of finite order if α is of finite order.

3 ⇒ 1: Let β be an automorphism of $\mathbb{P}^1$ such that $\beta(B) = B$ and $\beta(f(X(\mathbb{R}))) \neq f(X(\mathbb{R}))$. Then β maps $f(X(\mathbb{R}))$ into $f^-(X^-(\mathbb{R}))$. This means that the two curves (X, f) and (X^-, f^-) are isomorphic (see Lemma 2.1). □

When the hypothesis of Theorem 2.3 hold, we say that (X, f) is a Gaussian curve and that α is an imaginary automorphism for (X, f). Observe that such an automorphism α for f does neither commute with $\varphi(\sigma)$, nor with $\psi(\sigma)$. To put it otherwise, an imaginary automorphism α for f is not the complexification of an automorphism of X or X^-. This justifies the terminology "imaginary automorphism", in case the formula $\overline{\alpha} = [-1]_{\mathbb{C}} \circ \alpha$ is not striking enough.

The following statement implies that a Gaussian curve is, up to isomorphism, determined by its branch locus.

Corollary 2.4. *Let (X, f) be a Gaussian curve. Let B be the branch locus of (X, f). Let (X', f') be another real hyperelliptic curve, with branch locus B'. Then (X', f') is isomorphic to (X, f) if and only if there is an automorphism β of $\mathbb{P}^1$ such that $\beta(B') = B$.*

Proof. Suppose that there is an isomorphism (γ, β) from (X', f') onto (X, f). Then, obviously, $\beta(B') = B$.

Conversely, suppose that there is an automorphism β of $\mathbb{P}^1$ that satisfies $\beta(B') = B$. Then, $(X', \beta \circ f')$ is a ramified double cover of $\mathbb{P}^1$ of branch locus B. By Lemma 2.2, $(X', \beta \circ f')$ is either isomorphic to (X, f) or to (X^-, f^-). Since (X, f) is Gaussian, $(X', \beta \circ f')$ is isomorphic to (X, f). Hence (X', f') is isomorphic to (X, f). □

3 Moduli of Gaussian real curves

Let $\mathcal{H}$ be the moduli space of all real hyperelliptic curves. As a set, $\mathcal{H}$ consists of all isomorphism classes of ramified double covers of the real projective line $\mathbb{P}^1$. The set $\mathcal{H}$ is easily seen to have a natural structure of a semianalytic variety (see [4] for the definition of a seminanalytic variety). Of course, $\mathcal{H}$ has infinitely many connected components. Indeed, let (X, f) and (X', f') be real hyperelliptic curves. Then, (X, f) and (X', f') belong to the same connected component of the moduli space $\mathcal{H}$ of all real hyperelliptic curves if and only if X and X' have the same genus, and the continuous maps

$$f_{|X(\mathbb{R})}\colon X(\mathbb{R}) \longrightarrow \mathbb{P}^1(\mathbb{R}) \quad \text{and} \quad f'_{|X'(\mathbb{R})}\colon X'(\mathbb{R}) \longrightarrow \mathbb{P}^1(\mathbb{R})$$

are homeomorphic.

Now, let $\mathcal{G}$ be the locus in $\mathcal{H}$ of all Gaussian curves. Let us show that $\mathcal{G}$ is a real analytic subvariety of $\mathcal{H}$.

Define an involution ι on $\mathcal{H}$ by

$$\iota(X, f) = (X^-, f^-).$$

It is clear that $\iota^2 = \text{id}$. Since $\iota \neq \text{id}$, ι is an involution on $\mathcal{H}$. By definition of a Gaussian curve, the subset $\mathcal{G}$ of $\mathcal{H}$ of Gaussian curves is equal to the set of fixed points of ι on $\mathcal{H}$. Since ι is analytic, the set $\mathcal{G}$ is a real analytic subvariety of $\mathcal{H}$. In particular, $\mathcal{G}$ acquires a natural structure of a semianalytic variety. It is the moduli space of all Gaussian curves.

Let (X, f) be a real hyperelliptic curve. We say that (X, f) is *extraordinary* if f is unramified over the real points of $\mathbb{P}^1$. Otherwise, (X, f) is called *ordinary*. A connected component of $\mathcal{H}$ contains either only ordinary real hyperelliptic curves, or only extraordinary ones. Therefore, we define a connected component of $\mathcal{H}$ to be *ordinary* if all its elements are ordinary, and we define it to be *extraordinary* if all its elements are extraordinary.

Proposition 3.1. *The subset $\mathcal{G}$ of $\mathcal{H}$ of all Gaussian curves is contained in the union of all ordinary components of $\mathcal{H}$.* □

Proof. Let (X, f) be an extraordinary real hyperelliptic curve. The real locus of (X, f) is empty if and only if the real locus of (X^-, f^-) is not empty (see [1, Section 4.3]). So (X, f) and (X^-, f^-) are not isomorphic, i.e. (X, f) is not Gaussian. Then any Gaussian curve is ordinary. □

Let X be a real algebraic curve. Denote by $g(X)$ the genus of X. Denote by $r(X)$ the number of real components of X, i.e., the number of connected components of $X(\mathbb{R})$. Let g be a natural integer, and let r be a natural integer satisfying $1 \le r \le g+1$. Denote by $\mathcal{H}_{g,r}$ the subset of $\mathcal{H}$ of all ordinary real hyperelliptic curves (X,f) such that the genus of X is equal to g and the number of real components of X is equal to r. Then, $\mathcal{H}_{g,r}$ is an ordinary connected component of $\mathcal{H}$. Conversely, any ordinary connected component of $\mathcal{H}$ is of the form $\mathcal{H}_{g,r}$ for some natural integers g and r satisfying $1 \le r \le g+1$. Define

$$\mathcal{G}_{g,r} = \mathcal{G} \cap \mathcal{H}_{g,r},$$

i.e. $\mathcal{G}_{g,r}$ is the locus in $\mathcal{H}_{g,r}$ of Gaussian curves. The sequel of the paper is devoted to the study of $\mathcal{G}_{g,r}$.

Let (X,f) be a Gaussian curve. Let β be an automorphism of $\mathbb{P}^1$ of finite order satisfying $\beta(f(X(\mathbb{R}))) \neq f(X(\mathbb{R}))$ (cf. Theorem 2.3). Two cases can occur: either β has real fixed points, or β has not. In the former case we say that X is Gaussian of *type I*. In the latter case we say that X is Gaussian of *type II*. Note that a Gaussian curve can be of both types at the same time.

Let $\mathcal{G}^{\mathrm{I}}$ be the locus of Gaussian curves in $\mathcal{G}$ that are of type I, and let $\mathcal{G}^{\mathrm{II}}$ be the locus of Gaussian curves in $\mathcal{G}$ that are of type II. Then, $\mathcal{G}^{\mathrm{I}} \cup \mathcal{G}^{\mathrm{II}} = \mathcal{G}$. Let also $\mathcal{G}^{\mathrm{I}}_{g,r}$ be the locus of Gaussian curves of type I in $\mathcal{G}_{g,r}$, and let $\mathcal{G}^{\mathrm{II}}_{g,r}$ be the locus of Gaussian curves of type II in $\mathcal{G}_{g,r}$. Then, $\mathcal{G}_{g,r} = \mathcal{G}^{\mathrm{I}}_{g,r} \cup \mathcal{G}^{\mathrm{II}}_{g,r}$.

Define $\tilde{\mathcal{G}}$ to be the moduli space of triples (X,f,β), where (X,f) is a Gaussian curve and β is an automorphism of $\mathbb{P}^1$ of finite order that satisfies $\beta(B) = B$ and $\beta(f(X(\mathbb{R}))) \neq f(X(\mathbb{R}))$, where B is the branch locus of f. Here, two such triples (X,f,β) and (X',f',β') are said to be *isomorphic* if there is an isomorphism (h,k) from (X,f) into (X',f') that satisfies $\beta' \circ k = k \circ \beta$. As a set, $\tilde{\mathcal{G}}$ consists of all isomorphism classes of triples (X,f,β) as above. We will see that $\tilde{\mathcal{G}}$ has a natural structure of a semianalytic variety.

Define $\tilde{\mathcal{G}}^{\mathrm{I}}$ to be the locus in $\tilde{\mathcal{G}}$ of the isomorphism classes of (X,f,β) with β having real fixed points. Define $\tilde{\mathcal{G}}^{\mathrm{II}}$ to be the locus in $\tilde{\mathcal{G}}$ of the isomorphism classes of (X,f,β) with β not having real fixed points. Then, $\tilde{\mathcal{G}}$ is the disjoint union of $\tilde{\mathcal{G}}^{\mathrm{I}}$ and $\tilde{\mathcal{G}}^{\mathrm{II}}$. Let

$$\varphi\colon \tilde{\mathcal{G}} \longrightarrow \mathcal{G}$$

be the forgetful map. By Theorem 2.3, φ is surjective. Moreover, $\varphi(\tilde{\mathcal{G}}^{\mathrm{I}}) = \mathcal{G}^{\mathrm{I}}$ and $\varphi(\tilde{\mathcal{G}}^{\mathrm{II}}) = \mathcal{G}^{\mathrm{II}}$. By Theorem 2.3, φ has finite fibers.

4 Gaussian curves of type I

In this section we give an explicit description of the moduli space $\tilde{\mathcal{G}}^{\mathrm{I}}$. First, we show how to construct explicitly Gaussian curves of type I. After that, we will show that all Gaussian curves of type I are obtained by this construction (cf. Proposition 4.1).

For the explicit construction of Gaussian curves of type I, let k be a separable polynomial in $\mathbb{R}[x]$ such that $k(0) \neq 0$. Let $X = X_k$ be the smooth model of the real curve defined by the equation $y^2 = xk(x^2)$, and let $f = f_k \colon X \to \mathbb{P}^1$ be the morphism defined by $f(x, y) = x$. Then (X, f) is a real hyperelliptic curve, as defined in Section 2.

Let $\beta \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the automorphism $\beta(x) = -x$. Then, there is a commuting imaginary automorphism α for f inducing β by Theorem 2.3. Indeed, since the polynomial $xk(x^2)$ changes sign at $x = 0$, the automorphism β does not map the subset $f(X(\mathbb{R}))$ of $\mathbb{P}^1(\mathbb{R})$ into itself. Therefore, (X, f, β) is a Gaussian curve of type I.

Let Σ be the subset of $\mathbb{R}[x]$ of all separable polynomials k satisfying $k(0) \neq 0$. Let

$$\pi \colon \Sigma \longrightarrow \tilde{\mathcal{G}}^{\mathrm{I}}$$

be the map $\pi(k) = (X_k, f_k, x \mapsto -x)$.

Proposition 4.1. *The map π is surjective.*

Proof. Let (X, f, β) be an element of $\tilde{\mathcal{G}}^{\mathrm{I}}$, i.e., X is a Gaussian curve of type I, and the automorphism β has real fixed points. Since $\beta \neq \mathrm{id}$, the automorphism β has exactly 2 real fixed points. Then, after a change of coordinates of $\mathbb{P}^1$, we may assume that β is the automorphism on $\mathbb{P}^1$ defined by $\beta(x) = -x$. In particular, the order of β is equal to 2.

Now, there is a nonzero separable polynomial h in $\mathbb{R}[x]$ such that X is a smooth model of the real curve defined by the equation $y^2 = h(x)$ and such that f is the mapping $(x, y) \mapsto x$. Since β respects the branch locus of f, $\beta^\star(h) = \lambda h$, for some nonzero real number λ. Since β does not map $f(X(\mathbb{R}))$ into itself, $\lambda < 0$. Since β has finite order, $\lambda = -1$. It follows that there is a separable polynomial k in $\mathbb{R}[x]$ with $k(0) \neq 0$ and such that $h(x) = xk(x^2)$. Hence, X is a smooth model of the curve defined by the equation $y^2 = xk(x^2)$, i.e., (X, f, β) is isomorphic to the Gaussian curve $(X_k, f_k, x \mapsto -x)$ of type I. □

The map π is, in fact, a quotient map. Indeed, the group $\mathbb{R}^\star$ acts left-handedly on Σ by multiplication, i.e, $\lambda \cdot k = \lambda k$. The group $\Gamma = \mathbb{R}^\star \rtimes \mu_2$ acts right-handedly on Σ as follows. The first factor of Γ acts by $\lambda \cdot k(x) = \lambda k(\lambda^2 x)$. The second factor of Γ acts by $(-1) \cdot k = k(x^{-1}) \cdot x^d$, where $d = \deg(k)$. It is clear that the actions of $\mathbb{R}^\star$ and Γ on Σ commute.

Since a Gaussian curve is determined by its branch locus (cf. Corollary 2.4), one has the following statement.

Theorem 4.2. *The map* π *above induces a bijection*

$$\overline{\pi}\colon \mathbb{R}^\star\backslash\Sigma/\Gamma \longrightarrow \tilde{\mathcal{G}}^{\mathrm{I}}.$$

In particular, $\tilde{\mathcal{G}}^{\mathrm{I}}$ *acquires a natural structure of a semianalytic variety as a quotient of* Σ. □

For $k \in \Sigma$, let $d(k)$ denote its degree, $p(k)$ the number of its positive real roots, and $q(k)$ the number of its negative real roots. Let d, p and q be natural integers. Let $\Sigma_{(d,p,q)}$ be the set of polynomials $k \in \Sigma$ such that $d(k) = d$, $p(k) = p$ and $q(k) = q$. Of course, $\Sigma_{(d,p,q)}$ is nonempty if and only if $p + q \leq d$ and $p + q \equiv d \pmod 2$. If nonempty, $\Sigma_{(d,p,q)}$ has exactly 2 connected components: two polynomials $k, \ell \in \Sigma_{(d,p,q)}$ belong to the same connected component of $\Sigma_{(d,p,q)}$ if and only if $k(0)$ and $\ell(0)$ have the same sign. Now, $\Sigma_{(d,p,q)}$ is stable for the actions of $\mathbb{R}^\star$ and Γ on Σ. The quotient $\mathbb{R}^\star\backslash\Sigma_{(d,p,q)}/\Gamma$ is connected. Therefore, the connected components of $\mathbb{R}^\star\backslash\Sigma/\Gamma$ are $\mathbb{R}^\star\backslash\Sigma_{(d,p,q)}/\Gamma$, where d, p, q satisfy the conditions $p+q \leq d$ and $p+q \equiv d \pmod 2$.

It follows that the connected components of $\tilde{\mathcal{G}}^{\mathrm{I}}$ are the subsets $\pi(\Sigma_{(d,p,q)})$.

Theorem 4.3. *Let* g *and* r *be natural integers satisfying* $1 \leq r \leq g+1$. *The connected component* $\mathcal{H}_{g,r}$ *of the moduli space* $\mathcal{H}$ *of real hyperelliptic curves contains Gaussian curves of type I. More precisely, the irreducible components of the locus of Gaussian curves of type I in the moduli space* $\mathcal{H}_{g,r}$ *are the subsets* $\varphi(\pi(\Sigma_{(g,r-1,q)}))$, *where* q *is a natural integer satisfying* $q \leq g - r + 1$ *and* $q \equiv g - r + 1 \pmod 2$. *In particular, all its irreducible components are of dimension* $g - 1$, *if* $g \geq 1$.

Proof. Since the connected components of $\tilde{\mathcal{G}}^{\mathrm{I}}$ are the subsets $\pi(\Sigma_{(d,p,q)})$, the irreducible components of $\mathcal{G}^{\mathrm{I}}$ are the subsets $\varphi(\pi(\Sigma_{(d,p,q)}))$, where $p + q \leq d$ and $p+q \equiv d \pmod 2$. In order to show the statement of the theorem, it suffices to show that a real hyperelliptic curve (X, f) belonging to $\varphi(\pi(\Sigma_{(d,p,q)}))$ has genus d and has $p + 1$ real components. But this is easy to check. □

5 Gaussian curves of type II

To study Gaussian curves of type II it is useful to introduce the following notation. Let B be a reduced effective divisor on $\mathbb{P}^1$, such that $B(\mathbb{R})$ is of positive even degree. Let I be a finite union of intervals of $\mathbb{P}^1(\mathbb{R})$ such that the boundary ∂I is equal to $B(\mathbb{R})$. We say that an automorphism $\beta \in \mathrm{Aut}(\mathbb{P}^1, B)$ is of signature -1 if $\beta(I) \neq I$.

Let $\tilde{\mathcal{B}}$ be the moduli space of all pairs (B, β) where B is as above, and $\beta \in \mathrm{Aut}(\mathbb{P}^1, B)$ is an automorphism of finite order and of signature -1. As a set, $\tilde{\mathcal{B}}$ consists of the isomorphism classes of all pairs (B, β). Two such pairs (B, β) and (B', β') are isomorphic if there is an automorphism k of $\mathbb{P}^1$ such that $k(B) = B'$ and $k \circ \beta = \beta' \circ k$. It is clear that $\tilde{\mathcal{B}}$ has a natural structure of a semianalytic variety.

Let

$$\rho\colon \tilde{\mathcal{G}} \longrightarrow \tilde{\mathcal{B}}$$

be the map that associates to an element (X, f, β) of $\tilde{\mathcal{G}}$, the element (B, β) of $\tilde{\mathcal{B}}$, where B is the branch locus of f. Then, ρ is a bijection by Theorem 2.3.

Define the subset $\tilde{\mathcal{B}}^{\mathrm{I}}$ of $\tilde{\mathcal{B}}$ consisting of all pairs (B, β), where β has real fixed points. Define also the subset $\tilde{\mathcal{B}}^{\mathrm{II}}$ of $\tilde{\mathcal{B}}$ consisting of all pairs (B, β), where β has no real fixed points. Then, $\tilde{\mathcal{B}}$ is the disjoint union of $\tilde{\mathcal{B}}^{\mathrm{I}}$ and $\tilde{\mathcal{B}}^{\mathrm{II}}$. In fact, $\tilde{\mathcal{B}}^{\mathrm{I}}$ and $\tilde{\mathcal{B}}^{\mathrm{II}}$ are open subsets of $\tilde{\mathcal{B}}$. In particular, $\tilde{\mathcal{B}}^{\mathrm{I}}$ and $\tilde{\mathcal{B}}^{\mathrm{II}}$ come along with a natural structure of a semianalytic variety. Again, the restrictions ρ^{I} and ρ^{II} of ρ to the subsets $\tilde{\mathcal{G}}^{\mathrm{I}}$ and $\tilde{\mathcal{G}}^{\mathrm{II}}$, respectively, are isomorphisms onto $\tilde{\mathcal{G}}^{\mathrm{I}}$ and $\tilde{\mathcal{G}}^{\mathrm{II}}$, respectively.

Now we are ready to give an explicit description of the moduli space $\tilde{\mathcal{G}}^{\mathrm{II}}$. It will give rise to an explicit description of each irreducible component of $\mathcal{G}^{\mathrm{II}}$.

Let (X, f, β) be an element of $\tilde{\mathcal{G}}^{\mathrm{II}}$, i.e., X is a Gaussian curve of type II, and β is an automorphism of $\mathbb{P}^1$ having no real fixed points and of even nonzero order, by Theorem 2.3. Let n be the order of β. Of course, the nonzero natural integer n is uniquely determined by (X, f, β). Let us call n the *order* of (X, f, β). Denote by $\tilde{\mathcal{G}}^{\mathrm{II},n}$ the subset of $\tilde{\mathcal{G}}^{\mathrm{II}}$ of all (X, f, β) of order n. Of course, $\tilde{\mathcal{G}}^{\mathrm{II}}$ is the disjoint union of the subsets $\tilde{\mathcal{G}}^{\mathrm{II},n}$, for $n \in 2\mathbb{N}$, $n \neq 0$. Let $\mathcal{G}^{\mathrm{II},n}$ be the image of $\tilde{\mathcal{G}}^{\mathrm{II},n}$ in $\mathcal{H}$ by the forgetful map φ. We say that a Gaussian curve (X, f) of type II is of order n if it belongs to the subset $\mathcal{G}^{\mathrm{II},n}$ of $\mathcal{H}$. Of course, a Gaussian curve of type II can be of several orders at the same time, i.e., the intersection $\mathcal{G}^{\mathrm{II},n} \cap \mathcal{G}^{\mathrm{II},k}$ is not necessarily empty.

As for the case of Gaussian curves of type I, we need to determine the genus and the number of real components that can have a Gaussian curve of type II. More precisely, given $g, r \in \mathbb{N}$ with $1 \leq r \leq g + 1$, we determine a necessary and sufficient condition on g, r for $\mathcal{G}^{\mathrm{II}}_{g,r} = \mathcal{G}^{\mathrm{II}} \cap \mathcal{H}_{g,r}$ to be empty or not.

Lemma 5.1. *Let r and ℓ be nonzero natural integers. Then, ℓ is the order of an odd element of $\mathbb{Z}/2r\mathbb{Z}$, if and only if ℓ divides $2r$ and the quotient $\frac{2r}{\ell}$ is odd.*

Proof. Note that it makes sense to speak about odd elements of $\mathbb{Z}/2r\mathbb{Z}$ since $2r$ is even. It is clear that ℓ is the order of an element of $\mathbb{Z}/2r\mathbb{Z}$ if and only if ℓ divides $2r$. Moreover, the elements of $\mathbb{Z}/2r\mathbb{Z}$ of order ℓ are the generators of the subgroup $k\mathbb{Z}/2r\mathbb{Z}$ of $\mathbb{Z}/2r\mathbb{Z}$, where $k = \frac{2r}{\ell}$. These generators are either all odd, or all even. Since k is of order ℓ in $\mathbb{Z}/2r\mathbb{Z}$, the integer ℓ is the order of an odd element of $\mathbb{Z}/2r\mathbb{Z}$ if and only if ℓ divides $2r$ and the quotient $\frac{2r}{\ell}$ is odd. $\square$

Lemma 5.2. *Let r and ℓ be nonzero natural integers. Let $h\colon S^1 \to S^1$ be the homeomorphism defined by $h(x) = \xi x$, where ξ is a primitive ℓth root of unity. Then, there is a disjoint union I of r closed intervals in S^1 such that $h(\partial I) \subseteq \partial I$ if and only if ℓ divides $2r$. Moreover, in that case, $h(I) \not\subseteq I$ if and only if $\frac{2r}{\ell}$ is odd.*

Proof. The first statement is clear. In order to show the second, suppose that ℓ divides $2r$, and let I be a disjoint union of r closed intervals such that $h(\partial I) \subseteq \partial I$. The complement $S^1 \setminus \partial I$ has exactly $2r$ connected components. Moreover, the set of connected components of $S^1 \setminus \partial I$ is cyclically ordered. Choose a bijection between $\mathbb{Z}/2r\mathbb{Z}$ and the set of connected components of $S^1 \setminus \partial I$ that respects cyclic orderings. Then h induces a bijection τ from $\mathbb{Z}/2r\mathbb{Z}$ into itself that respects the cyclic ordering. It follows that there is an element $s \in \mathbb{Z}$ such that $\tau(x) = x + s$ for all $x \in \mathbb{Z}/2r\mathbb{Z}$. Since the order of τ is equal to ℓ, the integer s is of order ℓ in $\mathbb{Z}/2r\mathbb{Z}$.

Now, $h(I) \not\subseteq I$ if and only if s is odd. By the preceding lemma, $h(I) \not\subseteq I$ if and only if $\frac{2r}{\ell}$ is odd. $\square$

Theorem 5.3. *Let $g, r \in \mathbb{N}$ such that $1 \leq r \leq g + 1$. Let $n \in 2\mathbb{N}$, $n \neq 0$. Then there is a Gaussian curve in $\mathcal{H}_{g,r}$ of type II and of order n if and only if the following two conditions are satisfied.*

1. *n divides $2r$ and the quotient $\frac{2r}{n}$ is odd, and*
2. *n divides $g - r + 1$ or n divides $g - r$.*

Proof. Suppose that (X, f, β) is a Gaussian curve of type II and of order n such that X is of genus g and has r real components. As before, we may assume that the fixed points of β are the points $\pm\sqrt{-1}$. Since β does not map $f(X(\mathbb{R}))$ into itself, the order n of β divides $2r$, and $\frac{2r}{n}$ is odd, by Lemma 5.2. Moreover, f has $2g + 2$ ramification points. Exactly $2r$ of them are real. Hence, exactly $2g + 2 - 2r$ of them are nonreal. The set of nonreal ramification points of f is stable for the action of β. Since the complex conjugate of a nonreal ramification point of f is also a ramification point, and since β only has $\pm\sqrt{-1}$ as fixed points, either $2g + 2 - 2r$ is divisible by $2n$, or $2g - 2r$ is divisible by $2n$. It follows that n divides $g - r + 1$, or n divides $g - r$.

Conversely, suppose that $n \in 2\mathbb{N} \setminus \{0\}$ satisfies conditions 1 and 2. Let us show that there is a Gaussian curve (X, f) in $\mathcal{H}_{g,r}$ of type II and of order n. First, choose an automorphism β of $\mathbb{P}^1$ of order n having $\pm\sqrt{-1}$ as fixed points. Since n divides $2r$, one can choose $2r$ distinct real points $P_1, \ldots, P_{2r}$ of $\mathbb{P}^1$ such that the set $\{P_1, \ldots, P_{2r}\}$ is stable for the action of β. Now, there are two cases to consider: the case that $2n$ divides $2g - 2r + 2$, and the case that $2n$ divides $2g - 2r$. In the first case, one can choose $2g - 2r + 2$ nonreal complex points $P_{2r+1}, \ldots, P_{2g+2}$ of $\mathbb{P}^1$ different from $\pm\sqrt{-1}$ such that the set $\{P_{2r+1}, \ldots, P_{2g+2}\}$ is stable for complex conjugation, as well as for the action of β. In the second case, one can choose $2g - 2$ of such points $P_{2r+1}, \ldots, P_{2g}$, and one lets $P_{2g+1} = \sqrt{-1}$ and $P_{2g+2} = -\sqrt{-1}$. In both cases, one ends up with a set of $2g + 2$ complex points that is stable for complex conjugation, as well as for the action of β. Moreover, precisely $2r$ of them are real. It follows that each ramified double covering of $\mathbb{P}^1$ ramified at these $2g + 2$ complex points defines a real hyperelliptic curve (X, f) of genus g having r real components. By Lemma 5.2, β does not map $f(X(\mathbb{R}))$ into itself. Therefore, (X, f, β) is a Gaussian curve of type II and of order n. $\square$

We need the following notation. For an integer k, denote by $\mathrm{ord}_2(k)$ the 2-valuation of k, i.e., the unique natural integer i such that $k = 2^i k'$ with k' an odd integer.

Corollary 5.4. *Let $g, r \in \mathbb{N}$ such that $1 \le r \le g + 1$. There is a Gaussian curve in $\mathcal{H}_{g,r}$ of type II if and only if $\mathrm{ord}_2(g) = \mathrm{ord}_2(r)$ or $\mathrm{ord}_2(g + 1) = \mathrm{ord}_2(r)$. Moreover, in that case, $\mathcal{H}_{g,r}$ contains Gaussian curves of type II and of order 2^{i+1}, where $i = \mathrm{ord}_2(r)$. Furthermore, the order of any Gaussian curve of type II in $\mathcal{H}_{g,r}$ is a multiple of 2^{i+1}.*

Proof. Suppose that there is a Gaussian curve (X, f, β) of type II of genus g and having r real components. Let us show that $\mathrm{ord}_2(g) = \mathrm{ord}_2(r)$ or $\mathrm{ord}_2(g+1) = \mathrm{ord}_2(r)$. Let n be the order of β. Let $i = \mathrm{ord}_2(r)$ and write $r = 2^i r'$ with r' an odd integer. By Theorem 5.3, $\frac{2r}{n}$ is odd. Hence, $\mathrm{ord}_2(n) = \mathrm{ord}_2(2r) = i + 1$. Write $n = 2^{i+1} n'$, with n' an odd integer. By Theorem 5.3, there are two cases to consider: the case where n divides $g - r + 1$ and the case where n divides $g - r$.

In the first case, one necessarily has that 2^{i+1} divides $g - 2^i r' + 1$. Then, 2^i divides $g + 1$. Write $g + 1 = 2^i g'$, where g' is an integer. Then, 2 divides $g' - r'$. Since r' is odd, g' is odd as well. It follows that $\mathrm{ord}_2(g + 1) = i = \mathrm{ord}_2(r)$.

In the second case, one obtains similarly that $\mathrm{ord}_2(g) = \mathrm{ord}_2(r)$. This shows that $\mathrm{ord}_2(g) = \mathrm{ord}_2(r)$ or $\mathrm{ord}_2(g + 1) = \mathrm{ord}_2(r)$.

In order to show the converse, suppose that $\mathrm{ord}_2(g) = \mathrm{ord}_2(r)$ or $\mathrm{ord}_2(g+1) = \mathrm{ord}_2(r)$. Let $i = \mathrm{ord}_2(r)$ and let $n = 2^{i+1}$. Then, n satisfies the conditions of Theorem 5.3. It follows that there is a Gaussian curve of type II and of genus g having r real components.

The last two statements follow immediately from Theorem 5.3. $\square$

Let g and r be natural integers such that $1 \leq r \leq g+1$. Let $\tilde{\mathcal{B}}^{\mathrm{II}}_{g,r}$ be the subset of $\tilde{\mathcal{B}}^{\mathrm{II}}$ consisting of all pairs (B,β), where $\deg(B) = 2g+2$ and $\deg(B \cap \mathbb{P}^1(\mathbb{R})) = 2r$. Of course, $\tilde{\mathcal{B}}^{\mathrm{II}}_{g,r}$ is a nonempty open subset of $\tilde{\mathcal{B}}^{\mathrm{II},n}$. Moreover, $\tilde{\mathcal{B}}^{\mathrm{II}}$ is the disjoint union of the subsets of the form $\tilde{\mathcal{B}}^{\mathrm{II}}_{g,r}$.

Let n be a nonzero even natural integer. Let $\tilde{\mathcal{B}}^{\mathrm{II},n}$ be the subset of $\tilde{\mathcal{B}}^{\mathrm{II}}$ of all pairs (B,β) for which the order of β is equal to n. Of course, $\tilde{\mathcal{B}}^{\mathrm{II},n}$ is an open subset of $\tilde{\mathcal{B}}^{\mathrm{II},n}$. Moreover, $\tilde{\mathcal{B}}^{\mathrm{II}}$ is the disjoint union of the subsets $\tilde{\mathcal{B}}^{\mathrm{II},n}$, where n runs through all nonzero even natural integers.

Put

$$\tilde{\mathcal{B}}^{\mathrm{II},n}_{g,r} = \tilde{\mathcal{B}}^{\mathrm{II}}_{g,r} \cap \tilde{\mathcal{B}}^{\mathrm{II},n}.$$

According to Theorem 5.3, $\tilde{\mathcal{B}}^{\mathrm{II},n}_{g,r}$ is nonempty if and only if

1. n divides $2r$ and the quotient $\frac{2r}{n}$ is odd, and

2. n divides $g-r+1$ or n divides $g-r$.

In that case, it is easy to see that

$$\dim \tilde{\mathcal{B}}^{\mathrm{II},n}_{g,r} = \begin{cases} \frac{2g+2}{n} - 1 & \text{if } n \text{ divides } g-r+1, \text{ and} \\ \frac{2g}{n} - 1 & \text{if } n \text{ divides } g-r. \end{cases}$$

Theorem 5.5. *Let g and r be natural integers satisfying $1 \leq r \leq g+1$. Suppose that $\mathrm{ord}_2(g) = \mathrm{ord}_2(r)$ or $\mathrm{ord}_2(g+1) = \mathrm{ord}_2(r)$. Then, the locus $\mathcal{G}^{\mathrm{II}}_{g,r}$ of Gaussian curves of type II in $\mathcal{H}_{g,r}$ is equal to*

$$\mathcal{G}^{\mathrm{II}}_{g,r} = \varphi \circ \rho^{-1}(\tilde{\mathcal{B}}^{\mathrm{II},2^{i+1}}_{g,r}),$$

where $i = \mathrm{ord}_2(r)$. In particular, $\mathcal{G}^{\mathrm{II}}_{g,r}$ is irreducible and

$$\dim \mathcal{G}^{\mathrm{II}}_{g,r} = \begin{cases} \frac{g+1}{2^i} - 1 & \text{if } \mathrm{ord}_2(g+1) = \mathrm{ord}_2(r), \text{ and} \\ \frac{g}{2^i} - 1 & \text{if } \mathrm{ord}_2(g) = \mathrm{ord}_2(r). \end{cases} \qquad \square$$

6 Example: Gaussian curves of genus 2

In the present Section we apply our results to real curves of genus 2. In particular, by means of our general theory, we recover results of [2, Proposition 7.2]. All real curves of genus 2 are hyperelliptic and, by Proposition 3.1, we are interested only to the ordinary ones, that split into 3 homeomorphism classes: $\mathcal{H}_{2,1}$, $\mathcal{H}_{2,2}$ and $\mathcal{H}_{2,3}$.

Example 6.1. We start with the so-called M-curves, that is the curves of genus 2 whose real locus consists of 3 connected components. There are the following 3 types of Gaussian curves.

1. First of all, by Theorem 5.5 there is an irreducible 2-dimensional family of Gaussian curves of type II and order 2 (we denote it by $\mathcal{G}_{2,3}^{\mathrm{II},2}$, and similar notations will be used in the present section). The curves of this family may be represented by an affine plane equation of the form:

$$y^2 = x(x-a)(x-b)(ax+1)(bx+1).$$

In fact, by [6, pag. 347], for the general $X \in \mathcal{G}_{2,3}^{\mathrm{II},2}$, the automorphism group of $X_{\mathbb{C}}$ is the non-cyclic group of order 4, generated by the hyperelliptic involution and by the automorphism α_1 defined as:

$$\alpha_1(x,y) = \left(-\frac{1}{x}, \frac{y\sqrt{-1}}{x^3}\right).$$

$\mathrm{Aut}X_{\mathbb{C}}$ consists of 2 real automorphisms (the trivial one and the hyperelliptic involution) and 2 imaginary automorphisms (α_1 and its conjugate). The real automorphism β_1 induced on $\mathbb{P}^1$ by α_1 and $\overline{\alpha_1}$ is of order 2 and it has no real fixed points; so, as expected, it provides X of a structure of Gaussian curve of type II and order 2.

2. Next, by Theorem 4.3, there is an irreducible 1-dimensional family of Gaussian curves of type I, represented by affine plane equations of the form

$$y^2 = x(x^2 - a^2)(a^2x^2 - 1).$$

For the general curve X of this family the group $\mathrm{Aut}X_{\mathbb{C}}$ has order 8 and it is generated by the automorphism α_1 above and the automorphism α_2 defined as:

$$\alpha_2(x,y) = (-x, y\sqrt{-1}).$$

α_1, α_2 and their conjugate are the 4 imaginary automorphisms of X. The pair $\alpha_1, \overline{\alpha_1}$, as we have already seen, give to the curve the structure of Gaussian curve of type II and order 2. On the other hand, the pair $\alpha_2, \overline{\alpha_2}$ provides X of a structure of Gaussian curve of type I, as we have seen in Proposition 4.1.

3. Finally, by Theorem 5.3 there is a single Gaussian curve of type II and order 6. An easy computation (we omit the details) shows that it is the curve X of equation

$$y^2 = 3x^5 - 10x^3 + 3x$$

and that $\mathrm{Aut}X_{\mathbb{C}}$ has order 24 and it is generated by the automorphism α_2 above and by the automorphism α_3 defined as:

$$\alpha_3(x,y) = \left(\frac{\sqrt{3}x+1}{-x+\sqrt{3}}, \frac{8y\sqrt{-1}}{(-x+\sqrt{3})^3}\right).$$

So there are 12 imaginary automorphisms on X: $\alpha_1 = \alpha_3^3$, α_2, α_3, $\alpha_4 = \alpha_3^5$, $\alpha_5 = \alpha_2\alpha_3^2$, $\alpha_6 = \alpha_2\alpha_3^4$ and their conjugate. We have already described the automorphisms $\alpha_1, \overline{\alpha_1}$ and $\alpha_2, \overline{\alpha_2}$: they give to the curve the structure of Gaussian curve of type II and order 2 and of type I, respectively. The automorphisms β_3 and β_4 induced by the pairs $\alpha_3, \overline{\alpha_3}$ and $\alpha_4, \overline{\alpha_4}$ are of order 6 and so they provided X of a structure of Gaussian curve of type II and order 6. Finally, it is immediate that the automorphism $\alpha_5, \overline{\alpha_5}$ and $\alpha_6, \overline{\alpha_6}$ are additional imaginary automorphisms of type I.

We remark that there is a chain of inclusions

$$\mathcal{G}_{2,3}^{\mathrm{II},6} \subset \mathcal{G}_{2,3}^{\mathrm{I}} \subset \mathcal{G}_{2,3}^{\mathrm{II},2} = \mathcal{G}_{2,3} \subset \mathcal{H}_{2,3}$$

of irreducible spaces having dimension 0,1,2 and 3, respectively. We also remark that, for the curves seen in the present example, all the automorphisms are either real or imaginary: this is a very particular case, because in general a complex non-real automorphism is not necessarily "purely imaginary" in the sense of our definition!

Example 6.2. For real curves of genus 2 having 2 real connected components, Theorems 4.3 and 5.3 ensure that there are only 2 types of Gaussian curves: there is a 1-dimensional irreducible family of Gaussian curves of type I, consisting of the curves of equation

$$y^2 = x(x^2 + 1)(x^2 - a^2),$$

and one curve of this family has also a structure of Gaussian curve of type II and order 4: one easily sees that it is the curve of equation

$$y^2 = x^5 - x.$$

So there is a chain of inclusions

$$\mathcal{G}_{2,2}^{\mathrm{II},4} \subset \mathcal{G}_{2,2}^{\mathrm{I}} = \mathcal{G}_{2,2} \subset \mathcal{H}_{2,2}$$

of irreducible spaces having dimension 0,1 and 3, respectively. Note that there are no Gaussian curves of type II and order 2: this imply that the locus of Gaussian curves has lower dimension than for the M-curves and in particular it does not disconnect the entire space $\mathcal{H}_{2,2}$.

Example 6.3. Finally we consider the case of real curves of genus 2 having 1 real connected component. By Theorem 5.5 there is a 2-dimensional irreducible family of Gaussian curves of type II, consisting of curves of equation

$$y^2 = x(x^2 - 2ax + a^2 + b^2)((a^2 + b^2)x^2 + 2ax + 1).$$

There is also a family of Gaussian curves of type I, but the main difference with respect to previous cases is that it is not irreducible: as explained in Theorem 4.3, it splits into two 1-dimensional families, consisting of the curves of equations

$$y^2 = x(x^4 + a^2)$$

and

$$y^2 = x(x^2 + 1)(x^2 + a^2).$$

As above we have the following kind of stratification:

$$\mathcal{G}^{\mathrm{I}}_{2,1} \subset \mathcal{G}^{\mathrm{II},2}_{2,1} = \mathcal{G}_{2,1} \subset \mathcal{H}_{2,1}.$$

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Typeset by $\mathcal{AMS}$-LaTeX