Heights on abelian varieties

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1 Height on projective space

We will follow to a great extend Chapter 6 of [5].

If K is a number field and v a finite place of K, that is, v corresponds to a prime ideal $\mathfrak{p}$ of the ring of integers of K, then we define a norm $\|\cdot\|_v$ on K by

$$\|x\|_v = \left(\frac{1}{N\mathfrak{p}}\right)^{\mathrm{ord}_{\mathfrak{p}}(x)},$$

where $N\mathfrak{p}$ is the absolute norm of $\mathfrak{p}$. If v is an infinite place, that is, v corresponds to an embedding σ of K in $\mathbb{R}$ or v corresponds to a conjugate pair $\{\sigma, \overline{\sigma}\}$ of embeddings of K in $\mathbb{C}$, then we define a norm $\|\cdot\|_v$ on K by

$$\|x\|_v = \begin{cases} |\sigma(x)|, & \text{if } v \text{ is real,} \\ |\sigma(x)|^2, & \text{if } v \text{ is complex.} \end{cases}$$

Clearly, for any place v of K, the homothety $y \mapsto xy$ transforms a Haar measure μ on the completion K_v of K at v into $\|x\|_v \cdot \mu$. Let M_K be the set of places of K and M_K^∞ the set of infinite places. Then we have the *product formula*

$$\prod_{v \in M_K} \|x\|_v = 1,$$

for every $x \in K^\star$. This can be easily seen as follows (cf. A. Weil: Basic Number Theory, Ch. IV, Sect. 4, Theorem 5). Let A be the ring of adèles of K. The product formula will follow if we prove that any Haar measure on

A is invariant under the homothety $\lambda_x : y \mapsto xy$ of A, for any $x \in K^\star$. Since A/K is a compact topological group and λ_x induces an isomorphism $\overline{\lambda_x}$ of A/K, any Haar measure on A/K is invariant under $\overline{\lambda_x}$. Moreover, since K is discrete and the restriction $\widehat{\lambda_x}$ of λ_x to K is an isomorphism of K as a topological group, any Haar measure on K is invariant under $\widehat{\lambda_x}$. Therefore, any Haar measure on A is invariant under λ_x.

If $P = (x_0 : \cdots : x_n)$ is in $\mathbb{P}^n(K)$ then we define the *height of P relative to K* by

$$H_K(P) = \prod_{v \in M_K} \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}.$$

Observe that, by the product formula, this is well defined.

Example 1. If $K = \mathbb{Q}$ and $P \in \mathbb{P}^n(\mathbb{Q})$ then we may assume $P = (x_0 : \cdots : x_n)$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$. Then

$$H_{\mathbb{Q}}(P) = \max\{|x_0|, \ldots, |x_n|\}.$$

$\square$

It is clear that $H_K(P) \geq 1$, for every $P \in \mathbb{P}^n(K)$. If L is a finite extension of K and $P \in \mathbb{P}^n(K)$ then

$$H_L(P) = H_K(P)^{[L:K]}.$$

Hence, we can define the *absolute height* H on $\mathbb{P}^n(\overline{K})$ by

$$H(P) = H_L(P)^{\frac{1}{[L:\mathbb{Q}]}},$$

where L is some number field containing the coordinates of P. It will be convenient to define the *(logarithmic) height* h on $\mathbb{P}^n(\overline{K})$ by

$$h(P) = \log H(P).$$

Theorem 2 (Northcott) *Let C and d be constants. Then*

$$\{P \in \mathbb{P}^n(\overline{K}) \mid H(P) \leq C[K(P) : K] \leq d\}$$

is a finite set.

For a proof the reader is refered to [5], Chapter 6 or [32].

2 Heights on projective varieties

We will define height functions on a projective algebraic variety V over a number field K, using morphisms from V into projective space. Suppose

$$f: V \longrightarrow \mathbb{P}^n$$

is a morphism of algebraic varieties over K. Then one defines the *(logarithmic) height on V relative to f* by

$$\begin{array}{rcl} h_f: V(\overline{K}) & \longrightarrow & \mathbb{R} \\ P & \longmapsto & h(f(P)). \end{array}$$

Let us call real-valued functions h and h' on the set $V(\overline{K})$ *equivalent*, denoted by $h \sim h'$, if $|h-h'|$ is bounded on $V(\overline{K})$. It turns out that the height h_f depends only, up to equivalence, on the invertible sheaf $\mathcal{L} = f^\star \mathcal{O}_{\mathbb{P}^n}(1)$.

Theorem 3 *Let V be a projective algebraic variety over K. If $f: V \to \mathbb{P}^n$ and $g: V \to \mathbb{P}^m$ are morphisms over K such that*

$$f^\star \mathcal{O}_{\mathbb{P}^n}(1) \cong g^\star \mathcal{O}_{\mathbb{P}^m}(1)$$

then h_f and h_g are equivalent.

Proof. Recall that a morphism $\varphi: V \to \mathbb{P}^k$ is uniquely determined by (the isomorphism class of) the invertible sheaf $\mathcal{L} = \varphi^\star \mathcal{O}(1)$ and the global sections $s_i = \varphi^\star x_i \in \Gamma(V, \mathcal{L})$. Therefore, it suffices to prove the theorem in the case that $m \geq n$ and $f = \pi \circ g$, where π is the rational map

$$\begin{array}{rcl} \pi: \mathbb{P}^m & \cdots\longrightarrow & \mathbb{P}^n \\ (x_0: \cdots : x_m) & \longmapsto & (x_0: \cdots : x_n). \end{array}$$

Clearly, $h_g - h_f \geq 0$. To prove that $h_g - h_f$ is bounded from above, observe that

$$g(V) \cap \mathfrak{V}(\mathfrak{X}_0, \ldots, \mathfrak{X}_n)$$

is empty. Since $g(V)$ is closed, let $I \subseteq K[X_0, \ldots, X_m]$ be its defining homogeneous ideal. Then

$$\sqrt{I + (X_0, \ldots, X_n)} = (X_0, \ldots, X_m)$$

in $K[X_0,\dots,X_m]$. Therefore, there exist a positive integer q and $F_{ij} \in K[X_0,\dots,X_m]$ such that

$$X_{n+i}^q - \sum_{j=0}^{n} F_{ij}X_j \in I, \quad i=0,\dots,m-n.$$

We may assume F_{ij} to be homogeneous of degree $q-1$. Denote the coefficients of F_{ij} by a_{ijk}. If $L \subseteq \overline{K}$ is a finite extension of K and w is a place of L then we define

$$\varepsilon_w = \begin{cases} 0 & \text{if } w \text{ is finite,} \\ 1 & \text{if } w \text{ is real,} \\ 2 & \text{if } w \text{ is complex} \end{cases}$$

and put

$$c_w = (n+1)^{\varepsilon_w} \binom{q-1+m}{m}^{\varepsilon_w} \cdot \max \|a_{ijk}\|_w.$$

Choose $P \in g(V)(L)$, say $P = (x_o : \cdots : x_m)$ with $x_i \in L$. It is easy to see that

$$\|x_{n+i}\|_w^q \leq c_w \cdot \max_{j\leq m} \|x_j\|_w^{q-1} \cdot \max_{j\leq n} \|x_j\|_w,$$

for $i=0,\dots,m-n$. Put

$$c'_w = \max\{1, c_w^{\frac{1}{q}}\},$$

then

$$\max_{i\leq m} \|x_i\|_w \leq c'_w \cdot \max_{j\leq n} \|x_j\|_w.$$

In particular,

$$\begin{aligned} H_L(x_0 : \cdots : x_m) &= \prod_{w\in M_L} \max_{i\leq m} \|x_i\|_w \\ &\leq \left(\prod_{w\in M_L} c'_w\right)\left(\prod_{w\in M_L} \max_{j\leq n} \|x_j\|_w\right) \\ &= \left(\prod_{v\in M_K} c'_v\right)^d H_L(x_0 : \cdots : x_n), \end{aligned}$$

where $d = [L\colon K]$. Therefore

$$h(x_0\colon \cdots \colon x_m) \leq h(x_0\colon \cdots \colon x_n) + c,$$

where $c = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} c'_v$ which neither depends on P nor on L.. Hence $h_g - h_f$ is bounded from above. $\square$

As a consequence, we can define, up to equivalence, a height function $h_{\mathcal{L}}$ for every invertible sheaf $\mathcal{L}$ on V which is basepoint-free. For, choose a morphism f over K from V into $\mathbb{P}^n$ such that

$$\mathcal{L} \cong f^\star \mathcal{O}_{\mathbb{P}^n}(1).$$

(Such a morphism exists since $\mathcal{L}$ is basepoint-free.) Then, by Theorem 3,

$$h_{\mathcal{L}} = h_f$$

depends only, up to equivalence, on $\mathcal{L}$. More precisely, one defines $h_{\mathcal{L}}$ as the equivalence class of h_f. That is, if $\mathcal{H}(V(\overline{K}))$ is the group of equivalence classes of real-valued functions on $V(\overline{K})$, we have $h_{\mathcal{L}} \in \mathcal{H}(V(\overline{K}))$. However, often we will treat $h_{\mathcal{L}}$ as a real-valued function, keeping in mind that $h_{\mathcal{L}}$ is only defined up to equivalence. It is easy to prove that, for any basepoint-free invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on V,

$$h_{\mathcal{L} \otimes \mathcal{M}} \quad \sim \quad h_{\mathcal{L}} + h_{\mathcal{M}}. \tag{1}$$

As a consequence, for any invertible sheaf $\mathcal{L}$ on V, we can define, up to equivalence, a height function $h_{\mathcal{L}}$ by

$$h_{\mathcal{L}} = h_{\mathcal{L}_1} - h_{\mathcal{L}_2},$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are basepoint-free invertible sheaves on V such that

$$\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}.$$

(Such sheaves always exist; see [HAG], p. 121.) By (1), this does not depend on $\mathcal{L}_1$, $\mathcal{L}_2$. Hence the following result due to A. Weil.

Theorem 4 *Let V be a projective algebraic variety over K. There exists a unique homomorphism*

$$h\colon \operatorname{Pic} V \longrightarrow \mathcal{H}(V(\overline{K}))$$

such that

(i) *if* $V = \mathbb{P}^n$ *then* $h_{\mathcal{O}_{\mathbb{P}^n}(1)}$ *is the usual height* h *on projective space.*

(ii) *if* W *is a projective algebraic variety over* K *and* $f\colon V \to W$ *is a morphism over* K *then*

$$h_{f^*\mathcal{L}} = h_{\mathcal{L}} \circ f,$$

for any $\mathcal{L} \in \operatorname{Pic} W$.

It is then easy to prove, using Theorem 2, the following finiteness theorem.

Theorem 5 *Let* V *be a projective algebraic variety over* K. *If* $\mathcal{L}$ *is an ample sheaf on* V *then, for all constants* C *and* d, *the set*

$$\{P \in V(\overline{K}) \mid h_{\mathcal{L}}(P) \leq C[K(P):K] \leq d\}$$

is a finite set.

Observe that it makes sense to call an element of $\mathcal{H}(V)$ bounded from below (or above).

Theorem 6 *Let* V *be a projective algebraic variety over* K. *If* $\mathcal{L}$ *is an invertible sheaf on* V *and* s *is a global section then* $h_{\mathcal{L}}$ *is bounded from below on the set*

$$\{P \in V(\overline{K}) \mid s(P) \neq 0\}.$$

Proof. Choose basepoint-free invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ on V such that

$$\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}.$$

Let $s_0, \ldots, s_n$ be global sections of $\mathcal{L}_2$ that generate $\mathcal{L}_2$. Choose global sections $s_{n+1}, \ldots, s_m$ of $\mathcal{L}_1$ such that

$$s \otimes s_0, \ldots, s \otimes s_n, s_{n+1}, \ldots, s_m$$

generate $\mathcal{L}_1$. Then, whenever $P \in V(\overline{K})$ with $s(P) \neq 0$,

$$\begin{aligned} h_{\mathcal{L}_1}(P) &= h(s \otimes s_0(P), \ldots, s \otimes s_n(P), s_{n+1}(P), \ldots, s_m(P)) \\ &\geq h(s \otimes s_0(P), \ldots, s \otimes s_n(P)) \\ &= h(s_0(P), \ldots, s_n(P)) \\ &= h_{\mathcal{L}_2}(P). \end{aligned}$$

Therefore, $h_{\mathcal{L}}$ is bounded from below on the set of $P \in V(\overline{K})$ such that $s(P) \neq 0$. □

3 Heights on abelian varieties

We will need the Theorem of the Cube.

Theorem 7 *Let X_1, X_2, X_3 be complete algebraic varieties over the field K and let $P_i \in X_i(K)$. Then, an invertible sheaf $\mathcal{L}$ on $X_1 \times X_2 \times X_3$ is trivial whenever its restrictions to $\{P_1\} \times X_2 \times X_3$, $X_1 \times \{P_2\} \times X_3$ and $X_1 \times X_2 \times \{P_3\}$ are trivial.*

Proof. Let us give a proof when $\mathrm{char}(K) = 0$, since this is the case we are interested in. Then it suffices to prove the theorem for $K = \mathbb{C}$.

Before we continue the proof let us recall the following definition. A contravariant functor F from the category of complete complex algebraic varieties with basepoints into the category of abelian groups is called of order $\leq n$ if for all complete complex algebraic varieties $X_0, \ldots, X_n$ with basepoints, the natural mapping

$$F(\prod_{i=0}^{n} X_i) \longrightarrow \prod_{j=0}^{n} F(\prod_{i \neq j} X_i)$$

is an isomorphism. As an example, the Theorem of the Cube states that the functor Pic is of order ≤ 2.

To finish the proof we switch to an analytic point of view. Let $\mathcal{O}_{X,h}$ denote the sheaf of analytic functions on X. According to the GAGA-principle,

$$\mathrm{Pic}\, X \cong H^1(X, \mathcal{O}_{X,h}^{\star}),$$

for any complete complex algebraic variety X. The long exact sequence associated to

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X,h} \stackrel{\exp}{\longrightarrow} \mathcal{O}_{X,h}^{\star} \longrightarrow 0$$

implies the existence of an exact sequence

$$H^1(X, \mathcal{O}_{X,h}) \longrightarrow H^1(X, \mathcal{O}_{X,h}^{\star}) \longrightarrow H^2(X, \mathbb{Z}).$$

Since both $H^1(\cdot, \mathcal{O}_{X,h})$ and $H^2(\cdot, \mathbb{Z})$ are functors of order ≤ 2, the functor $H^1(\cdot, \mathcal{O}_{X,h}^{\star})$ is of order ≤ 2. This proves the theorem. □

Corollary 8 *Let X be an abelian variety over the field K and let $p_i: X^3 \to X$ be the projection on the ith factor. Let $p_{ij} = p_i + p_j$ and $p_{ijk} = p_i + p_j + p_k$. Then, for any invertible sheaf $\mathcal{L}$ on X, the invertible sheaf*

$$p_{123}^\star \mathcal{L} \otimes p_{12}^\star \mathcal{L}^{-1} \otimes p_{13}^\star \mathcal{L}^{-1} \otimes p_{23}^\star \mathcal{L}^{-1} \otimes p_1^\star \mathcal{L} \otimes p_2^\star \mathcal{L} \otimes p_3^\star \mathcal{L}$$

on X^3 is trivial.

Proof. Taking restrictions of this sheaf to $O \times X \times X$, $X \times O \times X$ and $X \times X \times O$ yields a trivial sheaf. The conclusion follows from the Theorem of the Cube. □

Since this corollary expresses a relation between sheaves on X^3, we have immediately, by Theorem 4, the following fact about heights on abelian varieties.

Theorem 9 *If X is an abelian variety over a number field K then, for any invertible sheaf $\mathcal{L}$ on X,*

$$h_\mathcal{L}(P+Q+R)-h_\mathcal{L}(P+Q)-h_\mathcal{L}(P+R)-h_\mathcal{L}(Q+R)+h_\mathcal{L}(P)+h_\mathcal{L}(Q)+h_\mathcal{L}(R) \sim 0,$$

as functions on $X(\overline{K})^3$.

Let us denote for an abelian variety X over K the multiplication-by-n mapping from X into itself by $[n]$, for any integer n. Recall that an invertible sheaf $\mathcal{L}$ is called symmetric (resp. antisymmetric) whenever $[-1]^\star \mathcal{L} \cong \mathcal{L}$ (resp. $[-1]^\star \mathcal{L} \cong \mathcal{L}^{-1}$). As a consequence of Corollary 8, one can prove the following.

Corollary 10 *If X is an abelian variety over the field K and $\mathcal{L}$ is an invertible sheaf on X then*

$$[n]^\star \mathcal{L} \cong \mathcal{L}^{(n^2+n)/2} \otimes [-1]^\star \mathcal{L}^{(n^2-n)/2},$$

for any integer n. In particular,

$$[n]^\star \mathcal{L} \cong \begin{cases} \mathcal{L}^{n^2}, & \text{if } \mathcal{L} \text{ is symmetric,} \\ \mathcal{L}^n, & \text{if } \mathcal{L} \text{ is antisymmetric.} \end{cases}$$

Again, this translates into properties of heights on abelian varieties.

Theorem 11 *If X is an abelian variety over the number field K and $\mathcal{L}$ is an invertible sheaf on X then*

$$h_{\mathcal{L}} \circ [n] \sim \tfrac{n^2+n}{2}\, h_{\mathcal{L}} + \tfrac{n^2-n}{2}\, h_{\mathcal{L}} \circ [-1],$$

for any integer n. In particular,

$$h_{\mathcal{L}} \circ [n] \sim \begin{cases} n^2 h_{\mathcal{L}}, & \text{if } \mathcal{L} \text{ is symmetric,} \\ n h_{\mathcal{L}}, & \text{if } \mathcal{L} \text{ is antisymmetric.} \end{cases}$$

The property of $h_{\mathcal{L}}$ in Theorem 9 will imply the existence of a canonical height function, the *Néron-Tate height relative to $\mathcal{L}$*,

$$\hat{h}_{\mathcal{L}}: X(\overline{K}) \longrightarrow \mathbb{R},$$

as stated in the following theorem.

Theorem 12 *If X is an abelian variety over the number field K and $\mathcal{L}$ is an invertible sheaf on X then there exist a unique symmetric bilinear mapping $b_{\mathcal{L}}: X(\overline{K}) \times X(\overline{K}) \to \mathbb{R}$ and a unique linear mapping $l_{\mathcal{L}}: X(\overline{K}) \to \mathbb{R}$, such that*

$$\hat{h}_{\mathcal{L}}: X(\overline{K}) \longrightarrow \mathbb{R},$$

defined by

$$\hat{h}_{\mathcal{L}}(P) = \tfrac{1}{2} b_{\mathcal{L}}(P,P) + l_{\mathcal{L}}(P),$$

is equivalent to $h_{\mathcal{L}}$. Moreover, if $\mathcal{L}$ is symmetric then $l_{\mathcal{L}} = 0$ and if $\mathcal{L}$ is ample then $b_{\mathcal{L}}$ is positive definite on $X(\overline{K}) \otimes \mathbb{R}$.

Proof. The existence and uniqueness of $b_{\mathcal{L}}$ and $l_{\mathcal{L}}$ follow from the lemma below, whose proof is left to the reader.

If $\mathcal{L}$ is symmetric then in virtue of Theorem 11

$$\tfrac{1}{2} n^2 b_{\mathcal{L}}(P,P) + n l_{\mathcal{L}}(P) = \hat{h}_{\mathcal{L}}(nP) = n^2 \hat{h}_{\mathcal{L}}(P) = \tfrac{1}{2} n^2 b_{\mathcal{L}}(P,P) + n^2 l_{\mathcal{L}}(P),$$

for any integer n. Hence $l_{\mathcal{L}} = 0$.

If $\mathcal{L}$ is ample then $[-1]^\star \mathcal{L}$ is ample too. Hence, $\mathcal{M} = \mathcal{L} \otimes [-1]^\star \mathcal{L}$ is ample. Moreover by uniqueness

$$b_{\mathcal{L}} = \tfrac{1}{2} b_{\mathcal{L}} + \tfrac{1}{2} b_{[-1]^\star \mathcal{L}} = \tfrac{1}{2} b_{\mathcal{M}}.$$

Therefore it suffices to prove that $b_{\mathcal{L}}$ is positive definite on $X(\overline{K}) \otimes \mathbb{R}$ for any symmetric ample invertible sheaf $\mathcal{L}$.

Since then $\hat{h}_{\mathcal{L}} = \frac{1}{2} b_{\mathcal{L}}$, it follows from Theorem 5 that for any finitely generated subgroup A of $X(\overline{K})$ and for any $C \in \mathbb{R}$ the cardinality of the set

$$\{P \in A \mid b_{\mathcal{L}}(P,P) \leq C\}$$

is finite. It is not difficult to prove that this implies that $b_{\mathcal{L}}$ is positive definite on $X(\overline{K}) \otimes \mathbb{R}$. $\square$

Lemma 13 *Let G be an abelian group and $h: G \to \mathbb{R}$ a function such that*

$$h(P+Q+R) - h(P+Q) - h(P+R) - h(Q+R) + h(P) + h(Q) + h(R) \sim 0,$$

as functions on G^3. Then there exists a unique symmetric bilinear mapping $b: G \times G \to \mathbb{R}$ and a unique homomorphism $l: G \to \mathbb{R}$ such that $h \sim \hat{h}$, where

$$\hat{h}(P) = \tfrac{1}{2} b(P,P) + l(P).$$

4 Metrized line bundles

In this section K will be a number field, R its ring of integers and X a projective scheme over R.

If $\mathcal{L}$ is a line bundle (more precisely, an invertible sheaf) on $\operatorname{Spec} R$ the R-module $\Gamma(\operatorname{Spec} R, \mathcal{L})$ is projective of rank 1. A *metrized line bundle on* $\operatorname{Spec} R$ is a line bundle $\mathcal{L}$ together with v-adic metrics $\|\cdot\|_v$ on $\Gamma(\operatorname{Spec} R, \mathcal{L}) \otimes_R K_v$, for any infinite place v of K. As an example, the trivial sheaf $\widetilde{R}$ together with the standard norms $\|\cdot\|_v$ on K_v is a metrized line bundle on $\operatorname{Spec} R$. If $\mathcal{L}$ is a metrized line bundle on $\operatorname{Spec} R$ then the *degree of* $\mathcal{L}$ is the real number

$$\deg \mathcal{L} = \log \#(M/Rs) - \sum_{v \in M_K^\infty} \log \|s\|_v,$$

for any nonzero $s \in M = \Gamma(\operatorname{Spec} R, \mathcal{L})$. (It is easy to check that the right-hand side is independent of s.)

If $\mathcal{L}$ is an invertible sheaf on X and v is a place of K then $\mathcal{L}$ is the sheaf of sections of some geometrical line bundle on X whose set of K_v-rational points will be denoted by $\mathcal{L}(K_v)$. Observe that $\mathcal{L}(K_v)$ is a topological line bundle on $X(K_v)$. A *v-adic metric on $\mathcal{L}$* is a continuously varying v-adic metric on each fibre of $\mathcal{L}(K_v)$. A *metrized line bundle on X* is a line bundle $\mathcal{L}$ on X together with v-adic metrics on $\mathcal{L}$, for every $v \in M_K^\infty$.

Example 14. Let $X = \mathbb{P}_R^n$ and $\mathcal{L} = \mathcal{O}(d)$, with $d \geq 0$. Then $\mathcal{L}$ is generated by global sections and $\Gamma(X, \mathcal{L})$ is just the R-module of homogeneous polynomials in $x_0, \ldots, x_n$ over R of degree d. If v is an infinite place of K then, for any $f \in \Gamma(X, \mathcal{L})$,

$$\begin{array}{rcl} X(K_v) & \longrightarrow & \mathbb{R} \\ P & \longmapsto & \dfrac{\|f(P)\|_v}{\max\limits_{i \leq n} \|x_i(P)\|_v^d} \end{array}$$

defines a v-adic metric on $\mathcal{L}$. □

Lemma 15 *Let $\mathcal{L}$ be a line bundle on X. If $v \in M_K^\infty$ and $\|\cdot\|_v$ and $\|\cdot\|_v'$ are v-adic metrics on $\mathcal{L}$ then there exist $c_1, c_2 > 0$ such that*

$$c_1 \|\cdot\|_v \leq \|\cdot\|_v' \leq c_2 \|\cdot\|_v$$

on $\mathcal{L}(K_v)$.

Proof. Follows from the fact that $X(K_v)$ is compact. □

If $\mathcal{L}$ is a metrized line bundle on X and P is an R-rational point of X, that is, P is a morphism of R-schemes

$$\operatorname{Spec} R \longrightarrow X,$$

then $P^*\mathcal{L}$ is a metrized line bundle on $\operatorname{Spec} R$. Observe that P_K is a K-rational point. We will prove the following theorem using Lemma 15.

Theorem 16 *If $\mathcal{L}$ is a metrized line bundle on X then*

$$\deg P^*\mathcal{L} \sim [K : \mathbb{Q}] h_{\mathcal{L}}(P_K)$$

as real-valued functions on $X(R)$.

Proof. It suffices to prove the theorem for $\mathcal{L} \cong \mathcal{O}(1)$. In virtue of Lemma 15 we may choose a convenient metric on $\mathcal{L}$. Let us define, for any infinite place v of K, the v-adic metric $\|\cdot\|_v$ on $\mathcal{O}(1)$ as in Example 14.

We will compute $\deg P^\star\mathcal{O}(1)$, where P is an R-rational point of $X \subseteq \mathbb{P}^n_R$. We may assume that $P^\star x_0$ is a nonzero section of $\Gamma(\operatorname{Spec} R, P^\star\mathcal{O}(1))$. Then,

$$P^\star\mathcal{O}(1)/RP^\star x_0 \cong (\sum Rx_i(P))/Rx_0(P) \cong (\sum R\frac{x_i}{x_0}(P))/R.$$

Hence,

$$\begin{aligned} \# P^\star\mathcal{O}(1)/RP^\star x_0 &= \prod_{v\notin M_K^\infty} \max_{i\le n} \|\frac{x_i}{x_0}(P)\|_v \\ &= \left(\prod_{v\notin M_K^\infty} \max_{i\le n} \|x_i(P)\|_v\right) \cdot \prod_{v\in M_K^\infty} \|x_0(P)\|_v. \end{aligned}$$

Therefore,

$$\begin{aligned} \deg P^\star\mathcal{O}(1) &= \sum_{v\notin M_K^\infty} \log \max_{i\le n} \|x_i(P)\|_v + \sum_{v\in M_K^\infty} \log \|x_0(P)\|_v + \\ & \qquad - \sum_{v\in M_K^\infty} \log \frac{\|x_0(P)\|_v}{\max_{i\le n}\|x_i(P)\|_v} \\ &= [K\colon \mathbb{Q}]h_{\mathcal{L}}(P_K) \end{aligned}$$

$\square$