THE GROUP OF ALGEBRAIC DIFFEOMORPHISMS OF A REAL RATIONAL SURFACE IS n-TRANSITIVE

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Abstract. Let X be a rational nonsingular compact connected real algebraic surface. Denote by $\mathrm{Diff}_{\mathrm{alg}}(X)$ the group of algebraic diffeomorphisms of X into itself. The group $\mathrm{Diff}_{\mathrm{alg}}(X)$ acts diagonally on X^n, for any natural integer n. We show that this action is transitive, for all n.

As an application we give a new and simpler proof of the fact that two rational nonsingular compact connected real algebraic surfaces are algebraically diffeomorphic if and only if they are homeomorphic as topological surfaces.

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1. Introduction

Let X be a nonsingular compact connected real algebraic manifold, i.e., X is a compact connected submanifold of $\mathbb{R}^n$ defined by real polynomial equations, where n is some natural integer. We study the group of algebraic diffeomorphisms of X into itself. Let us make precise what we mean by an algebraic diffeomorphism.

An *algebraic map* φ of X into itself is a map of the form

$$\varphi(x) = \left(\frac{p_1(x)}{q_1(x)}, \ldots, \frac{p_n(x)}{q_n(x)}\right) \tag{1.1}$$

where $p_1, \ldots, p_n, q_1, \ldots, q_n$ are real polynomials in the variables $x_1, \ldots, x_n$, with $q_i(x) \neq 0$ for any $x \in X$. An algebraic map from X into itself is also called a *regular map* [2]. Note that an algebraic map is necessarily of class C^∞. An algebraic map φ is an *algebraic diffeomorphism* if φ is algebraic, bijective and φ^{-1} is algebraic. An algebraic diffeomorphism from X into itself is also called a *biregular map* [2]. We denote by $\mathrm{Diff}_{\mathrm{alg}}(X)$ the group of algebraic diffeomorphism of X into itself.

For a general real algebraic manifold, the group $\mathrm{Diff}_{\mathrm{alg}}(X)$ tends to be rather small. For example, if X admits a complexification $\mathcal{X}$ of general type then $\mathrm{Diff}_{\mathrm{alg}}(X)$ is finite. Indeed, any algebraic diffeomorphism of X into itself is the restriction to X of a birational automorphism of $\mathcal{X}$. The group of birational automorphisms of $\mathcal{X}$ is known to be finite [7]. Therefore, $\mathrm{Diff}_{\mathrm{alg}}(X)$ is finite for such real algebraic manifolds.

In the current paper, we study the group $\mathrm{Diff}_{\mathrm{alg}}(X)$ when X is a compact connected real algebraic surface, i.e., a compact connected real algebraic

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manifold of dimension 2. By what is said above, the group of algebraic diffeomorphisms of such a surface is most interesting when the Kodaira dimension of X is equal to $-\infty$, and, in particular, when X is geometrically rational. By a result of Comessatti, a connected geometrically rational real surface is rational (see Theorem IV of [4] and the remarks thereafter, or [8, Corollary VI.6.5]). Therefore, we will concentrate our attention to the group $\mathrm{Diff}_{\mathrm{alg}}(X)$ when X is a rational compact connected real algebraic surface.

Recall that a real algebraic surface X is *rational* if there are a nonempty Zariski open subset U of $\mathbb{R}^2$, and a nonempty Zariski open subset V of X, such that U and V are algebraically diffeomorphic. In particular, this means that X contains a nonempty Zariski open subset V that admits a parametrization by real rational functions in two variables.

Examples of rational real algebraic surfaces are the following:

- the unit sphere S^2 defined by the equation $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$,
- the real algebraic torus $S^1 \times S^1$, where S^1 is the unit circle defined by the equation $x^2 + y^2 = 1$ in $\mathbb{R}^2$,
- the real projective plane $\mathbb{P}^2(\mathbb{R})$ (refer to [2, Theorem 3.4.4] for an explicit realization of $\mathbb{P}^2(\mathbb{R})$ as a real algebraic surface), and
- any real algebraic surface obtained from one of the above ones by repeatedly blowing up a real point.

The following conjecture has attracted our attention.

Conjecture 1.2 ([1, Conjecture 1.4])**.** *Let X be a rational nonsingular compact connected real algebraic surface. Let n be a natural integer. Then the group $\mathrm{Diff}_{\mathrm{alg}}(X)$ acts n-transitively on X.*

The conjecture seems known to be true only in the case where X is algebraically diffeomorphic to $S^1 \times S^1$:

Theorem 1.3 ([1, Theorem 1.3])**.** *The group $\mathrm{Diff}_{\mathrm{alg}}(S^1 \times S^1)$ acts n-transitively on $S^1 \times S^1$, for any natural integer n.* □

The object of the paper is to prove Conjecture 1.2 for all rational surfaces:

Theorem 1.4. *The group $\mathrm{Diff}_{\mathrm{alg}}(X)$ acts n-transitively on X, whenever X is a rational nonsingular compact connected real algebraic surface, and n is a natural integer.*

Our proof goes as follows. We first prove n-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(S^2)$ (see Theorem 2.3). For this, we need a large class of algebraic diffeomorphisms of S^2 into itself. Lemma 2.1 constructs such a large class. Once n-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(S^2)$ is established, we prove n-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(X)$, for any other rational surface X, by the following argument.

If X is algebraically diffeomorphic to $S^1 \times S^1$ then the n-transitivity has been proved in [1, Theorem 1.3]. Therefore, we may assume that X is not algebraically diffeomorphic to $S^1 \times S^1$. It follows from the Minimal Model Program for real algebraic surfaces, due to János Kollár [5, 6], that X is isomorphic to a blowing-up of S^2 in m points, for some natural integer m (see Theorems 4.1 and 4.3 for precise statements). The n-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(X)$ will then be proved by induction on m.

Theorem 1.4 shows that the group of algebraic diffeomorphisms of a rational real algebraic surface is big. It would, therefore, be particularly interesting to study the dynamics of algebraic diffeomorphisms of rational real surfaces, as is done for K3-surfaces in [3], for example.

As an application of Theorem 1.4, we present in Section 5 a simplified proof of the following result.

Theorem 1.5 ([1, Theorem 1.2])**.** *Let X and Y be rational nonsingular compact connected real algebraic surfaces. Then the following statements are equivalent.*

(1) *The real algebraic surfaces X and Y are algebraically diffeomorphic.*
(2) *The topological surfaces X and Y are homeomorphic.*

Indeed, the Minimal Model Program for real algebraic surfaces and the n-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ suffice to deduce that result (see the remark following Theorem 1.2 of [1]).

2. n-Transitivity of $\mathrm{Diff}_{\mathrm{alg}}(S^2)$

We need to slightly extend the notion of an algebraic map between real algebraic manifolds. Let X and Y be real algebraic submanifolds of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Let A be any subset of X. An *algebraic map* from A into Y is a map φ as in (1.1), where $p_1, \ldots, p_n, q_1, \ldots, q_n$ are real polynomials in the variables $x_1, \ldots, x_m$, with $q_i(x) \neq 0$ for any $x \in A$. To put it otherwise, a map φ from A into Y is *algebraic* if there is a Zariski open subset U of X containing A such that φ is the restriction of an algebraic map from U into Y.

We will consider algebraic maps from a subset A of X into Y, in the special case where X is algebraically diffeomorphic to the real algebraic line $\mathbb{R}$, the subset A of X is a closed interval, and Y is algebraically diffeomorphic to the real algebraic group $\mathrm{SO}_2(\mathbb{R})$.

Denote by S^2 the 2-dimensional sphere defined in $\mathbb{R}^3$ by

$$x^2 + y^2 + z^2 = 1.$$

Lemma 2.1. *Let L be a line through the origin of $\mathbb{R}^3$ and denote by $I \subset L$ the closed interval whose boundary is $L \cap S^2$. Denote by $L^\perp$ the plane orthogonal to L containing the origin. Let $f\colon I \to \mathrm{SO}(L^\perp)$ be an algebraic map. Define $\varphi_f\colon S^2 \to S^2$ by*

$$\varphi_f(z, x) = (f(x)z, x)$$

where $(z, x) \in (L^\perp \oplus L) \cap S^2$. Then φ_f is an algebraic diffeomorphism of S^2.

Proof. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we may assume that $S^2 \subset \mathbb{C} \times \mathbb{R}$ is given by the equation $|z|^2 + x^2 = 1$, and L is the line $\{0\} \times \mathbb{R}$. Then $L^\perp = \mathbb{C} \times \{0\}$ and $\mathrm{SO}(L^\perp) = S^1$. The map φ_f is an algebraic map from S^2 into itself. Let $\overline{f}$ be the complex conjugate of f, i.e. $\forall x \in I$, $\overline{f}(x) = \overline{f(x)}$. We have $\varphi_{\overline{f}} \circ \varphi_f = \varphi_f \circ \varphi_{\overline{f}} = id$. Therefore φ_f is an algebraic diffeomorphism of S^2. □

Lemma 2.2. *Let $x_1, \ldots, x_n$ be n distinct points of the closed interval $[-1, 1]$, and let $\alpha_1, \ldots, \alpha_n$ be elements of $\mathrm{SO}_2(\mathbb{R})$. Then there is an algebraic map $f\colon [-1, 1] \to \mathrm{SO}_2(\mathbb{R})$ such that $f(x_j) = \alpha_j$ for $j = 1, \ldots, n$.*

Proof. Since $\mathrm{SO}_2(\mathbb{R})$ is algebraically diffeomorphic to the unit circle S^1, it suffices to prove the statement for S^1 instead of $\mathrm{SO}_2(\mathbb{R})$. Let P be a point of S^1 distinct from $\alpha_1, \dots, \alpha_n$. Since $S^1 \setminus \{P\}$ is algebraically diffeomorphic to $\mathbb{R}$, it suffices, finally, to prove the statement for $\mathbb{R}$ instead of $\mathrm{SO}_2(\mathbb{R})$. The latter statement is an easy consequence of Lagrange polynomial interpolation. □

Theorem 2.3. *Let n be a natural integer. The group $\mathrm{Diff}_{\mathrm{alg}}(S^2)$ acts n-transitively on S^2.*

Proof. We will need the following terminology. Let W be a point of S^2, let L be the line in $\mathbb{R}^3$ passing through W and the origin. The intersection of S^2 with any plane in $\mathbb{R}^3$ that is orthogonal to L is called a *parallel of S^2 with respect to W*.

Let $P_1, \dots, P_n$ be n distinct points of S^2, and let $Q_1, \dots, Q_n$ be n distinct points of S^2. We need to show that there is an algebraic diffeomorphism φ from S^2 into itself such that $\varphi(P_j) = Q_j$, for all j.

Up to a projective linear automorphism of $\mathbb{P}^3(\mathbb{R})$ fixing S^2, we may assume that all the points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ are in a sufficiently small neighborhood of the north pole $N := (0, 0, 1)$ of S^2. Indeed, we may assume that none of the points is contained in a small spherical disk D centered at N. Then the images of the points by the inversion with respect to the boundary of D are contained in D.

We can choose two points W and W' of S^2 in the xy-plane such that the angle WOW' is equal to $\pi/2$ and such that the following property holds. Any parallel with respect to W contains at most one of the points $P_1, \dots, P_n$, and any parallel with respect to W' contains at most one of $Q_1, \dots, Q_n$. Denote by Γ_j the parallel with respect to W that contains P_j, and by Γ'_j the one with respect to W' that contains Q_j.

Since the disk D has been chosen sufficiently small, $\Gamma_j \cap \Gamma'_j$ is nonempty for all $j = 1, \dots, n$. Let R_j be one of the intersection points of Γ_j and Γ'_j (see Figure 1). It is now sufficient to show that there is an algebraic diffeomorphism φ of S^2 such that $\varphi(P_j) = R_j$.

Let again L be the line in $\mathbb{R}^3$ passing through W and the origin. Denote by $I \subset L$ the closed interval whose boundary is $L \cap S^2$. Let x_j be the unique element of I such that $\Gamma_j = (x_j + L^\perp) \cap S^2$. Let $\alpha_j \in \mathrm{SO}(L^\perp)$ be such that $\alpha_j(P_j - x_j) = R_j - x_j$. According to Lemma 2.2, there is an algebraic map $f \colon I \to \mathrm{SO}(L^\perp)$ such that $f(x_j) = \alpha_j$. Let $\varphi := \varphi_f$ as in Lemma 2.1. By construction, $\varphi(P_j) = R_j$, for all $j = 1, \dots, n$. □

3. Contractible curves

Let Y be a real algebraic surface and let P be a nonsingular point of Y. We denote by $B_P(Y)$ the blow-up of Y at P.

Definition 3.1. *Let X be a projective real algebraic surface. Let C be a real algebraic curve contained in X. We say that C is* contractible *if there is a projective real algebraic surface Y, a nonsingular point $P \in Y$, and an algebraic diffeomorphism $\varphi\colon B_P(Y) \to X$ such that $\varphi^{-1}(C)$ is equal to the exceptional curve of $B_P(Y)$ over P. By abuse of language, we will then also say that* Y is obtained from X by contracting C to a point.

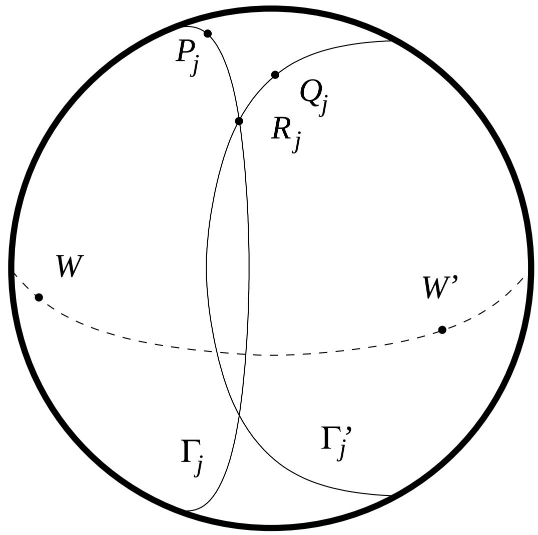


FIGURE 1. The sphere S^2 with the parallels Γ_j and Γ'_j.

If a curve C is contractible, then C is nonsingular, irreducible and rational. Moreover, C is contained in the set of nonsingular points of X. In this paper, we will only consider contractible curves in nonsingular surfaces.

Theorem 3.2. *Let X be a nonsingular projective real algebraic surface. Let C be a nonsingular rational irreducible real algebraic curve contained in X. Assume that*

(1) *X admits a nonsingular projective complexification $\mathcal{X}$ in which the Zariski closure $\mathcal{C}$ of C is nonsingular and such that the self-intersection $\mathcal{C}^2$ is greater than or equal to -1, and*
(2) *the normal bundle of C in X is nontrivial.*

Then C is contractible. Moreover, the surface Y obtained from X by contracting C to a point is nonsingular.

Proof. Let $\mathcal{X}$ be a nonsingular projective complexification of X such that the Zariski closure $\mathcal{C}$ of C in $\mathcal{X}$ is nonsingular and $\mathcal{C}^2 \geq -1$. Since C is rational, C is diffeomorphic to a circle. Since the normal bundle of C in X is nontrivial, the degree of $\mathcal{I}|_{\mathcal{C}}$ is odd, where $\mathcal{I}$ is the ideal sheaf of $\mathcal{C}$ in $\mathcal{X}$. It follows that the self-intersection of $\mathcal{C}$ is odd. Let k be an integer such that $\mathcal{C}^2 = 2k-1$. Since $\mathcal{C}^2 \geq -1$, one has $k \geq 0$. On $\mathcal{C}$, choose k pairs of complex conjugate points $P_1, \overline{P_1}, \ldots, P_{2k}, \overline{P_{2k}}$. Let $\widetilde{\mathcal{X}}$ be the blow-up of $\mathcal{X}$ at these points. The surface $\widetilde{\mathcal{X}}$ is again a nonsingular projective complexification of X. Moreover, the strict transform $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ in $\widetilde{\mathcal{X}}$ is a nonsingular rational algebraic curve defined over $\mathbb{R}$ whose self-intersection is equal to -1. Then there is a nonsingular projective algebraic surface $\mathcal{Y}$ defined over $\mathbb{R}$, a nonsingular real point $P \in \mathcal{Y}$, and an isomorphism $\Phi\colon B_P(\mathcal{Y}) \to \widetilde{\mathcal{X}}$ such that $\Phi^{-1}(\widetilde{\mathcal{C}})$ is equal to the exceptional curve of $B_P(\mathcal{Y})$ over P. To put it otherwise, the surface $\mathcal{Y}$ defined over $\mathbb{R}$ is obtained from $\widetilde{\mathcal{X}}$ by contracting $\mathcal{C}$ to a point. It follows that the set of real points Y of

$\mathcal{Y}$ is a nonsingular projective real algebraic surface obtained from X by contracting C to point. It is clear that Y is nonsingular. □

4. n-Transitivity of $\mathrm{Diff}_{\mathrm{alg}}(X)$

We reformulate a result of [1] and adapt it to our purposes:

Theorem 4.1 ([1, Theorem 3.1])**.** *Let X be a rational nonsingular compact connected real algebraic surface. Then,*

(1) *X is either algebraically diffeomorphic to $S^1 \times S^1$, or*
(2) *X is algebraically diffeomorphic to a real algebraic surface obtained from S^2 by successively blowing up.* □

It is in 4.1 that Kollár's Minimal Model Program for real algebraic surfaces is used.

If X is a rational surface algebraically diffeomorphic to a successive blowing-up of S^2, as in Theorem 4.1 above, then one can get rid of the adjective "successive" by using the following statement (compare [1, Lemma 4.1] and how it is used to prove [1, Lemma 4.3]).

Lemma 4.2. *Let $P \in S^2$ and let $C \subseteq S^2$ be an euclidean circle in S^2 containing P. Let $B_P(S^2)$ be the blowing-up of S^2 at P, and let E be the exceptional curve of $B_P(S^2)$ over P. Denote by $\widetilde{C} \subset B_P(S^2)$ the strict transform of C. Then there is an algebraic diffeomorphism φ of $B_P(S^2)$ into itself such that $\varphi(E) = \widetilde{C}$.*

Proof. The statement immediately follows from the fact that $B_P(S^2)$ is algebraically diffeomorphic to the real projective plane $\mathbb{P}^2(\mathbb{R})$, and that E and $\widetilde{C}$ are real projective lines on $\mathbb{P}^2(\mathbb{R})$. □

The following sharpened version of Theorem 4.1 follows:

Theorem 4.3. *Let X be a rational nonsingular compact connected real algebraic surface. Then,*

(1) *X is either algebraically diffeomorphic to $S^1 \times S^1$, or*
(2) *there are distinct points $R_1, \ldots, R_m$ of S^2 such that X is algebraically diffeomorphic to the real algebraic surface obtained from S^2 by blowing up $R_1, \ldots, R_m$.* □

Proof of Theorem 1.4. Let X be a rational surface. By Theorem 4.3, X is algebraically diffeomorphic to $S^1 \times S^1$ or to the blow-up of S^2 at a finite number of distinct points $R_1, \ldots, R_m$. If X is algebraically diffeomorphic to $S^1 \times S^1$ then $\mathrm{Diff}_{\mathrm{alg}}(X)$ acts n-transitively by [1, Theorem 1.3]. Therefore, we may assume that X is algebraically diffeomorphic to the blow-up $B_{R_1,\ldots,R_m}(S^2)$ of S^2 at $R_1, \ldots, R_m$. We will show that $\mathrm{Diff}_{\mathrm{alg}}(X)$ acts n-transitively on X for all n by induction on m.

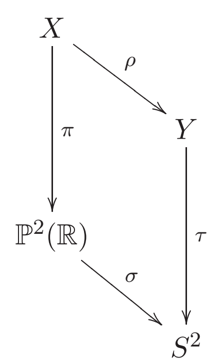
If $m = 0$, then $\mathrm{Diff}_{\mathrm{alg}}(X)$ is n-transitive, for all n, by Theorem 2.3. Let $m > 0$, and let X be $B_{R_1,\ldots,R_m}(S^2)$. Let $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ be two n-tuples of points of X where $P_j \neq P_k$ and $Q_j \neq Q_k$ whenever $j \neq k$. We want to show that there is an algebraic diffeomorphism φ of X such that $\varphi(P_j) = Q_j$ for all j.

We identify $\mathbb{P}^2(\mathbb{R})$ with $B_{R_m}(S^2)$ via an algebraic diffeomorphism. We may consider $R_1, \dots R_{m-1}$ as points of $\mathbb{P}^2(\mathbb{R})$ and the surface X is the surface $B_{R_1,\dots,R_{m-1}}(\mathbb{P}^2(\mathbb{R}))$. Let $\pi \colon X \to \mathbb{P}^2(\mathbb{R})$ be the blowing-up morphism. Let L be a line in $\mathbb{P}^2(\mathbb{R})$ that does not contain any of the points $R_k, \pi(P_j), \pi(Q_j)$. The inverse image $\widetilde{L}$ of L in X is a real algebraic curve in X. We show that $\widetilde{L}$ is contractible.

Since π is an algebraic diffeomorphism from a neighborhood of $\widetilde{L}$ in X onto a neighborhood of L in $\mathbb{P}^2(\mathbb{R})$, the inverse image $\widetilde{L}$ is a nonsingular rational real algebraic curve contained in X. Moreover, since the normal bundle of L in $\mathbb{P}^2(\mathbb{R})$ is nontrivial, the normal bundle of $\widetilde{L}$ in X is nontrivial.

A complexification of $\mathbb{P}^2(\mathbb{R})$ is the projective plane $\mathbb{P}^2$. Therefore, a complexification of X is the algebraic variety over $\mathbb{R}$ obtained from $\mathbb{P}^2$ by blowing up the real points $R_1, \dots, R_m$ of $\mathbb{P}^2$. Denote this complexification by $\mathcal{X}$. Let $\mathcal{L}$ be the Zariski closure of L in $\mathbb{P}^2$. Of course, $\mathcal{L}$ is a nonsingular algebraic curve over $\mathbb{R}$ whose self-intersection is equal to 1. Denote by $\widetilde{\pi}$ the blowing-up morphism from $\mathcal{X}$ into $\mathbb{P}^2$, and by $\widetilde{\mathcal{L}}$ the inverse image of $\mathcal{L}$ by $\widetilde{\pi}$. Since $\widetilde{\pi}$ is an isomorphism over a neighborhood of $\mathcal{L}$, the algebraic curve $\widetilde{\mathcal{L}}$ over $\mathbb{R}$ is a nonsingular complexification of $\widetilde{L}$, and its self-intersection is equal to 1.

It follows from Theorem 3.2 that $\widetilde{L}$ is contractible. Let Y be the resulting surface and let $\rho \colon X \to Y$ be the morphism that contracts $\widetilde{L}$ to a point P, see Definition 3.1. Let $\sigma \colon \mathbb{P}^2(\mathbb{R}) \to S^2$ be the morphism that contracts the line L of $\mathbb{P}^2(\mathbb{R})$ to a point. Then π induces a morphism $\tau \colon Y \to S^2$, i.e., one has the following diagram:



The morphism τ is the blow-up of S^2 at the points $R_1, \dots, R_{m-1}$. Since the real algebraic curve $\widetilde{L}$ does not contain any of the points P_j or Q_j of X, the points $\rho(P_1), \dots, \rho(P_n)$ are distinct, and the same holds for the points $\rho(Q_1), \dots, \rho(Q_n)$. Moreover, $P \neq \sigma(P_j)$ and $P \neq \sigma(Q_j)$ for all j. By the induction hypothesis, the group $\mathrm{Diff}_{\mathrm{alg}}(Y)$ acts $(n+1)$-transitively on Y. Therefore, there is an algebraic diffeomorphism ψ of Y such that $\psi(\rho(P_j)) = \rho(Q_j)$ and $\psi(P) = P$. Since X is the blow-up of Y at P, the map ψ induces an algebraic diffeomorphism φ of X with the required property. $\square$

5. Classification of rational real algebraic surfaces

Proof of Theorem 1.5. Let X and Y be a rational nonsingular compact connected real algebraic surfaces. Of course, if X and Y are algebraically diffeomorphic then X and Y are homeomorphic. In order to prove the converse, suppose that X and Y are homeomorphic. We show that there is an algebraic diffeomorphism from X onto Y.

By Theorem 4.3, we may assume that X and Y are not homeomorphic to $S^1 \times S^1$. Then, again by Theorem 4.3, X and Y are both algebraically diffeomorphic to a real algebraic surface obtained from S^2 by blowing up a finite number of distinct points. Hence, there are distinct points $P_1, \ldots, P_n$ of S^2 and distinct points $Q_1, \ldots, Q_m$ of S^2 such that

$$X \cong B_{P_1,\ldots,P_n}(S^2) \quad \text{and} \quad Y \cong B_{Q_1,\ldots,Q_m}(S^2).$$

Since X and Y are homeomorphic, $m = n$. By Theorem 2.3, there is an algebraic diffeomorphism φ from S^2 into S^2 such that $\varphi(P_i) = Q_i$ for all i. It follows that φ induces an algebraic diffeomorphism from X onto Y. □

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