The exponential sequence in real algebraic geometry and Harnack's Inequality for proper reduced real schemes

J. Huisman

Abstract

We introduce the analytification X^{an} of a scheme X locally of finite type over $\mathbb{R}$. On X^{an} one has an exponential sequence if X is reduced. This exponential sequence gives rise to an analytic description of the Picard group $\mathrm{Pic}(X)$ if X is proper and reduced. Using this description we generalize Harnack's Inequality for real algebraic curves to arbitrary proper reduced real schemes.

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1 Introduction

Topological considerations have shown to be fruitful in real algebraic geometry. One of the most striking examples of statements obtained by such considerations is the Harnack-Thom Inequality: for a nonsingular projective real algebraic variety X one has

$$\dim H^{\cdot}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \leq \dim H^{\cdot}(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}),$$

where $H^{\cdot}$ denotes the total singular cohomology. This inequality can be proven using the cohomology of the Smith sequence on the quotient space $X(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, or using $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant cohomology of $X(\mathbb{C})$ (see [14, §1.3]).

The Harnack-Thom Inequality can be considered as an attempt to generalize to higher dimension Harnack's Inequality for real algebraic curves [8]:

given a nonsingular connected projective real algebraic curve X of genus g, the number $\#\pi_0(X(\mathbb{R}))$ of connected components of the space $X(\mathbb{R})$ of real points of X satisfies

$$\#\pi_0(X(\mathbb{R})) \leq g+1.$$

This inequality is sharp in the sense that for every natural integer g there is a nonsingular connected projective real algebraic curve X of genus g such that the space $X(\mathbb{R})$ has $g+1$ connected components.

The Harnack-Thom Inequality is, indeed, a generalization of Harnack's Inequality. It is, however, unsatisfactory that the Harnack-Thom Inequality involves the total cohomology $H^{\cdot}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ of $X(\mathbb{R})$, so that, in general, one gets out of that inequality weak bounds on the number of connected components of the space of real points of a real algebraic variety. For example, one does not know the maximum number of connected components that can have the space of real points of a nonsingular surface of degree 5 in $\mathbb{P}^3_{\mathbb{R}}$ [10].

On a general basis, one may expect that topological considerations will not suffice to push further our comprehension of real algebraic varieties. In this paper we introduce and study the real analytic structure on a real scheme. More precisely, let X be a scheme locally of finite type over $\mathbb{R}$. Let $X_{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Then, the analytification $X_{\mathbb{C}}^{\mathrm{an}}$ of the complex scheme $X_{\mathbb{C}}$ is a locally ringed space on which the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts naturally. Note that we consider here the algebraic action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $X_{\mathbb{C}}^{\mathrm{an}}$, i.e. the antilinear action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the structure sheaf $\mathcal{O}_{X_{\mathbb{C}}^{\mathrm{an}}}$ of analytic functions on $X_{\mathbb{C}}^{\mathrm{an}}$ that lies over the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the topological space $X_{\mathbb{C}}^{\mathrm{an}}$. This algebraic action seems to be more interesting than the geometric action which is more usually considered—for example in the theory of Klein surfaces [1].

The quotient of the locally ringed space $X_{\mathbb{C}}^{\mathrm{an}}$ by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is again a locally ringed space. We denote this locally ringed space by X^{an}. We call it the *real analytic space* associated to X or the *analytification* of X or also the *real analytic structure* on X. Note that the underlying set of X^{an} is nothing but the set of closed points of the scheme X.

More generally, one can define the category of *real analytic spaces* in much the same way as the one of complex analytic spaces [6]: as local models one takes locally ringed spaces of the form $(\mathrm{supp}(\mathcal{O}_{|U}/\mathcal{I}), \mathcal{O}_{|U}/\mathcal{I})$, where $\mathcal{I}$ is a sheaf of $\mathcal{O}_{|U}$-ideals of finite type, U is an open subset of $\mathbb{C}^n/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and $\mathcal{O}$ is the structure sheaf on $\mathbb{C}^n/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

It is clear from its definition that a real analytic space admits a complexification which is a complex analytic space. One then has a commutative diagram:

$$\begin{array}{ccc}
\begin{array}{c}\text{schemes locally}\\ \text{of finite type }/\mathbb{R}\end{array} & \xrightarrow[\text{fication}]{\text{analyti-}} & \begin{array}{c}\text{real analytic}\\ \text{spaces}\end{array}\\
\text{complexi}\Big\downarrow\text{fication} & & \text{complexi}\Big\downarrow\text{fication}\\
\begin{array}{c}\text{schemes locally}\\ \text{of finite type }/\mathbb{C}\end{array} & \xrightarrow[\text{fication}]{\text{analyti-}} & \begin{array}{c}\text{complex analytic}\\ \text{spaces}\end{array}
\end{array}$$

One can say that real analytic spaces were, somehow, a missing link in the above diagram; the topological quotient $X(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ has been studied before, but, as far as I know, it has not been studied as a real analytic space.

Having real analytic spaces at our disposal, we develop a transcendental approach to real algebraic geometry. One of its main ingredients is the exponential sequence. As in the complex case, it will give rise to a transcendental description of the Picard group of a proper reduced real scheme. It is this description that allows a sharp generalization of Harnack's Inequality.

2 The GAGA-principle for coherent sheaves on real schemes

Let X be a proper real scheme. Using the GAGA-principle for coherent sheaves on $X_{\mathbb{C}}$, it is easily seen that the GAGA-principle also holds for coherent sheaves on X [13, 7]. More precisely, for a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, denote by $\mathcal{F}^{\mathrm{an}}$ the induced sheaf of $\mathcal{O}_{X^{\mathrm{an}}}$-modules on X^{an}. The GAGA-principle for coherent sheaves on X is the following statement:

Theorem 2.1. *Let X be a proper real scheme. The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is an equivalence from the category of coherent $\mathcal{O}_X$-modules into the category of coherent $\mathcal{O}_{X^{\mathrm{an}}}$-modules. Moreover, the canonical morphism*

$$H^p(X,\mathcal{F}) \longrightarrow H^p(X^{\mathrm{an}},\mathcal{F}^{\mathrm{an}})$$

is an isomorphism. □

Let X be a scheme locally of finite type over $\mathbb{R}$. Recall that the *Picard group* $\mathrm{Pic}(X)$ of X is the group of isomorphism classes of invertible sheaves on X. Similarly, one can define the *Picard group* $\mathrm{Pic}(X^{\mathrm{an}})$ of X^{an} to be the group of isomorphism classes of invertible sheaves on the locally ringed space X^{an}. One has canonical isomorphisms

$$\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}_X^\star) \quad \text{and} \quad \mathrm{Pic}(X^{\mathrm{an}}) \cong H^1(X^{\mathrm{an}},\mathcal{O}_{X^{\mathrm{an}}}^\star).$$

Applying the GAGA-principle to the subcategory of invertible sheaves of the category of coherent sheaves, one has the following consequence:

Corollary 2.2. *Let X be a proper real scheme. Then, the canonical morphism from* $\mathrm{Pic}(X)$ *into* $\mathrm{Pic}(X^{\mathrm{an}})$ *is an isomorphism.* □

3 The exponential sequence in real algebraic geometry

Let X be a reduced scheme locally of finite type over $\mathbb{R}$. Then, the complex analytic space $X_{\mathbb{C}}^{\mathrm{an}}$ associated to X is reduced. In particular, the canonical map from $\mathcal{O}_{X_{\mathbb{C}}^{\mathrm{an}}}$ into the sheaf of continuous complex-valued functions on $X_{\mathbb{C}}^{\mathrm{an}}$ is injective [6, Proposition 4.2.5]. Hence, one can consider the exponential morphism

$$\exp\colon \mathcal{O}_{X_{\mathbb{C}}^{\mathrm{an}}} \longrightarrow \mathcal{O}_{X_{\mathbb{C}}^{\mathrm{an}}}^{\star}$$

on the complex analytic space $X_{\mathbb{C}}^{\mathrm{an}}$. Clearly, this morphism is Σ-equivariant, where $\Sigma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Hence, it induces a morphism $\exp/\Sigma$ from $\mathcal{O}_{X^{\mathrm{an}}}$ into $\mathcal{O}_{X^{\mathrm{an}}}^{\star}$. The morphism $\exp/\Sigma$ is not surjective in general. In order to have a surjective exponential morphism into the sheaf $\mathcal{O}_{X^{\mathrm{an}}}^{\star}$ we modify the morphism $\exp/\Sigma$ in the following way:

Denote by μ_2 the multiplicative group $\{\pm 1\}$. The locally constant sheaf on X^{an} with fiber μ_2 is again denoted by μ_2. Of course, one can consider μ_2 as a subsheaf of the sheaf $\mathcal{O}_{X^{\mathrm{an}}}^{\star}$. The *exponential morphism* on the real analytic structure X^{an} of X is the morphism

$$\exp\colon \mu_2 \oplus \mathcal{O}_{X^{\mathrm{an}}} \longrightarrow \mathcal{O}_{X^{\mathrm{an}}}^{\star}$$

defined by $\exp(\varepsilon \oplus f) = \varepsilon \cdot (\exp/\Sigma)(f)$. Now, the morphism exp is a surjective morphism.

In order to determine the kernel of the exponential morphism on X^{an}, let $\mathbb{Z}$ be the locally constant sheaf on $X_{\mathbb{C}}^{\mathrm{an}}$ with fiber $\mathbb{Z}$. Let p be the quotient map from $X_{\mathbb{C}}^{\mathrm{an}}$ onto X^{an}. One has an induced action of Σ on the sheaf $p_{\star}\mathbb{Z}$ on X^{an}. The sheaf $p_{\star}\mathbb{Z}$ contains the locally constant sheaf $\mathbb{Z}$ as a subsheaf. In fact, the sheaf $(p_{\star}\mathbb{Z})^{\Sigma}$ of Σ-invariant sections of $p_{\star}\mathbb{Z}$ is equal to the sheaf $\mathbb{Z}$. Let tr be the trace map $s \mapsto (1+\sigma)\cdot s$ from $p_{\star}\mathbb{Z}$ into $\mathbb{Z}$, where σ is the non trivial element of Σ. Denote by $\mathcal{K}$ the kernel of the trace map tr. Then, $\mathcal{K}$ is a sheaf of Abelian groups on X^{an}. We exhibit an isomorphism of $\mathcal{K}$ with the kernel of the exponential morphism on X^{an}.

Consider the slightly modified exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \iota\ } \mu_2 \oplus \mathcal{O}_{X_{\mathbb{C}}^{\mathrm{an}}} \xrightarrow{\exp'} \mathcal{O}_{X_{\mathbb{C}}^{\mathrm{an}}}^{\star} \longrightarrow 0 \qquad (1)$$

on the complex analytic space $X_{\mathbb{C}}^{\rm an}$. The morphism ι associates to an integer n the section $(-1)^n \oplus n\pi\sqrt{-1}$. The morphism $\exp'$ associates to a section $\varepsilon \oplus f$ of $\mu_2 \oplus \mathcal{O}_{X_{\mathbb{C}}^{\rm an}}$ the section $\varepsilon \cdot \exp(f)$ of $\mathcal{O}^\star_{X_{\mathbb{C}}^{\rm an}}$. It is clear that the above short sequence is exact.

Since the quotient map p from $X_{\mathbb{C}}^{\rm an}$ into $X^{\rm an}$ has zero-dimensional fibers, the induced short sequence

$$0 \longrightarrow p_\star\mathbb{Z} \xrightarrow{p_\star(\iota)} (p_\star\mu_2) \oplus (p_\star\mathcal{O}_{X_{\mathbb{C}}^{\rm an}}) \xrightarrow{p_\star(\exp')} p_\star\mathcal{O}^\star_{X_{\mathbb{C}}^{\rm an}} \longrightarrow 0$$

is exact as well. Now, considering the twisted Σ-action $(p_\star\mathbb{Z})(1)$ on $p_\star\mathbb{Z}$, all its morphisms are Σ-equivariant. Observe that the sheaf $(p_\star\mathbb{Z})(1)^\Sigma$ is equal to $\mathcal{K}$. Hence, taking the sheaves of Σ-invariant sections, one has an induced exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{p_\star(\iota)^\Sigma} \mu_2 \oplus \mathcal{O}_{X^{\rm an}} \xrightarrow{p_\star(\exp')^\Sigma} \mathcal{O}^\star_{X^{\rm an}}.$$

Now, the morphism $p_\star(\exp')^\Sigma$ is nothing but the exponential morphism exp on $X^{\rm an}$. Therefore, the kernel of the exponential morphism on $X^{\rm an}$ is isomorphic to $\mathcal{K}$. Let κ be the morphism $p_\star(\iota)^\Sigma$ from $\mathcal{K}$ into $\mu_2 \oplus \mathcal{O}_{X^{\rm an}}$. Then, one has the following statement:

Theorem 3.1. *Let X be a reduced scheme locally of finite type over $\mathbb{R}$. Then, the following short sequence is exact:*

$$0 \longrightarrow \mathcal{K} \xrightarrow{\kappa} \mu_2 \oplus \mathcal{O}_{X^{\rm an}} \xrightarrow{\exp} \mathcal{O}^\star_{X^{\rm an}} \longrightarrow 0. \quad \square$$

We call the exact sequence of Theorem 3.1 the *exponential sequence* on the real analytic structure $X^{\rm an}$ on X. It plays a similar role in real algebraic geometry as its counterpart in complex algebraic geometry.

Let X be a proper reduced real scheme. By the GAGA-principle (cf. Corollary 2.2), one can identify the Picard group $\mathrm{Pic}(X)$ of X with the cohomology group $H^1(X^{\rm an}, \mathcal{O}^\star_{X^{\rm an}})$. Then, the exponential sequence induces the exact sequence

$$0 \longrightarrow H^1(X^{\rm an}, \mathcal{K}) \longrightarrow H^1(X^{\rm an}, \mu_2 \oplus \mathcal{O}_{X^{\rm an}}) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^2(X^{\rm an}, \mathcal{K}). \quad (2)$$

We denote by c_1 the morphism from $\mathrm{Pic}(X)$ into $H^2(X^{\rm an}, \mathcal{K})$. We call the image $c_1(\mathcal{L})$ of an invertible sheaf $\mathcal{L}$ on X the *first Chern class* of $\mathcal{L}$. We define $\mathrm{Pic}^0(X)$ to be the kernel of the morphism c_1. One has a short exact sequence

$$0 \longrightarrow H^1(X^{\rm an}, \mathcal{K}) \longrightarrow H^1(X^{\rm an}, \mu_2 \oplus \mathcal{O}_{X^{\rm an}}) \longrightarrow \mathrm{Pic}^0(X) \longrightarrow 0. \quad (3)$$

We call the image of the first-Chern-class map the *Néron-Severi* group of X and denote it by $\mathrm{NS}(X)$. One has a short exact sequence

$$0\longrightarrow \mathrm{Pic}^0(X)\longrightarrow \mathrm{Pic}(X)\longrightarrow \mathrm{NS}(X)\longrightarrow 0.$$

It is clear from the exact sequences (2) and (3) that the groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}^0(X)$ have a natural structure of a not necessarily connected commutative real Lie group. Moreover, the neutral component $\mathrm{Pic}(X)^0$ of $\mathrm{Pic}(X)$ is equal to the neutral component $\mathrm{Pic}^0(X)^0$ of $\mathrm{Pic}^0(X)$. It follows from the exact sequence (3) that the group of connected components $\pi_0(\mathrm{Pic}^0(X))$ of $\mathrm{Pic}^0(X)$ is an Abelian group of exponent 2.

We will show in Section 7 that the real Lie group $\mathrm{Pic}^0(X)$ is, under certain conditions, canonically isomorphic to the set of real points $\mathrm{Pic}^0_X(\mathbb{R})$ of the Jacobian Pic^0_X of the real scheme X.

4 The plus-exponential sequence

Let X be a reduced scheme locally of finite type over $\mathbb{R}$. As we have seen in Section 3, one has an exponential morphism

$$\exp/\Sigma\colon \mathcal{O}_{X^{\mathrm{an}}}\longrightarrow \mathcal{O}^*_{X^{\mathrm{an}}}.$$

Let $\mathcal{O}^+_{X^{\mathrm{an}}}$ be the image sheaf of the morphism $\exp/\Sigma$. Let f be a section of $\mathcal{O}^*_{X^{\mathrm{an}}}$ over the open subset U of X^{an}. It is clear that f is a section of the subsheaf $\mathcal{O}^+_{X^{\mathrm{an}}}$ if and only if $f>0$ on $U\cap X(\mathbb{R})$. Denote by $\exp^+$ the morphism $\exp/\Sigma$ considered as a morphism into $\mathcal{O}^+_{X^{\mathrm{an}}}$, i.e.,

$$\exp^+\colon \mathcal{O}_{X^{\mathrm{an}}}\longrightarrow \mathcal{O}^+_{X^{\mathrm{an}}}.$$

It is clear that $\exp^+$ is equal to the restriction to $\mathcal{O}_{X^{\mathrm{an}}}$ of the exponential morphism $\exp$ on X^{an}. Since the kernel of the morphism

$$\mathrm{pr}_1\circ\kappa\colon \mathcal{K}\longrightarrow \mu_2$$

is equal to $2\mathcal{K}$, one has a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \xrightarrow{2\kappa} & \mathcal{O}_{X^{\mathrm{an}}} & \xrightarrow{\exp^+} & \mathcal{O}^+_{X^{\mathrm{an}}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 2} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \xrightarrow{\kappa} & \mu_2\oplus\mathcal{O}_{X^{\mathrm{an}}} & \xrightarrow{\exp} & \mathcal{O}^*_{X^{\mathrm{an}}} & \longrightarrow & 0
\end{array}\qquad (4)$$

In particular, one has the following statement:

Theorem 4.1. *Let X be a reduced scheme locally of finite type over $\mathbb{R}$. Then, the following short sequence is exact:*

$$0 \longrightarrow \mathcal{K} \xrightarrow{2\kappa} \mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{\exp^+} \mathcal{O}^+_{X^{\mathrm{an}}} \longrightarrow 0. \quad \square$$

We call the exact sequence of Theorem 4.1 the *plus-exponential sequence* on X^{an}. We define the *plus-Picard group* $\mathrm{Pic}^+(X)$ of X to be the cohomology group $H^1(X^{\mathrm{an}}, \mathcal{O}^+_{X^{\mathrm{an}}})$. If X is a proper reduced real scheme, the plus-exponential sequence gives rise to an exact sequence

$$0 \longrightarrow H^1(X^{\mathrm{an}}, \mathcal{K}) \longrightarrow H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \longrightarrow \mathrm{Pic}^+(X) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{K}).$$

We denote the morphism from $\mathrm{Pic}^+(X)$ into $H^2(X^{\mathrm{an}}, \mathcal{K})$ by c_1^+. We denote the kernel of c_1^+ by $\mathrm{Pic}^{+,0}(X)$ and its image by $\mathrm{NS}^+(X)$. The groups $\mathrm{Pic}^+(X)$ and $\mathrm{Pic}^{+,0}(X)$ have a natural structure of a commutative real Lie group. It is clear from the above exact sequence that $\mathrm{Pic}^{+,0}(X)$ is the neutral component of $\mathrm{Pic}^+(X)$. Moreover, since the torsion subgroup of $H^2(X^{\mathrm{an}}, \mathcal{K})$ is contained in the kernel of he morphism from $H^2(X^{\mathrm{an}}, \mathcal{K})$ into $H^2(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$, the torsion subgroup of $\mathrm{NS}^+(X)$ is equal to the torsion subgroup of $H^2(X^{\mathrm{an}}, \mathcal{K})$. In particular,

$$\mathrm{NS}^+(X)_2 = H^2(X^{\mathrm{an}}, \mathcal{K})_2,$$

where A_2 denotes the 2-torsion subgroup of an Abelian group A. Note, furthermore, that by definition, one has a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^{+,0}(X) \longrightarrow \mathrm{Pic}^+(X) \longrightarrow \mathrm{NS}^+(X) \longrightarrow 0.$$

5 Comparing the Picard group and the plus-Picard group

Let X be a proper reduced real scheme. Let i be the inclusion of the space of real points $X(\mathbb{R})$ of X into X^{an}. The inclusion of $\mathcal{O}^+_{X^{\mathrm{an}}}$ into $\mathcal{O}^\star_{X^{\mathrm{an}}}$ gives rise to a short exact sequence

$$0 \longrightarrow \mathcal{O}^+_{X^{\mathrm{an}}} \longrightarrow \mathcal{O}^\star_{X^{\mathrm{an}}} \xrightarrow{\mathrm{sign}} i_\star\mu_2 \longrightarrow 0\,,$$

where sign is the signature map. Since X is proper, one has an induced exact sequence

$$0 \longrightarrow H^0(X(\mathbb{R}), \mu_2)/H^0(X^{\mathrm{an}}, \mu_2) \longrightarrow \mathrm{Pic}^+(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^1(X(\mathbb{R}), \mu_2).$$

Here, the quotient $H^0(X(\mathbb{R}), \mu_2)/H^0(X^{\mathrm{an}}, \mu_2)$ denotes the quotient of the group $H^0(X(\mathbb{R}), \mu_2)$ by the image of $H^0(X^{\mathrm{an}}, \mu_2)$ in $H^0(X(\mathbb{R}), \mu_2)$. The morphism from $\mathrm{Pic}(X)$ into $H^1(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ is denoted by w_1. Let $\mathcal{L}$ be an invertible sheaf on X. Let $V(\mathcal{L})$ be the associated geometric line bundle over the scheme X. Then, $w_1(\mathcal{L})$ is nothing but the first Stiefel-Whitney class of the topological line bundle $V(\mathcal{L})(\mathbb{R})$ over $X(\mathbb{R})$ [11]. The image of w_1 is usually denoted by $H^1_{\mathrm{alg}}(X(\mathbb{R}),\mu_2)$ [2]. Hence, one has the following statement:

Lemma 5.1. *Let X be a proper reduced real scheme. Then, the following sequence is exact:*

$$0 \longrightarrow H^0(X(\mathbb{R}),\mu_2)/H^0(X^{\mathrm{an}},\mu_2) \longrightarrow \mathrm{Pic}^+(X) \longrightarrow$$
$$\longrightarrow \mathrm{Pic}(X) \xrightarrow{w_1} H^1_{\mathrm{alg}}(X(\mathbb{R}),\mu_2) \longrightarrow 0. \quad \square$$

The morphism of short exact sequences (4) gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^+(X) & \xrightarrow{c_1^+} & H^2(X^{\mathrm{an}},\mathcal{K}) \\ \downarrow & & \downarrow 2 \\ \mathrm{Pic}(X) & \xrightarrow{c_1} & H^2(X^{\mathrm{an}},\mathcal{K}) \end{array}$$

Since $\mathrm{NS}^+(X)$ ($\mathrm{NS}(X)$, resp.) is the image of c_1^+ (c_1, resp.), one has gets the following statement:

Lemma 5.2. *Let X be a proper reduced real scheme. Then, the subgroups* $\mathrm{NS}(X)$ *and* $\mathrm{NS}^+(X)$ *of* $H^2(X^{\mathrm{an}},\mathcal{K})$ *satisfy*

$$2\mathrm{NS}^+(X) \subseteq \mathrm{NS}(X). \quad \square$$

It follows from the preceding lemma that one has a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^{+,0}(X) & \longrightarrow & \mathrm{Pic}^+(X) & \xrightarrow{c_1^+} & \mathrm{NS}^+(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow 2 & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{c_1} & \mathrm{NS}(X) & \longrightarrow & 0 \end{array} \qquad (5)$$

Let K be the kernel and C the cokernel of the morphism from $\mathrm{Pic}^{+,0}(X)$ into $\mathrm{Pic}^0(X)$. Applying the Snake Lemma to (5) and using Lemma 5.1, one has the following statement:

Proposition 5.3. *Let X be a proper reduced real scheme. Then, the following sequence is exact:*

$$0 \longrightarrow K \longrightarrow H^0(X(\mathbb{R}),\mu_2)/H^0(X,\mu_2) \longrightarrow \mathrm{NS}^+(X)_2 \longrightarrow$$
$$\longrightarrow C \longrightarrow H^1_{\mathrm{alg}}(X(\mathbb{R}),\mu_2) \longrightarrow \mathrm{NS}(X)/2\mathrm{NS}^+(X) \longrightarrow 0. \quad \square$$

In order to determine the kernel K and cokernel C of the morphism from $\mathrm{Pic}^{+,0}(X)$ into $\mathrm{Pic}^0(X)$, we consider the square-map sq from $\mathcal{O}^\star_{X^{\mathrm{an}}}$ into $\mathcal{O}^+_{X^{\mathrm{an}}}$. One has a short exact sequence

$$0 \longrightarrow \mu_2 \longrightarrow \mathcal{O}^\star_{X^{\mathrm{an}}} \xrightarrow{\mathrm{sq}} \mathcal{O}^+_{X^{\mathrm{an}}} \longrightarrow 0 \tag{6}$$

Taking cohomology, one obtains the following statement:

Lemma 5.4. *Let X be a proper reduced real scheme. Then, the following sequence is exact:*

$$\begin{aligned} 0 \longrightarrow H^1(X^{\mathrm{an}}, \mu_2) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\mathrm{sq}} \mathrm{Pic}^+(X) \longrightarrow \\ \longrightarrow H^2(X^{\mathrm{an}}, \mu_2) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}^\star_{X^{\mathrm{an}}}). \quad \square \end{aligned}$$

One has a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \xrightarrow{\kappa} & \mu_2 \oplus \mathcal{O}_{X^{\mathrm{an}}} & \xrightarrow{\exp} & \mathcal{O}^\star_{X^{\mathrm{an}}} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle \mathrm{sq}} & & \\ 0 & \longrightarrow & \mathcal{K} & \xrightarrow{2\kappa} & \mathcal{O}_{X^{\mathrm{an}}} & \xrightarrow{\exp^+} & \mathcal{O}^+_{X^{\mathrm{an}}} & \longrightarrow & 0 \end{array}$$

This morphism gives rise to a commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow{c_1} & H^2(X^{\mathrm{an}}, \mathcal{K}) \\ \downarrow{\scriptstyle \mathrm{sq}} & & \| \\ \mathrm{Pic}^+(X) & \xrightarrow{c_1^+} & H^2(X^{\mathrm{an}}, \mathcal{K}) \end{array}$$

Since $\mathrm{NS}(X)$ ($\mathrm{NS}^+(X)$, resp.) is the image of c_1 (c_1^+, resp.), one gets the following statement:

Lemma 5.5. *Let X be a proper reduced real scheme. Then, the subgroups $\mathrm{NS}(X)$ and $\mathrm{NS}^+(X)$ of $H^2(X^{\mathrm{an}}, \mathcal{K})$ satisfy*

$$\mathrm{NS}(X) \subseteq \mathrm{NS}^+(X). \quad \square$$

One also gets a morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{c_1} & \mathrm{NS}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{sq}^0} & & \downarrow{\scriptstyle \mathrm{sq}} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^{+,0}(X) & \longrightarrow & \mathrm{Pic}^+(X) & \xrightarrow{c_1^+} & \mathrm{NS}^+(X) & \longrightarrow & 0 \end{array} \tag{7}$$

Since the map from $\mathrm{NS}(X)$ into $\mathrm{NS}^+(X)$ is injective, one has the following statement:

Lemma 5.6. *Let X be a proper reduced real scheme. Then,*

$$\mathrm{sq}^{-1}(\mathrm{Pic}^{+,0}(X)) = \mathrm{Pic}^0(X). \quad \square$$

By Lemma 5.4, the kernel of sq is equal to the image of $H^1(X^{\mathrm{an}}, \mu_2)$ in $\mathrm{Pic}(X)$. Since the map from $\mathrm{NS}(X)$ into $\mathrm{NS}^+(X)$ is injective, the image of $H^1(X^{\mathrm{an}}, \mu_2)$ in $\mathrm{Pic}(X)$ is contained in $\mathrm{Pic}^0(X)$ and is equal to the kernel of sq^0. Since $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^{+,0}(X)$ are real Lie groups of the same dimension, sq^0 is an isogeny. Now, $\mathrm{Pic}^{+,0}(X)$ is connected, hence sq^0 is surjective. This proves the following statement:

Lemma 5.7. *Let X be a proper reduced real scheme. Then, the following sequence is exact:*

$$0 \longrightarrow H^1(X^{\mathrm{an}}, \mu_2) \longrightarrow \mathrm{Pic}^0(X) \xrightarrow{\mathrm{sq}^0} \mathrm{Pic}^{+,0}(X) \longrightarrow 0. \quad \square$$

Applying the Snake Lemma to (7), and using Lemmas 5.4 and 5.7, one gets the following statement:

Lemma 5.8. *Let X be a proper reduced real scheme. Then, the following sequence is exact:*

$$0 \longrightarrow \mathrm{NS}(X) \longrightarrow \mathrm{NS}^+(X) \longrightarrow H^2(X^{\mathrm{an}}, \mu_2) \longrightarrow H^2(X^{\mathrm{an}}, \mathcal{O}^\star_{X^{\mathrm{an}}}). \quad \square$$

We will identify the group $H^1(X^{\mathrm{an}}, \mu_2)$ with its image in $\mathrm{Pic}^0(X)$. Let $\mathrm{Pic}^0(X)^0$ be the neutral component of the real Lie group $\mathrm{Pic}^0(X)$. Put

$$H^1(X^{\mathrm{an}}, \mu_2)^0 = H^1(X^{\mathrm{an}}, \mu_2) \cap \mathrm{Pic}^0(X)^0.$$

Then, the group of connected components of $\mathrm{Pic}^0(X)$ is canonically isomorphic to the quotient $H^1(X^{\mathrm{an}}, \mu_2)/H^1(X^{\mathrm{an}}, \mu_2)^0$. Now, since $\mathrm{Pic}^{+,0}(X)$ is a connected real Lie group, the image of the morphism from $\mathrm{Pic}^{+,0}(X)$ into $\mathrm{Pic}^0(X)$ is equal to the neutral component $\mathrm{Pic}^0(X)^0$ of $\mathrm{Pic}^0(X)$. This proves the following statement:

Lemma 5.9. *The cokernel C of the morphism from* $\mathrm{Pic}^{+,0}(X)$ *into* $\mathrm{Pic}^0(X)$ *fits into a short exact sequence*

$$0 \longrightarrow H^1(X^{\mathrm{an}}, \mu_2)^0 \longrightarrow H^1(X^{\mathrm{an}}, \mu_2) \longrightarrow C \longrightarrow 0. \quad \square$$

Since the multiplication-by-2 map on $\mathrm{Pic}^{+,0}(X)$ is equal to the composition of the morphism from $\mathrm{Pic}^{+,0}(X)$ into $\mathrm{Pic}^0(X)$ and the restriction of the square map to $\mathrm{Pic}^0(X)$, one also gets the following statement:

Lemma 5.10. *The kernel K of the morphism from* $\mathrm{Pic}^{+,0}(X)$ *into* $\mathrm{Pic}^0(X)$ *fits into a short exact sequence*

$$0 \longrightarrow K \longrightarrow \mathrm{Pic}^{+,0}(X)_2 \longrightarrow H^1(X^{\mathrm{an}}, \mu_2)^0 \longrightarrow 0. \quad \square$$

6 Generalized Harnack's Inequality

Let A be a not necessarily connected commutative real Lie group. The *compact dimension* of A is the greatest natural integer $c = \mathrm{cdim}(A)$ such that A admits a compact real Lie subgroup of dimension c. Observe that c is equal to the dimension of A_2 as an $\mathbb{F}_2$-vector space, i.e., the compact dimension of A is just a convenient way of referring to the $\mathbb{F}_2$-dimension of its 2-torsion subgroup. Note also that $c \leq \dim(A)$.

Our generalization of Harnack's Inequality is then the following statement:

Theorem 6.1. *Let X be a connected proper reduced real scheme. Let s be the number of connected components of $X(\mathbb{R})$. Let c be the compact dimension of $\mathrm{Pic}(X)$ and let n be the dimension of the $\mathbb{F}_2$-vector space $\mathrm{NS}^+(X)_2$. Then*

$$s \leq c + n + 1.$$

Proof. By Lemma 5.10, K is a $\mathbb{F}_2$-vector space of dimension at most c. The statement then follows from Proposition 5.3. □

Theorem 6.1 is, indeed, a generalization of Harnack's Inequality: it is easy to see that the cohomology group $H^2(X^{\mathrm{an}}, \mathcal{K})$ is torsion free if X is a connected proper reduced real scheme of pure dimension 1. Since $\mathrm{NS}^+(X)_2 = H^2(X^{\mathrm{an}}, \mathcal{K})_2$, the group $\mathrm{NS}^+(X)$ is 2-torsion free. By Theorem 6.1,

$$s \leq c + 1 \leq g + 1,$$

where g is the dimension of $H^1(X, \mathcal{O}_X)$. Hence, Theorem 6.1 implies Harnack's Inequality. In fact, one has the stronger inequality $s \leq c + 1$ for connected proper reduced real schemes of pure dimension 1. For example, one has the following statement:

Corollary 6.2. *Let X be a connected proper reduced real scheme of pure dimension 1. Suppose that the Jacobian Pic^0_X of X is isomorphic to a product of multiplicative groups and additive groups over $\mathbb{R}$, i.e.,*

$$\mathrm{Pic}^0_X \cong \mathbb{G}^k_{m/\mathbb{R}} \times \mathbb{G}^l_{a/\mathbb{R}}.$$

Then, $X(\mathbb{R})$ is connected.

Proof. Let Pic_X be the Picard group scheme of X [4, Theorem 8.2.3]. The natural map from $\mathrm{Pic}(X)$ into $\mathrm{Pic}_X(\mathbb{R})$ is injective [4, Proposition 8.1.4]. Since both real Lie groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}_X(\mathbb{R})$ are of the same dimension, the neutral components $\mathrm{Pic}(X)^0$ and $\mathrm{Pic}_X(\mathbb{R})^0$ are isomorphic. In particular, the compact dimension of $\mathrm{Pic}_X(\mathbb{R})$ is equal to the compact dimension of $\mathrm{Pic}(X)$. But, the compact dimension of $\mathrm{Pic}_X(\mathbb{R})$ is equal to the compact dimension of $\mathrm{Pic}_X(\mathbb{R})^0$, and the latter compact dimension is equal to 0. It follows from Theorem 6.1 that $X(\mathbb{R})$ is connected since $\mathrm{NS}^+(X)$ is torsion free. □

We will show in Section 8 that the bound in Theorem 6.1 on the number of connected components of $X(\mathbb{R})$ is sharp in the following sense: for all natural integers c, n and d, with $d \geq 2$, there is a nonsingular connected proper reduced real scheme X of dimension d such that the number of connected components of $X(\mathbb{R})$ is equal to $c+n+1$, where c is the compact dimension of $\mathrm{Pic}(X)$ and n is the dimension of $\mathrm{NS}^+(X)_2$.

Observe that the proof of Theorem 6.1 only uses the exactness of the first line of the sequence of Proposition 5.3. When one uses the full statement of Proposition 5.3, one gets the following more precise statement:

Theorem 6.3. *Let X be a connected proper reduced real scheme such that $X(\mathbb{R}) \neq \emptyset$. Define the following integers*

$$\begin{aligned} s &= \#\pi_0(X(\mathbb{R})) \\ n &= \dim \mathrm{NS}^+(X)_2 \\ b &= \dim H^1_{\mathrm{alg}}(X(\mathbb{R}), \mu_2) \\ m &= \dim \mathrm{NS}(X)/2\mathrm{NS}^+(X) \\ u &= \dim H^1(X^{\mathrm{an}}, \mu_2) \\ c &= \mathrm{cdim}\,\mathrm{Pic}(X). \end{aligned}$$

Then,

$$s+u+m = b+c+n+1.$$

Proof. By Lemmas 5.9 and 5.10, all groups that appear in the exact sequence of Proposition 5.3 are $\mathbb{F}_2$-vector spaces. Taking dimensions one gets:

$$\dim(K) - (s-1) + n - \dim(C) + b - m = 0.$$

By Lemma 5.9,

$$\dim(C) = u - \dim H^1(X^{\mathrm{an}}, \mu_2)^0,$$

and by Lemma 5.10,

$$\dim(K) = c - \dim H^1(X^{\mathrm{an}}, \mu_2)^0.$$

After substituting in the above equation, one obtains the result. □

It is interesting that the formula of Theorem 6.3 relates several invariants of real schemes that have been studied in different places in the literature. For example, the number $\#\pi_0(X(\mathbb{R}))$ of connected components of the space $X(\mathbb{R})$ for real surfaces X has been studied in the monograph [14]; the paper [3] contains a survey on results concerning the group $H^1_{\text{alg}}(X(\mathbb{R}), \mu_2)$; the group $H^1(X^{\text{an}}, \mu_2)$ is isomorphic to the *algebraically defined* cohomology group $H^1(X_b, \mu_2)$ which has been introduced and studied in the monograph [12] (see Section 15 of *op. cit.*); and the paper [5] contains a survey of results on the Picard group of real schemes.

As a consequence of Theorem 6.3, one can, in certain cases, measure how far the generalized Harnack's Inequality is from being an equality:

Corollary 6.4. *Let X be a connected proper reduced real scheme such that $X(\mathbb{R}) \neq \emptyset$ and $H^1_{\text{alg}}(X(\mathbb{R}), \mu_2) = 0$. Then, with notation as in Theorem 6.3,*

$$s + u = c + n + 1.$$

Proof. By hypothesis, $H^1_{\text{alg}}(X(\mathbb{R}), \mu_2) = 0$. It follows from Proposition 5.3 that $\text{NS}(X)/2\text{NS}^+(X) = 0$. Hence, with notation as in Theorem 6.3, $b = m = 0$ and the statement follows. ☐

Since $H^1_{\text{alg}}(X(\mathbb{R}), \mu_2)$ is a subgroup of $H^1(X(\mathbb{R}), \mu_2)$, the preceding corollary applies, in particular, to real schemes X such that $H^1(X(\mathbb{R}), \mu_2) = 0$.

As can be seen by inspecting its proof, the formula of Theorem 6.3 does not come directly from an exact sequence. However, if $H^1(X, \mathcal{O}_X) = 0$ then one does have an exact sequence behind the scene:

Theorem 6.5. *Let X be a connected proper reduced real scheme such that $H^1(X, \mathcal{O}_X) = 0$. Then, there is an exact sequence:*

$$\begin{aligned} 0 &\longrightarrow H^0(X(\mathbb{R}), \mu_2)/H^0(X, \mu_2) \longrightarrow \text{NS}^+(X)_2 \longrightarrow \\ &\longrightarrow H^1(X^{\text{an}}, \mu_2) \longrightarrow H^1_{\text{alg}}(X(\mathbb{R}), \mu_2) \longrightarrow \text{NS}(X)/2\text{NS}^+(X) \longrightarrow 0. \end{aligned}$$

If, moreover, $X(\mathbb{C})$ is simply connected and $X(\mathbb{R}) \neq \emptyset$ then, there are isomorphisms

$$H^0(X(\mathbb{R}), \mu_2)/H^0(X, \mu_2) \cong \text{NS}^+(X)_2$$

and

$$H^1_{\text{alg}}(X(\mathbb{R}), \mu_2) \cong \text{NS}(X)/2\text{NS}^+(X).$$

Proof. Since $H^1(X, \mathcal{O}_X) = 0$, the groups $\text{Pic}^{+,0}(X)$ and $\text{Pic}^0(X)^0$ are trivial. By Lemmas 5.9 and 5.10, the group C is isomorphic to $H^1(X^{\text{an}}, \mu_2)$ and the group K is trivial. The first assertion now follows from Proposition 5.3.

In order to show the second assertion, assume that $X(\mathbb{C})$ is simply connected. Since $X(\mathbb{R}) \neq \emptyset$, the quotient $X(\mathbb{C})/\Sigma$ is simply connected as well [9]. In particular, $H^1(X^{\mathrm{an}}, \mu_2) = 0$. Therefore, the group C is trivial and the second assertion follows from the first. □

Corollary 6.6. *Let X be a connected proper reduced real scheme such that $X(\mathbb{C})$ is simply connected, $H^1(X, \mathcal{O}_X) = 0$ and $X(\mathbb{R}) \neq \emptyset$. Let s be the number of connected components of $X(\mathbb{R})$ and let n be the $\mathbb{F}_2$-dimension of $\mathrm{NS}^+(X)_2$. Then,*

$$s = n + 1. \quad \square$$

The preceding corollary applies to nonsingular hypersurfaces X of dimension greater than 1 in projective space. It is now clear why it is hard to determine the maximum number of connected components that $X(\mathbb{R})$ can have, for a fixed degree: the group $\mathrm{NS}^+(X)$ is, in general, hard to determine. Indeed, we have only an analytic—and not an algebraic—description of the group $\mathrm{NS}^+(X)$. It would be interesting to have an algebraic description of the groups $\mathrm{NS}^+(X)$ or $\mathrm{Pic}^+(X)$.

Using Lemma 5.8 one can estimate from above the dimension of $\mathrm{NS}^+(X)_2$ by the dimension of $H^2(X^{\mathrm{an}}, \mu_2)$ if $\mathrm{NS}(X)$ is 2-torsion free. For example, one has the consequence out of Corollary 6.6:

Corollary 6.7. *Let X be a connected proper reduced real scheme such that $H^1(X, \mathcal{O}_X) = 0$, $X(\mathbb{C})$ is simply connected, $X(\mathbb{R}) \neq \emptyset$ and $\mathrm{NS}(X)$ is 2-torsion free. Let s be the number of connected components of $X(\mathbb{R})$, let b be the dimension of $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mu_2)$, let ρ be the rank of $\mathrm{NS}(X)$ and let v be the dimension of $H^2(X^{\mathrm{an}}, \mu_2)$. Then,*

$$s \leq v - \rho + b + 1.$$

Proof. By Theorem 6.5, $s \leq n + 1$. By Lemma 5.8, $\mathrm{NS}^+(X)$ is an extension of $\mathrm{NS}(X)$ by a subgroup of $H^2(X^{\mathrm{an}}, \mu_2)$. Since $\mathrm{NS}(X)$ is 2-torsion free, $\mathrm{NS}^+(X) = F \oplus \mathrm{NS}^+(X)_2$, where F is free Abelian of rank ρ. Since $\mathrm{NS}(X) \subseteq F$ and $2F = 2\mathrm{NS}^+(X)$. one has a short exact sequence

$$0 \longrightarrow \mathrm{NS}(X)/2\mathrm{NS}^+(X) \longrightarrow F/2F \longrightarrow F/\mathrm{NS}(X) \longrightarrow 0.$$

Hence, $\dim F/\mathrm{NS}(X) = \rho - b$. Since $\mathrm{NS}^+(X)/\mathrm{NS}(X)$ is the direct sum of $F/\mathrm{NS}(X)$ and $\mathrm{NS}^+(X)_2$, one has $\rho - b + n \leq v$. □

Since, under the same hypotheses, b is not greater than ρ, one has, in particular, the following statement:

Corollary 6.8. *Let X be a connected proper reduced real scheme such that $H^1(X, \mathcal{O}_X) = 0$, $X(\mathbb{C})$ is simply connected, $X(\mathbb{R}) \neq \emptyset$ and $\mathrm{NS}(X)$ is 2-torsion free. Let s be the number of connected components of $X(\mathbb{R})$ and let v be the dimension of $H^2(X^{\mathrm{an}}, \mu_2)$. Then,*

$$s \leq v + 1. \quad \square$$

The preceding 2 corollaries apply, for example, to nonsingular hypersurfaces of dimension at least 3 in real projective space, or to generic hypersurfaces in $\mathbb{P}^3_{\mathbb{R}}$. Indeed, for such real schemes X, one has $\mathrm{NS}(X_{\mathbb{C}}) = \mathrm{NS}(X_{\mathbb{C}})^{\Sigma} \cong \mathbb{Z}$. We will show in Section 7 that $\mathrm{NS}(X)$ then is isomorphic to $\mathrm{NS}(X_{\mathbb{C}})^{\Sigma}$. In particular, $\mathrm{NS}(X)$ is 2-torsion free, so that Corollaries 6.7 and 6.8 apply.

Unfortunately, the bound of Corollaries 6.7 and 6.8 on the number of connected components does not seem to improve our knowledge in the case of nonsingular hypersurfaces in $\mathbb{P}^3_{\mathbb{R}}$; it seems to be, therefore, more of theoretical than of practical interest.

7 Comparing the zero-Picard group and the Jacobian of a real algebraic variety

Let X be a proper reduced real scheme. Let Pic_X be the Picard group scheme of $X/\mathbb{R}$ [4, Theorem 8.2.3]. Since Pic_X is locally of finite type over $\mathbb{R}$, the set of real points $\mathrm{Pic}_X(\mathbb{R})$ is a not necessarily connected commutative real Lie group. One has a canonical morphism of real Lie groups

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_X(\mathbb{R}).$$

In fact, this morphism is injective [4, Proposition 8.1.4], and if X has a real point, for example, this morphism is an isomorphism.

The *Jacobian* of X is the neutral connected component Pic^0_X of the group scheme Pic_X. Its group of real points $\mathrm{Pic}^0_X(\mathbb{R})$ is also a not necessarily connected commutative real Lie group. Since Pic^0_X is a scheme of finite type over $\mathbb{R}$, the group of connected components of $\mathrm{Pic}^0_X(\mathbb{R})$ is finite. In fact, letting g be the dimension of Pic^0_X, it is well known that there is an integer r satisfying $0 \leq r \leq g$ such that the group of connected components of $\mathrm{Pic}^0_X(\mathbb{R})$ is isomorphic to μ_2^r.

Identifying all real Lie groups $\mathrm{Pic}^0(X)$, $\mathrm{Pic}(X)$ and $\mathrm{Pic}^0_X(\mathbb{R})$ with their images in $\mathrm{Pic}_X(\mathbb{R})$, we have the following diagram of inclusions:

$$\begin{array}{ccc} \mathrm{Pic}^0(X) & \hookrightarrow & \mathrm{Pic}(X) \\ & & \big\downarrow \\ \mathrm{Pic}^0_X(\mathbb{R}) & \hookrightarrow & \mathrm{Pic}_X(\mathbb{R}) \end{array}$$

The object of this section is to compare the real Lie groups $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0_X(\mathbb{R})$ as real Lie subgroups of $\mathrm{Pic}_X(\mathbb{R})$.

Theorem 7.1. *Let X be a proper reduced real scheme such that* $\mathrm{Pic}(X) = \mathrm{Pic}_X(\mathbb{R})$ *and such that* $\mathrm{NS}(X_\mathbb{C})^\Sigma$ *is 2-torsion-free. Then*

$$\mathrm{Pic}^0(X) = \mathrm{Pic}^0_X(\mathbb{R}).$$

Proof. The modified exponential sequence (1) on $X_\mathbb{C}$ gives rise to a modified first-Chern-class map:

$$c_1'\colon \mathrm{Pic}(X_\mathbb{C}) \longrightarrow H^2(X_\mathbb{C}^{\mathrm{an}}, \mathbb{Z}).$$

Restricting the exponential morphism $\exp'$ to the subsheaf $\mathcal{O}_{X_\mathbb{C}^{\mathrm{an}}}$ of $\mu_2 \oplus \mathcal{O}_{X_\mathbb{C}^{\mathrm{an}}}$ gives the usual exponential sequence on $X_\mathbb{C}^{\mathrm{an}}$. Moreover, one has a morphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\pi\sqrt{-1}} & \mathcal{O}_{X_\mathbb{C}^{\mathrm{an}}} & \xrightarrow{\exp} & \mathcal{O}^\star_{X_\mathbb{C}^{\mathrm{an}}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \mu_2 \oplus \mathcal{O}_{X_\mathbb{C}^{\mathrm{an}}} & \xrightarrow{\exp'} & \mathcal{O}^\star_{X_\mathbb{C}^{\mathrm{an}}} & \longrightarrow & 0
\end{array}$$

Hence, one has a commutative diagram

$$\begin{array}{ccc}
 & \overset{c_1}{\nearrow} & H^2(X_\mathbb{C}^{\mathrm{an}}, \mathbb{Z}) \\
\mathrm{Pic}(X_\mathbb{C}) & & \downarrow{\scriptstyle 2} \\
 & \underset{c_1'}{\searrow} & H^2(X_\mathbb{C}^{\mathrm{an}}, \mathbb{Z})
\end{array}$$

where the map c_1 is the usual first-Chern-class map from $\mathrm{Pic}(X_\mathbb{C})$ into $H^2(X_\mathbb{C}^{\mathrm{an}}, \mathbb{Z})$. It follows that $c_1' = 2c_1$.

The Galois group Σ of $\mathbb{C}/\mathbb{R}$ acts on $\mathrm{Pic}(X_\mathbb{C})$ and, by definition, $\mathrm{Pic}_X(\mathbb{R}) = \mathrm{Pic}(X_\mathbb{C})^\Sigma$. The action of Σ on $\mathrm{Pic}(X_\mathbb{C})$ induces an action on $\mathrm{Pic}^0(X_\mathbb{C})$ and $\mathrm{Pic}^0_X(\mathbb{R}) = \mathrm{Pic}^0(X_\mathbb{C})^\Sigma$. Define $\mathrm{Pic}^0(X_\mathbb{C}^{\mathrm{an}})'$ to be the kernel of the map c_1'. Then, the action of Σ on $\mathrm{Pic}(X_\mathbb{C})$ induces also an action on $\mathrm{Pic}^0(X_\mathbb{C})'$. Since $c_1' = 2c_1$ and $\mathrm{NS}(X_\mathbb{C})^\Sigma$ is 2-torsion-free,

$$(\mathrm{Pic}^0(X_\mathbb{C})')^\Sigma = \mathrm{Pic}^0_X(\mathbb{R}).$$

Now, let $p\colon X_\mathbb{C}^{\mathrm{an}} \to X^{\mathrm{an}}$ be the quotient map. The morphism $\mathrm{Pic}(p)$ from $\mathrm{Pic}(X)$ into $\mathrm{Pic}(X_\mathbb{C})$ induces a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{c_1} & \mathrm{NS}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Pic}^0(p)} & & \downarrow{\scriptstyle \mathrm{Pic}(p)} & & \downarrow{\scriptstyle \mathrm{NS}(p)} & & \\
0 & \longrightarrow & \mathrm{Pic}^0(X_\mathbb{C})' & \longrightarrow & \mathrm{Pic}(X_\mathbb{C}) & \xrightarrow{2c_1} & 2\mathrm{NS}(X_\mathbb{C}) & \longrightarrow & 0
\end{array}$$

The Galois group Σ of $\mathbb{C}/\mathbb{R}$ acts on the bottom exact sequence and, replacing $(\mathrm{Pic}^0(X_{\mathbb{C}})')^{\Sigma}$ by $\mathrm{Pic}^0_X(\mathbb{R})$ and $\mathrm{Pic}(X_{\mathbb{C}})^{\Sigma}$ by $\mathrm{Pic}_X(\mathbb{R})$, one gets an induced morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{c_1} & \mathrm{NS}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Pic}^0(p)} & & \downarrow{\scriptstyle \mathrm{Pic}(p)} & & \downarrow{\scriptstyle \mathrm{NS}(p)} & & \\ 0 & \longrightarrow & \mathrm{Pic}^0_X(\mathbb{R}) & \longrightarrow & \mathrm{Pic}_X(\mathbb{R}) & \xrightarrow{2c_1} & (2\mathrm{NS}(X_{\mathbb{C}}))^{\Sigma} & & \end{array} \tag{8}$$

This shows that $\mathrm{Pic}^0(X)$, considered as a subgroup of $\mathrm{Pic}_X(\mathbb{R})$ is contained in $\mathrm{Pic}^0_X(\mathbb{R})$.

In order to show the reverse inclusion, consider the trace map tr from $p_\star\mathcal{O}_{X^{\mathrm{an}}_{\mathbb{C}}}$ into $\mathcal{O}_{X^{\mathrm{an}}}$. It induces a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (p_\star\mathbb{Z})(1) & \longrightarrow & p_\star\mathcal{O}_{X^{\mathrm{an}}_{\mathbb{C}}} & \xrightarrow{p_\star\exp} & p_\star\mathcal{O}^\star_{X^{\mathrm{an}}_{\mathbb{C}}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle N} & & \\ 0 & \longrightarrow & \mathcal{K} & \xrightarrow{2\kappa} & \mathcal{O}_{X^{\mathrm{an}}} & \xrightarrow{\exp^+} & \mathcal{O}^+_{X^{\mathrm{an}}} & \longrightarrow & 0 \end{array}$$

where N denotes the norm map. One gets an induced morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(X_{\mathbb{C}}) & \longrightarrow & \mathrm{Pic}(X_{\mathbb{C}}) & \xrightarrow{c_1} & \mathrm{NS}(X_{\mathbb{C}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \mathrm{tr}} & & \\ 0 & \longrightarrow & \mathrm{Pic}^{+,0}(X) & \longrightarrow & \mathrm{Pic}^+(X) & \xrightarrow{c_1^+} & \mathrm{NS}^+(X) & \longrightarrow & 0 \end{array}$$

Taking Σ-invariant elements, one gets the following commutative diagram in which both rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0_X(\mathbb{R}) & \longrightarrow & \mathrm{Pic}_X(\mathbb{R}) & \xrightarrow{c_1} & \mathrm{NS}(X_{\mathbb{C}})^{\Sigma} & & \\ & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \mathrm{tr}} & & \\ 0 & \longrightarrow & \mathrm{Pic}^{+,0}(X) & \longrightarrow & \mathrm{Pic}^+(X) & \xrightarrow{c_1^+} & \mathrm{NS}^+(X) & \longrightarrow & 0 \end{array} \tag{9}$$

By hypothesis, $\mathrm{Pic}(X) = \mathrm{Pic}_X(\mathbb{R})$. The norm map N from $\mathrm{Pic}_X(\mathbb{R})$ into $\mathrm{Pic}^+(X)$ is equal to the square map sq from $\mathrm{Pic}(X)$ into $\mathrm{Pic}^+(X)$. The above diagram shows that

$$\mathrm{sq}(\mathrm{Pic}^0_X(\mathbb{R})) \subseteq \mathrm{Pic}^{+,0}(X).$$

By Lemma 5.6, $\mathrm{Pic}^0_X(\mathbb{R})$ is contained in $\mathrm{Pic}^0(X)$. $\square$

Corollary 7.2. *Let X be a proper reduced real scheme such that* $\mathrm{Pic}(X) = \mathrm{Pic}_X(\mathbb{R})$*, such that* $\mathrm{NS}(X_\mathbb{C})^\Sigma$ *is 2-torsion-free and such that the induced morphism from* $\mathrm{Pic}^0(X_\mathbb{C})^\Sigma$ *into* $\mathrm{NS}(X_\mathbb{C})^\Sigma$ *is surjective. Then, the morphism*

$$\mathrm{NS}(p)\colon \mathrm{NS}(X) \longrightarrow (2\mathrm{NS}(X_\mathbb{C}))^\Sigma$$

is an isomorphism. In particular, the Néron-Severi group $\mathrm{NS}(X)$ *of X is isomorphic to* $\mathrm{NS}(X_\mathbb{C})^\Sigma$*.*

Proof. By hypothesis, one has a short exact sequence:

$$0 \longrightarrow \mathrm{Pic}^0_X(\mathbb{R}) \longrightarrow \mathrm{Pic}_X(\mathbb{R}) \longrightarrow \mathrm{NS}(X_\mathbb{C})^\Sigma \longrightarrow 0.$$

Since $2(\mathrm{NS}(X_\mathbb{C}))^\Sigma$ is equal to $(2\mathrm{NS}(X_\mathbb{C}))^\Sigma$, the right-most morphism in the bottom row of diagram (8) is surjective. By Theorem 7.1, the morphisms $\mathrm{Pic}^0(p)$ and $\mathrm{Pic}(p)$ are isomorphisms. Hence, $\mathrm{NS}(p)$ is an isomorphism. □

8 Real schemes satisfying $s = c + n + 1$

In this section we show that the generalized Harnack's Inequality of Theorem 6.1 is sharp, i.e., we prove the following statement:

Proposition 8.1. *Let c, n and d be natural integers, where $d \geq 2$. Then, there is a connected nonsingular proper reduced real scheme X of dimension d such that*

$$s = c + n + 1,$$

where s is the number of connected components of $X(\mathbb{R})$, c is the compact dimension of $\mathrm{Pic}(X)$ *and n is the $\mathbb{F}_2$-dimension of* $\mathrm{NS}^+(X)_2$*.*

Proof. Let Y_0 be a nonsingular hypersurface in $\mathbb{P}^{d+1}_\mathbb{R}$ such that $Y_0(\mathbb{R})$ has $n+1$ connected components. By Corollary 6.6, the dimension of $\mathrm{NS}^+(Y_0)_2$ is equal to n. Replacing Y_0 by a blow-up of Y_0 in a real point, we may assume that there is a curve D_0 in Y_0 which is isomorphic to $\mathbb{P}^1_\mathbb{R}$. Let $Y_1, \ldots, Y_c$ be real schemes isomorphic to $\mathbb{P}^d_\mathbb{R}$ and let $D_i \subseteq Y_i$ be curves isomorphic to $\mathbb{P}^1_\mathbb{R}$. Choose nonreal closed points $P_i \in D_i$ and $Q_i \in D_{i+1}$ for $i = 0, \ldots, c-1$. Let Y the union of the schemes $Y_0, \ldots, Y_c$ with each pair of points (P_i, Q_i) identified to a point, for $i = 0, \ldots, c-1$. Let $D \subseteq Y$ be the union of the images of the curves D_i in Y, i.e., D is the union of the curves D_i with each pair of points (P_i, Q_i) identified to a point. Now, let (X, C) be a nonsingular deformation over $\mathbb{R}$ of the pair (Y, D). We show that X satisfies the conclusion of the proposition.

It is clear that the number of connected components of $X(\mathbb{R})$ is equal to $c + n + 1$. The real curve C is nonsingular and of genus c. The number of connected components of $C(\mathbb{R})$ is equal to $c+1$, i.e., C is an M-curve. Hence, C^{an} is topologically the complement of the union of $c+1$ disjoint open discs in the 2-sphere S^2. Since each Y_i^{an} is simply connected, the inclusion of C^{an} into X^{an} induces an isomorphism from $\pi_1(C^{\mathrm{an}})$ onto $\pi_1(X^{\mathrm{an}})$. It follows that the inclusion of C into X induces an isomorphism from $\mathrm{Pic}(X)$ onto $\mathrm{Pic}(C)$. In particular, the compact dimension of $\mathrm{Pic}(X)$ is equal to the compact dimension of $\mathrm{Pic}(C)$. Since C is a nonsingular real curve of genus c, the latter compact dimension is equal to c. Hence, the compact dimension of $\mathrm{Pic}(X)$ is equal to c.

In order to show that the dimension of $\mathrm{NS}^+(X)_2$ is equal to n, it suffices to show, by Proposition 5.3 and Lemma 5.9, that the natural morphism from $H^1(X^{\mathrm{an}}, \mu_2)$ into $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mu_2)$ is injective. By the preceding observations, the inclusion of C^{an} into X^{an} induces an isomorphism from $H^1(X^{\mathrm{an}}, \mu_2)$ onto $H^1(C^{\mathrm{an}}, \mu_2)$. Since C is an M-curve, the morphism from $H^1(C^{\mathrm{an}}, \mu_2)$ into $H^1_{\mathrm{alg}}(C(\mathbb{R}), \mu_2)$ is injective. Therefore, the morphism from $H^1(X^{\mathrm{an}}, \mu_2)$ into $H^1_{\mathrm{alg}}(X(\mathbb{R}), \mu_2)$ is injective. □

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Institut Mathématique de Rennes, Université de Rennes 1,
Campus de Beaulieu, 35042 Rennes Cedex, France.
E-mail: huisman@univ-rennes1.fr

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