EVERY CONNECTED SUM OF LENS SPACES IS A REAL COMPONENT OF A UNIRULED ALGEBRAIC VARIETY

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ABSTRACT. Let M be a connected sum of finitely many lens spaces, and let N be a connected sum of finitely many copies of $S^1 \times S^2$. We show that there is a uniruled algebraic variety X such that the connected sum $M \# N$ of M and N is diffeomorphic to a connected component of the set of real points $X(\mathbb{R})$ of X. In particular, any finite connected sum of lens spaces is diffeomorphic to a real component of a uniruled algebraic variety.

MSC 2000: 14P25

Keywords: uniruled algebraic variety, Seifert manifold, lens space, connected sum, equivariant line bundle, real algebraic model

1. Introduction

The Theorem of Nash-Tognoli states that any differentiable manifold is diffeomorphic to a real component of an algebraic variety. More precisely, for any compact connected differentiable manifold M, there is a nonsingular projective and geometrically irreducible real algebraic variety X, such that M is diffeomorphic to a connected component of the set of real points $X(\mathbb{R})$ of X. The question then naturally rises as to which differentiable manifolds actually occur as real components of algebraic varieties of a given class. For example, one may wonder which differentiable manifolds are diffeomorphic to a real component of an algebraic variety of Kodaira dimension $-\infty$. That specific question is the question we will address in the current paper for algebraic varieties of dimension 3.

In dimension ≤ 3, an algebraic variety X has Kodaira dimension $-\infty$ if and only if it is *uniruled*, i.e., if and only if there is a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$, where Y is a real algebraic variety of dimension $\dim(X) - 1$. Therefore, the question we study is the question as to which differentiable manifolds occur as a real component of a uniruled algebraic variety of dimension 3. In dimension 0 and 1, that question has a trivial answer. In dimension 2, the answer is due to Commessatti.

Theorem (Comessatti 1914 [1])**.** *Let X be a uniruled real algebraic surface. Then, a connected component of $X(\mathbb{R})$ is either nonorientable, or diffeomorphic to the sphere S^2 or the torus $S^1 \times S^1$. Conversely, a compact connected differentiable surface that is either nonorientable or diffeomorphic to S^2 or $S^1 \times S^1$, is diffeomorphic to a real component of a uniruled real algebraic surface.*

We have deliberately adapted the statement of Commessatti's Theorem for the purpose of the current paper. Commessatti stated the result for real surfaces that are geometrically rational, i.e., whose complexification is a complex rational surface. The more general statement above easily follows from that fact.

The authors are grateful to MSRI for financial support and excellent working conditions. The second author is member of the European Research Training Network RAAG (EC contract HPRN-CT-2001-00271).

In dimension 3, much progress has been made, due to Kollár, in classifying the differentiable manifolds that are diffeomorphic to a real component of a uniruled algebraic variety.

Theorem (Kollár 1998 [7, Th. 6.6])**.** *Let X be a uniruled real algebraic variety of dimension 3 such that $X(\mathbb{R})$ is orientable. Let M be a connected component of $X(\mathbb{R})$. Then, M is diffeomorphic to one of the following manifolds:*

(1) *a Seifert manifold,*
(2) *a connected sum of finitely many lens spaces,*
(3) *a locally trivial torus bundle over S^1, or doubly covered by such a bundle,*
(4) *a manifold belonging to an* a priori *given finite list of exceptions, or*
(5) *a manifold obtained from one of the above by taking the connected sum with a finite number of copies of $\mathbb{P}^3(\mathbb{R})$ and a finite number of copies of $S^1 \times S^2$.*

Recall that a *Seifert* manifold is a manifold admitting a differentiable foliation by circles. A *lens space* is a manifold diffeomorphic to a quotient of the 3-sphere S^3 by the action of a cyclic group. In case the set of real points of a uniruled algebraic variety is allowed not to be orientable, the results of Kollár are less precise due to many technical difficulties, but see [6, Theorem 8.3]. In order to complete the classification in the orientable case, Kollár proposed the following conjectures.

Conjecture (Kollár 1998 [7, Conj. 6.7])**.**
(1) *Let M be an orientable Seifert manifold. Then there is a uniruled algebraic variety X such that M is diffeomorphic to a connected component of $X(\mathbb{R})$.*
(2) *Let M be a connected sum of lens spaces. Then there is a uniruled algebraic variety X such that M is diffeomorphic to a connected component of $X(\mathbb{R})$.*
(3) *Let M be a locally trivial torus bundle over S^1 which is not a Seifert manifold. Then M is not diffeomorphic to a real component of a uniruled algebraic variety X.*
(4) *Let M be a manifold belonging to the* a priori *given list of exceptional manifolds. Then M is not diffeomorphic to a real component of a uniruled algebraic variety X.*

Let us also mention the following result of Eliashberg and Viterbo (unpublished).

Theorem (Eliashberg, Viterbo)**.** *Let X be a uniruled real algebraic variety. Let M be a connected component of $X(\mathbb{R})$. Then M is not hyperbolic.*

In an earlier paper, we proved Conjecture (1) above, i.e., that any orientable Seifert manifold M is diffeomorphic to a connected component of the set of real points of a uniruled real algebraic variety X [3, Th. 1.1]. Unfortunately, we do not know whether $X(\mathbb{R})$ is orientable, in general. Indeed, the uniruled variety X we constructed may have more real components than the one that is diffeomorphic to M, and we are not able to control the orientability of such additional components.

Recently, we realized that the methods used to prove Th. 1.1 of [3] can be generalized in order to obtain a similar statement concerning connected sums of lens spaces. In fact, we prove, in the current paper a slightly more general, statement.

To the best of our knowledge, Kollár did not conjecture which manifolds, that are connected sums of one of the above manifolds (1–4) with a finite number of copies of $\mathbb{P}^3(\mathbb{R})$ and a finite number of copies of $S^1 \times S^2$, are realizable as a real component of a uniruled algebraic variety. Of course, if M is realizable as a real component of a uniruled algebraic variety, the connected sum $M\#\mathbb{P}^3(\mathbb{R})$ is also realizable. But for $M\#(S^1 \times S^2)$, the question seems to be more delicate.

The main result of the paper is the following.

Theorem 1.1. *Let N_1 be an oriented connected sum of finitely many lens spaces, and let N_2 be an oriented connected sum of finitely many copies of $S^1 \times S^2$. Let M*

be the connected sum $N_1 \# N_2$*. Then, there is a uniruled real algebraic variety* X *such that* M *is diffeomorphic to a connected component of* $X(\mathbb{R})$*.*

Corollary 1.2. *Let* M *be a connected sum of finitely many lens spaces. Then, there is a uniruled real algebraic variety* X *such that* M *is diffeomorphic to a connected component of* $X(\mathbb{R})$*.*

This proves Conjecture (2) above. Conjectures (3) and (4) remain open. The proof of Theorem 1.1 has two parts.

Firstly, developing an idea of Kollár in [6], we prove the existence of a particular map $f\colon M \to S$ over a differentiable surface with boundary (see Theorem 2.6). Then along the same lines as in [3], we prove that we can suppose the existence of a finite ramified topological covering $\pi\colon \tilde{S} \to S$ such that the fiber product

$$\tilde{f}\colon \tilde{M} = M \times_S \tilde{S} \longrightarrow \tilde{S}$$

is a locally trivial differentiable circle bundle over the interior of $\tilde{S}$ (Theorem 2.7). Moreover, the covering $\tilde{f}$ is Galois, the Galois group G acting with fixed point-freely on $\tilde{M}$.

Secondly, we prove that there are

(1) a real algebraic surface $\tilde{S}'$, endowed with a real algebraic action of G,
(2) a real algebraic plane bundle (V, p) on $\tilde{S}'$, also endowed with a real algebraic action of G,
(3) a G-invariant real algebraic norm ν on V, and
(4) a G-invariant real algebraic function r on $\tilde{S}'$ with regular value 0,

such that the submanifold $\{r \geq 0\}$ of $\tilde{S}'$ is equivariantly diffeomorphic to $\tilde{S}$, and the submanifold $N = \{\nu^2 = r \circ p\}$ of V is equivariantly diffeomorphic to $\tilde{M}$. Since G acts fixed point-freely on the real algebraic variety N, the quotient N/G is a connected component of a real algebraic variety. Since M is diffeomorphic to N/G, it follows that M is a real component of a uniruled algebraic variety.

As one can notice, the proof of our main result, uses a generalization of the method of proof of Theorem 1.1 of [3]. Several people have pointed out to us work of Dovermann, Masuda and Suh [2], that would have been useful in realizing algebraically the equivariant set-up above. However, the results of Doverman *et al.* apply only to semi-free actions of a group, whereas here, the action of G is, more or less, arbitrary, in any case, not necessarily semi-free. Therefore, as a by-product of our methods, we can mention the following generalization of [2, Th. B] in the case of a certain finite group actions on a real plane bundle over a surface.

Theorem 1.3. *Let* S *be an orientable compact connected surface without boundary and let* G *be a finite group acting on* S*. Let* (V, p) *be an orientable differentiable real plane bundle over* S*, endowed with an action of* G *over the action on* S *such that*

(1) *S contains only finitely many fixed points, and*
(2) *G acts by orientable diffeomorphisms on V.*

Then there is a nonsingular real algebraic surface T *endowed with a real algebraic action of* G*, a strongly algebraic real plane bundle* (W, q) *over* T*, endowed with a real algebraic action of* G *over the action on* T*, such that there are* G*-equivariant diffeomorphisms* $\phi\colon S \to T$ *and* $\psi\colon V \to W$ *making the following diagram commutative.*

$$\begin{array}{ccc} V & \rightarrow & W \\ \downarrow & & \downarrow \\ S & \rightarrow & T \end{array}$$

For a proof, we refer to the paper [3], where this statement has not been stated explicitly.

Acknowledgement. The authors are grateful to S. Akbulut, J. Bochnak, H. King, W. Kucharz for bringing to the attention the above result of Dovermann *et al.* The authors thank K. H. Dovermann, J. Kollàr, O. Viro for helpful discussions and A. Marin for his interest.

2. Connected sums of lens spaces

Let $S^1 \times D^2$ be the *solid torus* where S^1 is the unit circle $\{u \in \mathbb{C} \mid |u| = 1\}$ and D^2 is the closed unit disc $\{z \in \mathbb{C}, |z| \leq 1\}$. A *Seifert fibration* of the solid torus is a differentiable map of the form

$$f_{p,q} : S^1 \times D^2 \to D^2 \ , (u, z) \mapsto u^q z^p \ ,$$

where p and q are natural integers, with $p \neq 0$ and $\gcd(p, q) = 1$. Let M be a 3-manifold. A *Seifert fibration* of M is a differentiable map f from M into a differentiable surface S having the following property. Every point $P \in S$ has a closed neighborhood U such that the restriction of f to $f^{-1}(U)$ is diffeomorphic to a Seifert fibration of the solid torus. Sometimes, nonorientable local models are also allowed in the literature, e.g. [8]. For our purposes, we do not need to include them in the definition of a Seifert fibration, since the manifolds we study are orientable.

Let C^2 be the *collar* defined by $C^2 = \{w \in \mathbb{C} \mid 1 \leq |w| < 2\}$. Let P be the differentiable 3-manifold defined by

$$P = \{((w, z) \in C^2 \times \mathbb{C} \mid |z|^2 = |w| - 1\}.$$

Let $\omega\colon P \to C^2$ be the projection defined by $\omega(w, z) = w$. It is clear that ω is a differentiable map, that ω is a trivial circle bundle over the interior of C^2, and that ω is a diffeomorphism over the boundary of C^2.

Definition 2.1. *Let $f\colon M \to S$ be a differentiable map from a 3-manifold M without boundary into a differentiable surface S with boundary. The map f is a* Werther map *if*

(1) *the restriction of f over the interior of S is a Seifert fibration, and*
(2) *the restriction of f over an open neighborhood of each boundary component of S is diffeomorphic to ω.*

Remarks 2.2. (1) Let M be a Seifert manifold which is not a connected sum of lens spaces, then for all Werther maps $M \to S$, we have $\partial S = \emptyset$, see [6, 3.7].
(2) Let M be a 3-manifold. A Werther map $M \to S$ is a Seifert fibration if and only if $\partial S = \emptyset$.

For an integer n, let μ_n be the multiplicative subgroup of $\mathbb{C}^\star$ of the n-th roots of unity. We agree that $\mu_0 = \{1\}$. Let p and q be relatively prime integers. The *lens space* $L_{p,q}$ is the quotient of the 3-sphere $S^3 = \{(w, z) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1\}$ by the action of μ_{pq} defined by

$$\xi \cdot (w, z) = (\xi^q w, \xi^p z),$$

for all $\xi \in \mu_{pq}$ and $(w, z) \in S^3$. A *lens space* is a differentiable manifold diffeomorphic to a manifold of the form $L_{p,q}$. It is clear that a lens space is an orientable compact connected differentiable manifold of dimension 3.

Lemma 2.3. *Let p and q be relatively prime integers. There is a Werther map $f\colon L_{p,q} \longrightarrow D^2$.*

Proof. Let $g\colon S^3 \longrightarrow D^2$ be the map $g(w,z) = w^p$ for all $(w,z) \in S^3$. Since g is constant on μ_{pq}-orbits, the map g induces a differentiable map $f\colon L_{p,q} \longrightarrow D^2$. It is easy to check that f is a Werther map. □

Lemma 2.4. *Let A^2 be the closed annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$. There is Werther map $f\colon S^1 \times S^2 \longrightarrow A^2$.*

Proof. Let S^2 be the 2-sphere in $\mathbb{C}\times\mathbb{R}$ defined by $|z|^2+t^2 = 1$. Let $f\colon S^1\times S^2 \longrightarrow A^2$ be the map defined by $f(w,z,t) = \frac{1}{2}(t+3)w$. It is easy to check that f is a Werther map. □

Lemma 2.5. *Let $f_1\colon M_1 \to S_1$ and $f_2\colon M_2 \to S_2$ be two Werther maps where M_1 and M_2 are oriented 3-manifolds without boundaries. Suppose that the boundaries ∂S_1 and ∂S_2 are nonempty. Then there is a differentiable surface S with nonempty boundary and a Werther map*

$$f\colon M_1\#M_2 \longrightarrow S\ .$$

Proof. Let $\gamma_i \subset S_i$, $i \in \{1,2\}$ be a simple path having its end points in the same boundary component of S_i, and whose interior is contained in the interior of S_i. One may assume that γ_i bounds a closed disc D_i in S_i, over the interior of which f_i is a trivial circle bundle. Let $T_i = \overline{S_i \setminus D_i}$ and let $N_i = \overline{M_i \setminus f_i^{-1}(D_i)}$. By construction, $f_i^{-1}(\gamma_i)$ is a 2-sphere in M_i bounded by the 3-ball $f_i^{-1}(D_i)$. The restriction of f_1 to $f_1^{-1}(D_1)$ is diffeomorphic to $f_2^{-1}(D_2)$. In particular, we have an orientation reversing diffeomorphism between $f_1^{-1}(\gamma_1)$ and $f_2^{-1}(\gamma_2)$ compatible with a diffeomorphism between γ_1 and γ_2. Therefore, the connected sum M of M_1 and M_2 is diffeomorphic to the manifold obtained from gluing N_1 and N_2 along the orientation reversing diffeomorphism between $f_1^{-1}(\gamma_1)$ and $f_2^{-1}(\gamma_2)$. Let S be the manifold obtained from gluing T_1 and T_2 along the diffeomorphism between γ_1 and γ_2. One has an induced differentiable map $f\colon M \to S$ that is a Werther map. □

Theorem 2.6. *Let N_1 be an oriented connected sum of finitely many lens spaces, and let N_2 be an oriented connected sum of finitely many copies of $S^1 \times S^2$. Let M be the connected sum $N_1\#N_2$. Then, there is a compact connected differentiable surface S with boundary and a Werther map $f\colon M \to S$.*

Proof. The statement follows from Lemmas 2.3, 2.4 and 2.5. □

Theorem 2.7. *Let N_1 be an oriented connected sum of finitely many lens spaces, and let N_2 be an oriented connected sum of finitely many copies of $S^1 \times S^2$. Let M be the connected sum $N_1\#N_2$. Then there is a Werther map $f\colon M \to S$ of M over a compact connected surface S, and a finite ramified topological covering $\pi\colon \tilde{S} \to S$ such that*

1. *$\tilde{S}$ is orientable,*
2. *π is unramified over the boundary of S,*
3. *π is Galois, i.e., π is a quotient map for the group of automorphisms of $\tilde{S}/S$,*
4. *the induced action of G on the fiber product $\tilde{M} = \tilde{S}\times_S M$ is fixed point-free,*
5. *the induced fibration $\tilde{f}\colon \tilde{M} \to \tilde{S}$ is a locally trivial circle bundle over the interior of $\tilde{S}$.*

Proof. If M is a Seifert manifold, the statement follows from Theorem 1.1 of [3]. If M is not Seifert, then by Theorem 2.6, there is a Werther map $f\colon M \to S$ of M over a compact connected differentiable surface S with nonempty boundary. The statement then follows as in the proof of Theorem 1.1 cited above. □

3. Algebraic models

Proof of Theorem 1.1. Let $f: M \to S$ be a Werther map of M as in Theorem 2.7. Let S' be a compact connected differentiable surface without boundary containing S, such that the complement of S in S' is a disjoint union of open discs. Similarly, there is such a surface $\tilde{S}'$ containing $\tilde{S}$. The ramified covering $\pi: \tilde{S} \to S$ of Theorem 2.7 extends to a ramified covering $\pi': \tilde{S}' \to S'$, introducing, if necessary, at most one ramification point, in each connected component of $S' \setminus S$. It is clear that the action of G extends to an action of $\tilde{S}'/S'$.

Let $r: S' \to \mathbb{R}$ be a differentiable function having 0 as a regular value, and such that $r^{-1}([0,\infty)) = S$. There is a differentiable real plane bundle (V,p) over $\tilde{S}'$ endowed with

(1) an action of G over the action of G on $\tilde{S}'$, and
(2) a G-invariant differentiable norm ν on V,

such that the 3-manifold $N = \{v \in V \mid \nu(v)^2 = r \circ \pi' \circ p(v)\}$ is G-equivariantly diffeomorphic to $\tilde{M}$.

Now, by Theorem 1.3, there are

(1) a structure of a real algebraic variety on $\tilde{S}'$ such that the action of G on $\tilde{S}'$ is algebraic, and
(2) a structure of a real algebraic vector bundle on V such that the action of G on V is algebraic.

Then we approximate ν by a real algebraic norm on V, again denoted by ν, and may assume that ν is G-equivariant. As usual in real algebraic geometry, the quotient surface $S' = \tilde{S}'/G$ is only a semialgebraic subset of a real algebraic surface Z. The surface Z has singularities at the image of points of $\tilde{S}'$ having even isotropy groups. Since r is nonzero at those singularities, one can approximate r be a real algebraic function, again denoted by r, such that $r^{-1}([0,\infty))$ is isotopic to S. Then, the corresponding 3-manifold N, as defined above, is a ruled real algebraic 3-fold, endowed with an algebraic action of G. As a differentiable manifold, N is still G-equivariantly diffeomorphic to $\tilde{M}$. In particular, the action of G is fixed point-free and the quotient N/G is a connected component of a uniruled real algebraic variety. Since N/G is diffeomorphic to M, the Theorem is proved. □

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