Cubic differential forms and the group law on the Jacobian of a real algebraic curve

J. Huisman

Sunto

In un precedente articolo [6] abbiamo descritto, in modo esplicito, la legge di gruppo sulla componente neutrale dell'insieme dei punti reali della Jacobiana di una quartica liscia. In questo articolo generalizziamo tale risultato a curve di genere superiore, dando una descrizione della legge di gruppo sulla componente neutrale dell'insieme dei punti reali della Jacobiana di una curva liscia, in termini di forme differenziali cubiche. Applicando tale risultato alle curve canoniche, si ottiene una descrizione geometrica esplicita della legge di gruppo, intersecando la curva con opportune ipersuperfici cubiche.

Abstract

In an earlier paper [6], we gave an explicit geometric description of the group law on the neutral component of the set of real points of the Jacobian of a smooth quartic curve. Here, we generalize this description to curves of higher genus. We get a description of the group law on the neutral component of the set of real points of the Jacobian of a smooth curve in terms of cubic differential forms. When applied to canonical curves, one gets an explicit geometric description of this group law by intersecting the curve with cubic hypersurfaces.

MSC 2000: 14H40, 14P99

Keywords: real curve, Jacobian, cubic differential form, canonical curve, cubic hypersurface

1 Introduction

In an earlier paper [6], we gave an explicit geometric description of the group law on the neutral component of the set of real points of the Jacobian of a smooth quartic curve. Here, we generalize this description to curves of higher genus. We get a description of the group law on the neutral component of the set of real points of the Jacobian of a smooth curve in terms of cubic differential forms (Theorem 3.2). When applied to canonical curves, one gets an explicit geometric description of this group law by intersecting the curve with cubic hypersurfaces (Corollary 4.1).

The paper is organized as follows. In Section 2, we will be more precise on the object of the paper. In Section 3, we state and prove the result on the geometric description of the neutral component of the Jacobian. In Section 4, we apply the preceding result to canonical curves. In Section 5, we compare the present description of the neutral component of the Jacobian with previous descriptions.

Acknowledgement. I am grateful to Michele Lattarulo for the translation of the abstract in Italian.

2 The object

Let C be smooth proper geometrically integral real algebraic curve. Let g be its genus. We assume that $g \geq 3$. A connected component of the set $C(\mathbb{R})$ of real points of C is called a *real branch* of C. By Harnack's Inequality [3], the number of real branches of C is less than or equal to $g+1$. We assume throughout the paper that C has at least g real branches. Choose, once and for all, g mutually distinct real branches $B_1, \ldots, B_g$ of C and put

$$B = B_1 \times \cdots \times B_g.$$

The Jacobian $\mathrm{Jac}(C)$ of C is a real Abelian variety of dimension g. Its set of real points $\mathrm{Jac}(C)(\mathbb{R})$ is a compact commutative real Lie group. We denote by $\mathrm{Jac}(C)(\mathbb{R})^0$ the connected component of $\mathrm{Jac}(C)(\mathbb{R})$ that contains 0. Let $O \in B$ be a base point. Define a map

$$\tau_O \colon B \longrightarrow \mathrm{Jac}(C)(\mathbb{R})$$

by

$$\tau_O(P) = \mathrm{cl}(\sum_{i=1}^{g} (P_i - O_i)),$$

for all $P \in B$, where cl denotes the class of a divisor. Since $\tau_O(O) = 0$ and since B is connected, τ_O maps B into $\mathrm{Jac}(C)(\mathbb{R})^0$. Applying the general result of [5] to the present situation, one has the following statement.

Theorem 2.1. *The map* $\tau_O \colon B \to \mathrm{Jac}(C)(\mathbb{R})^0$ *is a bijection.*

In particular, one gets, by transport of structure, a group law on B. The object of this paper is to give a geometric description of this group law in terms of cubic differential forms on C.

As it will be useful in the sequel as well, let us recall the principal result [4] that lead to the statement of Theorem 2.1. For a divisor D on C and for a real branch X of C, denote by $\deg_X(D)$ the degree of D on X.

Theorem 2.2. *Let D be a divisor on C. Let d be its degree and let k be the number of real branches X of C for which $\deg_X(D)$ is odd. If $d+k > 2g-2$ then D is nonspecial.* □

We refer to [6] for a short proof of Theorem 2.2.

For the convenience of the reader, we give a proof of Theorem 2.1 using Theorem 2.2.

Proof of Theorem 2.1. Suppose that $P, Q \in B$ are such that $\tau_O(P) = \tau_O(Q)$. Let D be the divisor $\sum P_i$ and let E be the divisor $\sum Q_i$ on C. Then, D and E are linearly equivalent. Now, D is of degree g and is of odd degree on exactly g real branches of C. Since $g+g > 2g-2$, the divisor D is nonspecial by Theorem 2.2. Then, by Riemann-Roch, the linear system $|D|$ is 0-dimensional, Since D and E belong to $|D|$, one has $D = E$. It follows that $P = Q$. This shows that τ_O is injective.

In order to show that τ_O is surjective, let δ be an element of $\mathrm{Jac}(C)(\mathbb{R})^0$. The element $\delta + \mathrm{cl}(\sum O_i)$ is a real point of the degree g part $\mathrm{Pic}^g_{C/\mathbb{R}}$ of the Picard scheme of C over $\mathbb{R}$. Since $\mathrm{Pic}^g_{C/\mathbb{R}}(\mathbb{R}) = \mathrm{Pic}^g(C)$ [1, Proposition 4.1.2 (i)], there is a divisor D on C such that $\mathrm{cl}(D) = \delta + \mathrm{cl}(\sum O_i)$. Since D is of degree g, we may assume that D is effective by Riemann-Roch. Since $\delta \in \mathrm{Jac}(C)(\mathbb{R})^0$, the divisor D has odd degree on the real branches $B_1, \ldots, B_g$ of C [1, §4.1]. In particular, there are points $P_i \in B_i$, for $i = 1, \ldots, g$, such that $D \geq P_1 + \ldots + P_g$. Since D is of degree g, this inequality is, in fact, an equality. Let $P = (P_1, \ldots, P_g) \in B$. Then, $\tau_O(P) = \mathrm{cl}(D - \sum O_i) = \delta$. This shows surjectivity of τ_O. □

3 Cubic differential forms

A cubic differential form on C is a global section ω of the third tensor power $\Omega^{\otimes 3}$ of the sheaf Ω of differential forms on C. The divisor $\mathrm{div}(\omega)$ of ω is the divisor of ω as a section of the invertible sheaf $\Omega^{\otimes 3}$.

The following statement says that any $5g-6$ real points of C that are well distributed over the real branches $B_1, \ldots, B_g$ of C, are in general position with respect to cubic differential forms.

Theorem 3.1. *Choose $5g-6$ real points X_j on C, for $j = g+1, \ldots, 6g-6$, such that each X_j belongs to one of the real branches B_i and such that each of the B_i contains an odd number of the points X_j. Then, there is a nonzero cubic differential form ω on C such that*

$$\operatorname{div}(\omega) \geq \sum_{j=g+1}^{6g-6} X_j.$$

Moreover, ω is unique up to multiplication by a nonzero scalar. Furthermore, there are unique points $X_j \in B_j$, for $j = 1, \ldots, g$, such that

$$\operatorname{div}(\omega) = \sum_{j=1}^{6g-6} X_j.$$

Proof. Let K be a canonical divisor on C. Let D be the divisor on C defined by

$$D = 3K - \sum_{j=g+1}^{6g-6} X_j.$$

We have to show that $h^0(D) = 1$. The degree of D is equal to $3(2g-2) - (5g-6) = g$. Moreover, D has odd degree on each of the real branches B_i. Indeed, the degree of K is even at each real branch of C [1, Corollary 4.2.2]. It follows, by Theorem 2.2, that D is nonspecial and, hence, that $h^0(D) = 1$. This shows the existence and uniqueness of ω.

Since the degree of $\operatorname{div}(\omega)$ is even on each of the real branches B_i [1, Corollary 4.2.2], and since the degree of the divisor $\sum_{j=g+1}^{6g-6} X_j$ is odd on each of the real branches B_i, there are points $X_j \in B_j$, for $j = 1, \ldots, g$ such that

$$\operatorname{div}(\omega) \geq \sum_{j=1}^{6g-6} X_j.$$

Now, both divisors that intervene in this inequality are of degree $6g-6$. Hence, the inequality is an equality. This also proves uniqueness of the points $X_j \in B_j$, for $j = 1, \ldots, g$. □

Choose $O \in B$ and choose $2g-6$ real points X_j, for $j = g+1, \ldots, 3g-6$, on C such that each X_j belongs to one of the real branches B_i and such that each of the B_i contains an even number of the points X_j. For example, one can choose all X_j to be equal to O_1. According to Theorem 3.1, there is a nonzero cubic differential form η on C such that

$$\operatorname{div}(\eta) = \sum_{i=1}^{g} 3O_i + \sum_{j=1}^{3g-6} X_j.$$

for some real points $X_1, \dots, X_g$ in $B_1, \dots B_g$, respectively.

Now, the geometric description of the group law of $\mathrm{Jac}(C)(\mathbb{R})^0$ is given in the following statement. Recall from Theorem 2.1 that τ_O is a bijective map from B onto $\mathrm{Jac}(C)(\mathbb{R})^0$, and is defined by sending $P \in B$ to $\mathrm{cl}(\sum(P_i - O_i))$.

Theorem 3.2. *Let $P, Q, R \in B$. Then,*

$$\tau_O(P) + \tau_O(Q) + \tau_O(R) = 0$$

in $\mathrm{Jac}(C)(\mathbb{R})^0$ if and only if there is a cubic differential form ω on C such that

$$\mathrm{div}(\omega) = \sum_{i=1}^{g}(P_i + Q_i + R_i) + \sum_{j=1}^{3g-6} X_j.$$

Proof. Suppose that $\tau_O(P) + \tau_O(Q) + \tau_O(R) = 0$ in $\mathrm{Jac}(C)(\mathbb{R})^0$. By Theorem 3.1, there is a nonzero cubic differential form ω on C such that

$$\mathrm{div}(\omega) \geq \sum_{i=1}^{g}(P_i + Q_i + R_i) + \sum_{j=g+1}^{3g-6} X_j.$$

Moreover, there are points $S_i \in B_i$, for $i = 1, \dots, g$, such that

$$\mathrm{div}(\omega) = \sum_{i=1}^{g}(P_i + Q_i + R_i + S_i) + \sum_{j=g+1}^{3g-6} X_j.$$

Then,

$$\mathrm{div}(\omega - \eta) = \sum_{i=1}^{g}(P_i - O_i) + (Q_i - O_i) + (R_i - O_i) + (S_i - X_i).$$

Taking divisor classes and using the hypothesis, one gets that

$$\mathrm{cl}(\sum_{i=1}^{g}(S_i - X_i)) = 0$$

in $\mathrm{Jac}(C)(\mathbb{R})^0$, i.e., $\tau_X(S) = 0$, where $X \in B$ is the point $(X_1, \dots, X_g)$. By Theorem 2.1, $S = X$ and it follows that there is a cubic differential form ω such that

$$\mathrm{div}(\omega) = \sum_{i=1}^{g}(P_i + Q_i + R_i) + \sum_{j=1}^{3g-6} X_j.$$

In order to show the converse, suppose that there is a cubic differential form ω satisfying the above equation. Then,

$$\operatorname{div}(\omega - \eta) = \sum_{i=1}^{g} (P_i - O_i) + (Q_i - O_i) + (R_i - O_i),$$

i.e., $\tau_O(P) + \tau_O(Q) + \tau_O(R) = 0$ in $\mathrm{Jac}(C)(\mathbb{R})^0$. □

4 Canonical curves

In this section we apply Theorem 3.2 to canonical curves, i.e., we suppose that C is canonically embedded in $\mathbb{P}^{g-1}$. In particular, we assume here that C is not hyperelliptic. Denote by f the embedding of C into $\mathbb{P}^{g-1}$. By Noether's Theorem, the natural map

$$f^\star \colon H^0(\mathbb{P}^{g-1}, \mathcal{O}(3)) \longrightarrow H^0(C, \Omega^{\otimes 3})$$

is surjective.

Choose, as before, $O \in B$ and $2g-6$ real points X_j, for $j = g+1, \ldots, 3g-6$, on C such that each X_j belongs to one of the real branches B_i and such that each of the B_i contains an even number of the points X_j. According to Theorem 3.1 and using Noether's Theorem, there is a real cubic hypersurface G in $\mathbb{P}^{g-1}$ not containing C such that

$$G \cdot C = \sum_{i=1}^{g} 3O_i + \sum_{j=1}^{3g-6} X_j.$$

for some real points $X_1, \ldots, X_g$ in $B_1, \ldots B_g$, respectively. Using Noether's Theorem again, Theorem 3.2 has as a consequence the following geometric description of the group law on $\mathrm{Jac}(C)(\mathbb{R})^0$.

Corollary 4.1. *Let $P, Q, R \in B$. Then,*

$$\tau_O(P) + \tau_O(Q) + \tau_O(R) = 0$$

in $\mathrm{Jac}(C)(\mathbb{R})^0$ *if and only if there is a real cubic hypersurface H in $\mathbb{P}^{g-1}$ not containing C such that*

$$H \cdot C = \sum_{i=1}^{g} (P_i + Q_i + R_i) + \sum_{j=1}^{3g-6} X_j. \tag{1}$$

Remark 4.2. If $g = 3$ then C is canonically embedded in $\mathbb{P}^2$ as a quartic. In particular, no cubic curve in $\mathbb{P}^2$ can contain C. Hence, Corollary 4.1 is a true generalization of Theorem 3.2 of [6].

Example 4.3. If $g = 4$ then C is canonically embedded in $\mathbb{P}^3$ as a sextic curve. Corollary 4.1 says that $P + Q + R = 0$ in B, for the induced group law on B, if and only if there is a cubic surface in $\mathbb{P}^3$ not containing C such that

$$H \cdot C = \sum_{i=1}^{4}(P_i + Q_i + R_i) + \sum_{j=1}^{6} X_j.$$

Remark 4.4. If one wants to construct explicitly the group law on B, it may be useful to choose a complementary subspace $V \subseteq H^0(\mathbb{P}^{g-1}, \mathcal{O}(3))$ of the kernel of $f^\star$, i.e., the restriction of $f^\star$ to V is an isomorphism onto $H^0(C, \Omega^{\otimes 3})$. Then, the statement of Corollary 4.1 can be sharpened as follows. For $P, Q, R \in B$, one has $P + Q + R = 0$ in B if and only if there is a cubic hypersurface H in the linear system $|V|$ such that the Equation (1) holds. Moreover, H is unique.

As a consequence, one can explicitly construct the group law on B. For $P \in B$, let $H \in |V|$ be the unique cubic hypersurface such that

$$H \cdot C \geq \sum_{i=1}^{g}(O_i + P_i) + \sum_{j=1}^{3g-6} X_j.$$

There are unique points $Q_i \in B_i$, for $i = 1, \ldots, g$ such that

$$H \cdot C = \sum_{i=1}^{g}(O_i + P_i + Q_i) + \sum_{j=1}^{3g-6} X_j.$$

Then, $-P = Q$ in B. For $P, Q \in B$, there are, similarly, a unique cubic hypersurface $H \in |V|$ and an element $R \in B$ such that

$$H \cdot C = \sum_{i=1}^{g}(P_i + Q_i + R_i) + \sum_{j=1}^{3g-6} X_j.$$

Then, $P + Q = -R$ in B.

It seems quite a miracle to me that the additive law $+$ on B defined as above gives rise to the structure of a commutative group on B.

5 Comparison with other descriptions

The current description of the neutral component of the Jacobian should be seen as lying between the one of [2] and the one of [5]. Let us compare the present description with the ones in [5, 2].

In [5], a real algebraic curve is embedded into $\mathbb{P}^{2g}$, where g is the genus of the curve. The group law of the neutral component of the Jacobian of the curve is then described by intersecting the curve with linear hypersurfaces of $\mathbb{P}^{2g}$. The disadvantage of this description is that it embeds the curve in a projective space of relatively high dimension. For example, it seems difficult to determine explicitly a system of generators of the ideal sheaf of the curve in $\mathbb{P}^{2g}$, and, therefore, it seems difficult to do explicit computations with that description of the Jacobian.

In [2], the curve is birationally embedded into the projective plane. The Jacobian is then described by intersecting the curve with plane curves of degree g. The advantage of this description, of course, is that the curve is embedded into a low-dimensional projective space. It certainly allows explicit computations on the Jacobian. However, if one is given a curve defined over a real number field, that description fails, in general, to give a description of the Mordell-Weil group of the curve.

The present description of the neutral component of the Jacobian of a canonical curve has the advantages of both descriptions above. Much is known about the ideal sheaf of a canonical curve. And, moreover, as in the case of genus 3 [6], if the curve is defined over a real number field, the present description allows to do computations in the Mordell-Weil group of the curve.

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Institut de Recherche Mathématique de Rennes
Université de Rennes 1
Campus de Beaulieu
35042 Rennes Cedex
France
E-mail: johannes.huisman@univ-rennes1.fr
Home page: http://name.math.univ-rennes1.fr/johannes.huisman/

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX