ALGEBRAIC MODULI OF REAL ELLIPTIC CURVES

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Abstract

We study algebraic moduli of real generalized elliptic curves. For this, one needs to study algebraic families of such curves. The most suitable class of parameter spaces seems to be the class of Nash manifolds. It turns out, however, that real generalized elliptic curves do not have fine Nash moduli. The somewhat more restricted moduli problem of so-called oriented real generalized elliptic curves does have fine Nash moduli. We prove this by explicitly constructing a universal family of oriented real generalized elliptic curves over a Nash manifold. It will follow that real generalized elliptic curves have coarse Nash moduli. In fact, the coarse moduli space is the Nash manifold $\mathbb{P}^1(\mathbb{R})$. As a consequence, every real generalized elliptic curve E has a real j-invariant $j_{\mathbb{R}}(E) \in \mathbb{R} \cup \{\infty\}$. Let E and F be real generalized elliptic curves. Then $j_{\mathbb{R}}(E) = j_{\mathbb{R}}(F)$ if and only if E and F are isomorphic as real curves. We also give an explicit formula for the real j-invariant of a real generalized elliptic curve defined by the Weierstrass equation $y^2 = x^3 + ax + b$.

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1 Introduction

Moduli of principally polarized real abelian varieties have been studied in several papers (e.g. [1, 2, 3]). The employed techniques were essentially analytic in nature, and gave rise to a semianalytic coarse moduli space of principally polarized real abelian varieties. This might be perceived as an unsatisfactory solution to the moduli problem. After all, there should exist *some* algebraic structure on that moduli space. Another disadvantage of the analytic approach is that it does not reveal anything about moduli of principally polarized abelian varieties over an arbitrary real closed field. The aim of this paper is to show how one could attack such moduli problems *algebraically*, at least for the case of 1-dimensional principally polarized real abelian varieties, i.e., real elliptic curves. The higher-dimensional case requires more complicated techniques and is therefore postponed.

It should also be said that all the results in this paper are stated for elliptic curves over the field of real numbers. Nevertheless, it is eminently clear that they hold over any real closed field.

In order to study moduli of a class of real algebraic varieties in an algebraic way, one has to study families of those varieties over algebraic parameter spaces. A good class of algebraic parameter spaces—at least for the class of real algebraic varieties that we are interested in—seems to be the class of Nash manifolds. Roughly speaking, a Nash manifold is a real analytic manifold where the transition maps are algebraic analytic maps. In fact, Nash manifolds were originally called real algebraic manifolds [4]. (We refer to [5] for the definition of Nash manifolds. By a Nash manifold we will mean C^ω-Nash manifold in the sens of loc. cit.).

We show that real elliptic curves have coarse Nash moduli. In fact, this coarse moduli space is the space $\mathbb{P}^1(\mathbb{R})\backslash\{0,\infty\}$. The whole space $\mathbb{P}^1(\mathbb{R})$ turns out to be the coarse moduli space of real generalized elliptic curves.

The paper is organized as follows. In Section 2 we recall some facts and definitions of the theory of elliptic curves. We show that the set of isomorphism classes of real generalized elliptic curves is in bijective correspondence with $\mathbb{P}^1(\mathbb{R})$. From that, one already gets a structure of a Nash manifold on the set of isomorphism classes of real generalized elliptic curves. We show in Section 6 that, in fact, this structure is natural. We make precise what that means in Section 4. For this, we need to introduce the notion of families of real generalized elliptic curves over Nash manifolds. These are defined in Section 3. We also need some auxilary moduli problems, those of linearized and oriented generalized real elliptic curves which are studied in Section 5. In Section 7, we give an explicit formula for the real j-invariant of a real generalized elliptic curve given by a Weierstrass equation.

2 Real elliptic curves

Let K be a field with $\mathrm{char}(K) \neq 2, 3$. An *elliptic curve over* K is an absolutely integral nonsingular complete algebraic curve over K of genus 1 endowed with a K-rational point.

Example 2.1. For any $a, b \in K$, let $\mathcal{W}_{(a,b)}$ be the projective curve over K in $\mathbb{P}^2_K$ defined by the Weierstrass equation equation

$$y^2 = x^3 + ax + b.$$

Then, the pointed curve $(\mathcal{W}_{(a,b)}, (0:1:0))$ is an elliptic curve over K if and only if its discriminant $\Delta = 4a^3 + 27b^2$ is nonzero.

A *generalized elliptic curve over* K is an absolutely integral complete algebraic curve over K of arithmetic genus 1 endowed with a nonsingular K-rational point and having at most one ordinary double point as singularities.

Example 2.2. The pointed projective curve $\mathcal{W}_{(a,b)}$ defined in Example 2.1, is a generalized elliptic curve over K for all $(a, b) \in K^{2\star} = K^2 \backslash \{(0, 0)\}$. The curve $\mathcal{W}_{(a,b)}$ is nonsingular if $\Delta \neq 0$. It is singular if $\Delta = 0$.

Proposition 2.3. *Let E be a generalized elliptic curve over K. Then, there is $(a, b) \in K^{2\star}$ such that $\mathcal{W}_{(a,b)}$ is isomorphic to E. If $\mathcal{W}_{(c,d)}$ is also isomorphic to E, for some $(c, d) \in K^{2\star}$, then there is a $\lambda \in K^\star$ such that $(c, d) = (\lambda^4 a, \lambda^6 b)$.* □

We will also need the notion of linearized generalized elliptic curves over K. Let E be a generalized elliptic curve over K. Denote its relative dualizing sheaf by ω. A *linearization of* E is a nonvanishing global section η of ω. A *linearized generalized elliptic curve* over K is a pair (E, η) consisting of a generalized elliptic curve E over K and a linearization η of E. Let (E, η) and (E', η') be linearized generalized elliptic curves over K. An isomorphism $\varphi\colon E \to E'$ is an isomorphism of linearized generalized elliptic curves if $\varphi^\star \eta' = \eta$.

Example 2.4. The differential form dx/y defines a nonvanishing global section of the relative dualizing sheaf ω on $\mathcal{W}_{(a,b)}$ for all $(a,b) \in K^{2\star}$. Hence dx/y defines a linearization of $\mathcal{W}_{(a,b)}$.

Proposition 2.5. *If (E, η) is a linearized generalized elliptic curve over K then there is a unique $(a,b) \in K^{2\star}$ such that $(\mathcal{W}_{(a,b)}, dx/y)$ is isomorphic to (E, η). Moreover, the isomorphism between $(\mathcal{W}_{(a,b)}, dx/y)$ and (E, η) is unique.* □

Define an action of $K^\star$ on $K^{2\star}$ by

$$\lambda \cdot (a,b) = (\lambda^4 a, \lambda^6 b).$$

According to Proposition 2.3, the isomorphism classes of generalized elliptic curves over K are parametrized by the quotient of $K^{2\star}$ by the group $K^\star$. In the case K is the field $\mathbb{R}$ of real numbers, we will show that this quotient exists in the category of Nash manifolds. In fact, we will show that $\mathbb{R}^\star \backslash \mathbb{R}^{2\star}$ is isomorphic to $\mathbb{P}^1(\mathbb{R})$ as a Nash manifold (Proposition 2.6).

First, we establish a morphism of Nash manifolds π from $\mathbb{R}^{2\star}$ into $\mathbb{P}^1(\mathbb{R})$ which is constant on $\mathbb{R}^\star$-orbits. Let (a,b) be an element of $\mathbb{R}^{2\star}$. For $\lambda \in \mathbb{R}^\star$ one has $\lambda \cdot (a,b) \in S^1$ if and only if

$$b^2 \lambda^{12} + a^2 \lambda^8 - 1 = 0. \tag{1}$$

Observe that this equation has a unique root $\rho = \rho(a,b)$ in $\mathbb{R}^+$. It can be easily checked that

$$\rho\colon \mathbb{R}^{2\star} \longrightarrow \mathbb{R}^+$$

is a Nash function. Indeed, let N be the Nash submanifold of $\mathbb{R}^+ \times \mathbb{R}^{2\star}$ defined by Equation (1). Consider the projection pr_2 from N onto $\mathbb{R}^{2\star}$. Since $\rho \circ pr_2(\lambda, a, b) = \lambda$ for all $(\lambda, a, b) \in N$, $\rho \circ pr_2$ is a Nash function on N. Moreover, pr_2 is étale and bijective, i.e., pr_2 is an isomorphism of Nash manifolds. Therefore $\rho = (\rho \circ pr_2) \circ pr_2^{-1}$ is a Nash function on $\mathbb{R}^{2\star}$. (In fact, we will establish in Section 7 an explicit expression for ρ.)

Define a morphism of Nash manifolds

$$\pi \colon \mathbb{R}^{2\star} \longrightarrow \mathbb{P}^1(\mathbb{R})$$

by

$$\pi(a, b) = \frac{\rho^4 a}{1 + \rho^6 b},$$

where $\rho = \rho(a, b)$. Then, π is constant on $\mathbb{R}^\star$-orbits since $\rho(\lambda \cdot (a, b)) = |\lambda|^{-1}\rho(a, b)$ for $(a, b) \in \mathbb{R}^{2\star}$ and $\lambda \in \mathbb{R}^\star$.

Proposition 2.6. *The morphism π from $\mathbb{R}^{2\star}$ onto $\mathbb{P}^1(\mathbb{R})$ is the quotient of $\mathbb{R}^{2\star}$ by the action of $\mathbb{R}^\star$ in the category of Nash manifolds.*

Proof. It suffices to observe that π is constant on $\mathbb{R}^\star$-orbits and that the inclusion of the unit circle S^1 into $\mathbb{R}^{2\star}$ is a section of π. □

3 Generalized elliptic curves over Nash manifolds

Let A be a commutative unitary ring. Let E be a proper flat scheme over A of finite presentation. Let $O \colon \mathrm{Spec}(A) \to E$ be an A-rational point of E. The scheme E over A is a *generalized elliptic curve over A* if for all geometric points $\mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(A)$ the fiber $(E_{\bar{K}}, O_{\bar{K}})$ is a generalized elliptic curve over the field $\bar{K}$.

Let $(X, \mathcal{N})$ be a Nash manifold. A *generalized elliptic curve over X* is a triple $(\mathcal{E}, \mathcal{U}, \varphi)$, often shortly denoted by $\mathcal{E}$ itself, satisfying the following conditions.

1. The set $\mathcal{U}$ is a finite covering of X by open semialgebraic subsets.

2. For every $U \in \mathcal{U}$, one has a generalized elliptic curve $\mathcal{E}(U)$ over the ring $\mathcal{N}(U)$ of Nash functions on U.

3. For all $U, V \in \mathcal{U}$, one has isomorphisms

$$\varphi_{V,U} \colon \mathcal{E}(U) \otimes_{\mathcal{N}(U)} \mathcal{N}(U \cap V) \longrightarrow \mathcal{E}(V) \otimes_{\mathcal{N}(V)} \mathcal{N}(U \cap V)$$

   of generalized elliptic curves over the ring $\mathcal{N}(U \cap V)$. These isomorphisms are supposed to be such that $\varphi_{W,V} \circ \varphi_{V,U} = \varphi_{W,U}$ over $U \cap V \cap W$, for all $U, V, W \in \mathcal{U}$.

One thinks of a generalized elliptic curve over a Nash manifold X as a family of real generalized elliptic curves, varying Nashly. This is justified by the following. If $\mathcal{E}$ is a generalized elliptic curve over X then for every $p \in X$ the fiber $\mathcal{E}_p$ is a real generalized elliptic curve, defined up to isomorphism. Indeed, choose $U \in \mathcal{U}$ such that $p \in U$. Then $\mathcal{E}_p = \mathcal{E}(U) \otimes_{\mathcal{N}(U)} k(p)$ is a generalized elliptic curve over the residue field $k(p)$ at p. In fact, $k(p) \cong \mathbb{R}$, canonically. Therefore, $\mathcal{E}_p$ is a real generalized elliptic curve. Clearly, the isomorphism class of $\mathcal{E}_p$ does not depend on the choice of U.

The following two examples of generalized elliptic curves over Nash manifolds will be of importance throughout the rest of the paper. They are both defined by global equations, i.e., of the form $(E, \{X\}, \mathrm{id}_E)$, where X is a Nash manifold and E is a generalized elliptic curve over the ring $\mathcal{N}(X)$ of global Nash functions on X.

Example 3.1. Consider the Nash manifold $\mathbb{R}^{2\star} = \mathbb{R}^2 \backslash \{(0,0)\}$. Let (a, b) be the canonical coordinates on $\mathbb{R}^{2\star}$. Then, the Weierstrass equation

$$y^2 = x^3 + ax + b.$$

defines a generalized elliptic curve $\mathcal{W}$ over $\mathbb{R}^{2\star}$. Its fibers $\mathcal{W}_{(a,b)}$ are the generalized elliptic curves of Example 2.2.

Example 3.2. Let $S^1 \subseteq \mathbb{R}^{2\star}$ be the unit circle. Then, the restriction $\mathcal{W}^1$ of $\mathcal{W}$ to the Nash manifold S^1 is a generalized elliptic curve over the Nash manifold S^1 (cf. Figure 1).

Let $\mathcal{E}$ be a generalized elliptic curve over a Nash manifold X. This will be formally denoted as if one had a morphism from $\mathcal{E}$ into X, e.g. $\pi\colon \mathcal{E} \to X$. The system O of specified $\mathcal{N}(U)$-rational points O_U of $\mathcal{E}(U)$, where U runs through the covering with respect to which $\mathcal{E}$ is defined, will be formally denoted as if it were a section of π, e.g. $O\colon N \to \mathcal{E}$. Furthermore, it is clear what should be understood by a sheaf on $\mathcal{E}$. Canonical constructions of sheaves on generalized elliptic curves over rings extend automatically to generalized elliptic curves over Nash manifolds. In particular, one has the sheaf $\mathcal{I}$ on $\mathcal{E}$ of vanishing ideals at O, and the relative dualizing sheaf $\omega = \omega_{\mathcal{E}/X}$ of $\mathcal{E}$ over X. Moreover, it is clear what should be understood by $\pi_\star\mathcal{F}$ for a sheaf $\mathcal{F}$ on $\mathcal{E}$.

A *linearized generalized elliptic curve* over a Nash manifold X is a pair $(\mathcal{E}, \eta)$ consisting of a generalized elliptic curve $\pi\colon \mathcal{E} \to X$ over X, and a nonvanishing global section η of $\pi_\star\omega$. Denote by $\omega^\star$ the subsheaf of ω of nonvanishing sections. Observe that the sheaf $\mathcal{N}^+$ of strictly positive Nash functions acts on $\pi_\star(\omega^\star)$. An *oriented generalized elliptic curve* over a Nash manifold X is a pair $(\mathcal{E}, \mu)$ consisting of a generalized elliptic curve $\pi\colon \mathcal{E} \to X$ over X, and a global section μ of the quotient sheaf $\pi_\star(\omega^\star)/\mathcal{N}^+$. Of course, a linearized generalized elliptic curve $(\mathcal{E}, \eta)$ gives canonically an oriented generalized elliptic curve $(\mathcal{E}, \bar{\eta})$ by considering the image $\bar{\eta}$ of η in the quotient sheaf $\pi_\star(\omega^\star)/\mathcal{N}^+$.

Example 3.3. Consider the generalized elliptic curve $\mathcal{W}$ over $\mathbb{R}^{2\star}$ of Example 3.1. Let ω be its relative dualizing sheaf. Observe that dx/y is a nonvanishing global section of $\pi_\star\omega$. Hence, $(\mathcal{W}, dx/y)$ is a linearized generalized elliptic curve over $\mathbb{R}^{2\star}$. This linearization is understood when $\mathcal{W} \to \mathbb{R}^{2\star}$ is considered as a linearized generalized elliptic curve. Its restriction $\mathcal{W}^1$ to S^1 is then also a linearized generalized elliptic curve. However, we will rather consider the induced structure of an oriented generalized elliptic curve over S^1 on $\mathcal{W}^1$.

Next, we need to make precise the notion of a morphism of generalized elliptic curves over a Nash manifold. Let $(\mathcal{E}, \mathcal{U}, \varphi)$ be a generalized elliptic

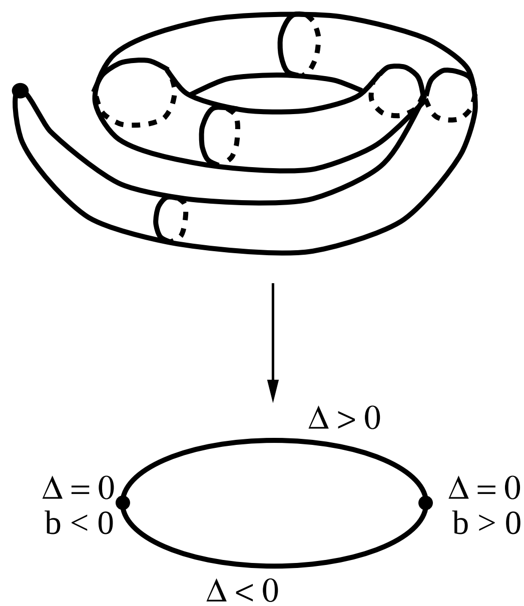

FIG. 1: The generalized elliptic curve $\mathcal{W}^1$ over the Nash manifold S^1. Of course, only the real points of $\mathcal{W}^1$ are depicted. The set of real points of the fiber $\mathcal{W}^1_{(a,b)}$ is connected if $\Delta > 0$, and has 2 connected components if $\Delta < 0$. The fiber $\mathcal{W}^1_{(a,b)}$ is singular, if $\Delta = 0$. In that case, this fiber is isomorphic to the curve $y^2 = x^2(x+1)$ if $b > 0$, and to the curve $y^2 = x^2(x-1)$ if $b < 0$. Although the figure might suggest the contrary, the total space of $\mathcal{W}^1$ is a *nonsingular* real scheme.

curve over the Nash manifold $(X,\mathcal{N})$. Let $\mathcal{V}$ be a finite covering of X by open semialgebraic subsets. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then one has an induced generalized elliptic curve $(\mathcal{F},\mathcal{V},\psi)$. Indeed, choose $i\colon \mathcal{V}\to\mathcal{U}$ such that $V\subseteq iV$ for every $V\in\mathcal{V}$, and let

$$\mathcal{F}(V)=\mathcal{E}(iV)\otimes_{\mathcal{N}(iV)}\mathcal{N}(V)$$

and

$$\psi_{V',V}=\varphi_{iV',iV}\otimes_{\mathcal{N}(iV\cap iV')}\mathcal{N}(V\cap V').$$

Then, obviously, $(\mathcal{F},\mathcal{V},\psi)$ is a generalized elliptic curve over X. This induced generalized elliptic curve will be denoted by $\mathcal{E}_{|\mathcal{V}}$. (In fact, it would have been more correct to denote it by $\mathcal{E}_{|i}$.)

Let $(\mathcal{E},\mathcal{U},\varphi)$ and $(\mathcal{F},\mathcal{U},\psi)$ be arbitrary generalized elliptic curves over X given on the same covering $\mathcal{U}$ of X. A *strict morphism* f from $\mathcal{E}$ into $\mathcal{F}$ is a system of morphisms of generalized elliptic curves $f_U\colon \mathcal{E}(U)\longrightarrow\mathcal{F}(U)$ over $\mathcal{N}(U)$ such that the diagram

$$\begin{array}{ccc}
\mathcal{E}(U)\otimes_{\mathcal{N}(U)}\mathcal{N}(U\cap V) & \xrightarrow{\varphi_{V,U}} & \mathcal{E}(V)\otimes_{\mathcal{N}(V)}\mathcal{N}(U\cap V)\\
\downarrow{\scriptstyle f_U\otimes 1} & & \downarrow{\scriptstyle f_V\otimes 1}\\
\mathcal{F}(U)\otimes_{\mathcal{N}(U)}\mathcal{N}(U\cap V) & \xrightarrow{\psi_{V,U}} & \mathcal{F}(V)\otimes_{\mathcal{N}(V)}\mathcal{N}(U\cap V)
\end{array}$$

commutes for every $U,V\in\mathcal{U}$. If $(\mathcal{E},\mathcal{U},\varphi)$ and $(\mathcal{F},\mathcal{V},\psi)$ are generalized elliptic curves over X then a *morphism* f from $\mathcal{E}$ into $\mathcal{F}$ is a strict morphism from $\mathcal{E}_{|\mathcal{W}}$ into $\mathcal{F}_{|\mathcal{W}}$ for some refinement $\mathcal{W}$ of both $\mathcal{U}$ and $\mathcal{V}$. Of course, the covering $\mathcal{W}$ is also supposed to be finite and to consist of open semialgebraic subsets of X. It should then be clear what it means to be isomorphic for linearized or oriented generalized elliptic curves over a Nash manifold.

If $f\colon Y\to X$ is a morphism of Nash manifolds and $\mathcal{E}$ is a generalized elliptic curve over X, then one clearly has an induced generalized elliptic curve $f^\star\mathcal{E}$ over Y. Of course, analogous statements hold for linearized and oriented generalized elliptic curves as well.

4 The moduli problems

A few words on moduli problems over non-algebraically closed fields is in order. In fact, we give a general definition of fine and coarse moduli spaces—or rather, moduli objects—for moduli problems on any category having a final object.

Let C be any category. Recall that the Yoneda functor h_M is defined by $h_M(X) = \mathrm{Hom}(X, M)$ and $h_M(f) = \mathrm{Hom}(f, M)$, for a C-object X and a C-morphism f. The map $h_M(f)$ is also denoted by $f^\star$. If $f\colon M \to M'$ is a C-morphism, then one has a natural transformation h_f of h_M into $h_{M'}$ defined by $h_f = \mathrm{Hom}(\cdot, f)$. Recall also that the Yoneda Lemma states that it is equivalent to give a natural transformation from h_M into a contravariant set-valued functor $\mathcal{M}$, or to give an element of $\mathcal{M}(M)$.

A *moduli problem* $\mathcal{M}$ on C is a contravariant functor $\mathcal{M}$ from the category C into the category of sets. A moduli problem $\mathcal{M}$ on C has *fine moduli* if the functor $\mathcal{M}$ is representable, i.e., if there is a C-object M such that the functor h_M is isomorphic to $\mathcal{M}$. A C-object M satisfying the fore-mentioned condition is called a *fine moduli space* for C. By the Yoneda Lemma, this is equivalent to the existence of an element ξ in $\mathcal{M}(M)$ such that for all C-objects X and for all elements $\zeta \in \mathcal{M}(X)$, there is a unique morphism $f\colon X \to M$ such that $f^\star\xi = \zeta$. The element ξ of $\mathcal{M}(M)$ is then called *universal.*

Example 4.1. Consider the following moduli problems on the category Nash of Nash manifolds. Define the functors $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$ on an object X of the category Nash by

$$\mathcal{M}(X) = \{\text{generalized elliptic curves over } X\}/\cong$$
$$\mathcal{M}'(X) = \{\text{oriented generalized elliptic curves over } X\}/\cong$$
$$\mathcal{M}''(X) = \{\text{linearized generalized elliptic curves over } X\}/\cong .$$

For a morphism $f\colon X \to Y$ in the category Nash, the maps $\mathcal{M}(f)$, $\mathcal{M}'(f)$ and $\mathcal{M}''(f)$ are simply the pull-back maps. Two of the main results of the paper are that the moduli problems $\mathcal{M}'$ and $\mathcal{M}''$ have fine moduli (Theorems 5.1 and 5.2).

Let C be a category with a final object denoted by $\star$. Let $\mathcal{M}$ be a moduli problem on C. One should think about the set $\mathcal{M}(\star)$ as the set of isomorphism classes of objects one wants to classify. The goal of a moduli problem is to give this set a structure of an object of C. For example, if M is a fine moduli space for $\mathcal{M}$, one has a bijection between $\mathcal{M}(\star)$ and the set of points of M, i.e., the set $h_M(\star) = \mathrm{Hom}(\star, M)$. Then, for any reasonable category C, the set $\mathcal{M}(\star)$ acquires a structure of a C-object by transport of structure.

Unfortunately, fine moduli spaces seem to be rather rare. Therefore, one introduces the notion of a coarse moduli space: The moduli problem $\mathcal{M}$ is said to have *coarse moduli* if there is a C-object M, together with a morphism of functors φ from $\mathcal{M}$ into the Yoneda functor h_M of M, satisfying the following conditions.

1. On the final object $\star$ of C, the morphism φ induces a bijection $\varphi_\star$ from $\mathcal{M}(\star)$ onto $h_M(\star)$.

2. For every C-object M' and for every morphism of functors $\psi\colon \mathcal{M} \to h_{M'}$ such that $\psi_\star$ is a bijection, there is a unique C-morphism $f\colon M \to M'$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\varphi} & h_M \\ & \overset{\psi}{\searrow} & \downarrow{\scriptstyle h_f} \\ & & h_{M'} \end{array}$$

The C-object M, or more precisely the pair (M, φ), is then called a *coarse moduli space* for $\mathcal{M}$. If it exists, a coarse moduli space is obviously unique up to unique isomorphism.

For "reasonably nice" moduli problems over, e.g., the category of schemes over an algebraically closed field, this definition of a coarse moduli space is equivalent to the usual definition [6], Definition 5.6. The advantage of our definition is that it applies to the category of Nash manifolds.

Example 4.2. One of our main results will be that the moduli problem $\mathcal{M}$ of generalized elliptic curves on the category Nash of Nash manifolds has coarse moduli (Theorem 6.4, see also [7] for another example of coarse moduli in real algebraic geometry). This means that, there is a structure of a Nash manifold on the set M of isomorphism classes of real generalized elliptic curves having the following property. Let X be any Nash manifold and let $\mathcal{E}$ be any generalized elliptic curve over X. Let f be the map from X into M defined by letting $f(p)$ be the isomorphism class of the fiber $\mathcal{E}_p$. Then, f is a morphism of Nash manifolds. Moreover, if M' is any other structure of a Nash manifold on the set M of isomorphism classes of real generalized elliptic curves satisfying this condition, then the identity map $\mathrm{id}\colon M \to M'$ is a morphism of Nash manifolds.

5 Universal properties of $\mathcal{W}$ and $\mathcal{W}^1$

Theorem 5.1. *The linearized generalized elliptic curve $\mathcal{W}$ over $\mathbb{R}^{2\star}$ is the universal linearized generalized elliptic curve. That is, for every Nash manifold X and for every linearized generalized elliptic curve $\pi\colon \mathcal{E} \to X$ over X there is a unique morphism $f\colon X \to \mathbb{R}^{2\star}$ such that*

$$f^\star\mathcal{W} \cong \mathcal{E}$$

as linearized generalized elliptic curves over X. In particular, the moduli problem $\mathcal{M}''$ of linearized generalized elliptic curves over the category Nash of Nash manifolds has fine moduli.

Proof. The uniqueness of f follows from Proposition 2.5. That proposition also implies that the identity is the only automorphism of a linearized generalized elliptic curve over a Nash manifold. Hence, in order to prove existence of f it suffices to show existence of f locally.

Let $O\colon X \to \mathcal{E}$ be the specified section of π. Denote by $\mathcal{I}$ the sheaf on $\mathcal{E}$ of vanishing ideals at O. Since, $\pi_\star \mathcal{I}^{-2}$ and $\pi_\star \mathcal{I}^{-3}$ are locally free sheaves of $\mathcal{N}$-modules on X, we may suppose that they are free over X.

Now, choose $x \in \Gamma(X, \pi_\star \mathcal{I}^{-2})$ and $y \in \Gamma(X, \pi_\star \mathcal{I}^{-3})$ such that $\{1, x, y\}$ is a basis of the $\Gamma(X, \mathcal{N})$-module $\Gamma(X, \pi_\star \mathcal{I}^{-3})$. By Riemann-Roch, there are $a_i \in \Gamma(X, \mathcal{N})$ such that

$$y^2 = a_0 x^3 + a_1 xy + a_2 x^2 + a_3 y + a_4 x + a_6.$$

Using the fact that $a_0 \neq 0$ on X, one can change coordinates such that $a_0 = 1$ and $a_1 = a_2 = a_3 = 0$. Then $g\colon X \to \mathbb{R}^{2\star}$ defined by $g(p) = (a_4(p), a_6(p))$ is a morphism of Nash manifolds. Moreover, we have an obvious isomorphism

$$\psi\colon \mathcal{E} \longrightarrow g^\star \mathcal{W}$$

of generalized elliptic curves over X.

Let η be the section of $\pi_\star \omega$ that linearizes $\mathcal{E}$. Since η is nonvanishing, there is a Nash function $\lambda\colon X \to \mathbb{R}^\star$ such that

$$\psi^\star \frac{dx}{y} = \lambda \eta.$$

Since dx/y is nonvanishing too, λ is nonvanishing. Then, we can let $f\colon X \to \mathbb{R}^{2\star}$ be the Nash morphism defined by

$$f = \lambda \cdot g = (\lambda^4 a_4, \lambda^6 a_6).$$

Define an isomorphism $\alpha\colon g^\star \mathcal{W} \to f^\star \mathcal{W}$ by defining α on each fiber $(g^\star \mathcal{W})_p = \mathcal{W}_{g(p)}$ by

$$\alpha_p(x, y) = (\lambda^2 x, \lambda^3 y).$$

and put $\varphi = \alpha \circ \psi$. Then φ is an isomorphism from $\mathcal{E}$ into $f^\star \mathcal{W}$ and, moreover,

$$\varphi^\star \frac{dx}{y} = \psi^\star \alpha^\star \frac{dx}{y} = \frac{1}{\lambda} \psi^\star \frac{dx}{y} = \eta.$$

Hence φ is an isomorphism of linearized generalized elliptic curves. $\square$

In order to prove that the moduli problem $\mathcal{M}'$ of oriented generalized elliptic curves has fine moduli too, it will be convenient to introduce a Grothendieck topology [8] on the category Nash of Nash manifolds. For a Nash manifold U, a *covering* of U is a collection $\{f_i\colon U_i \to U\}_{i\in I}$ of open embeddings of Nash manifolds, such that there is a finite subset I' of I with

$$U = \bigcup_{i\in I'} f_i(U_i).$$

One easily verifies that this indeed defines a Grothendieck topology on the category Nash. Of course, this Grothendieck topology is coarser than the canonical topology, so that all representable functors on Nash are sheaves.

Theorem 5.2. *The oriented generalized elliptic curve $\mathcal{W}^1$ over the Nash manifold S^1 is the universal oriented generalized elliptic curve. That is, for every Nash manifold X and for every oriented generalized elliptic curve $\mathcal{E}$ over X there is a unique morphism of Nash manifolds f from X into S^1 such that*

$$f^\star\mathcal{W}^1 \cong \mathcal{E}$$

as oriented generalized elliptic curves over X. In particular, the moduli problem $\mathcal{M}'$ of oriented generalized elliptic curves over the category Nash *of Nash manifolds has fine moduli.*

Proof. Let φ be the morphism of functors

$$\varphi\colon h_{\mathbb{R}^{2\star}} \longrightarrow \mathcal{M}''$$

defined by $\varphi_X(f) = f^\star\mathcal{W}$, where X is a Nash manifold and $f\colon X \to \mathbb{R}^{2\star}$ a morphism of Nash manifolds. By Theorem 5.1, φ is an isomorphism. The action of $\mathbb{R}^\star$ on $\mathbb{R}^{2\star}$ gives rise to an action of the functor $h_{\mathbb{R}^\star}$ of $\mathbb{R}^\star$ on $h_{\mathbb{R}^{2\star}}$. Via the isomorphism φ, we get an action of $h_{\mathbb{R}^\star}$ on $\mathcal{M}''$. Then, φ induces an isomorphism of sheaves on the site Nash

$$\bar{\varphi}\colon h_{\mathbb{R}^\star}\backslash h_{\mathbb{R}^{2\star}} \longrightarrow h_{\mathbb{R}^\star}\backslash\mathcal{M}''.$$

Here, the quotients are quotients in the category of sheaves on the site Nash. By Proposition 2.6, one has isomorphisms

$$h_{\mathbb{R}^\star}\backslash h_{\mathbb{R}^{2\star}} \cong h_{\mathbb{R}^\star\backslash\mathbb{R}^{2\star}} \cong h_{S^1}.$$

Moreover,

$$h_{\mathbb{R}^\star}\backslash\mathcal{M}'' \cong \mathcal{M}',$$

by definition of the moduli problem $\mathcal{M}'$ of oriented generalized elliptic curves. Hence, $\bar{\varphi}$ gives rise to an isomorphism

$$\psi\colon h_{S^1} \longrightarrow \mathcal{M}'.$$

In fact, $\psi_X(f) = f^\star\mathcal{W}^1$ for a Nash manifold X and a morphism $f\colon X \to S^1$. This proves that the oriented generalized elliptic curve $\mathcal{W}^1$ over S^1 is the universal oriented generalized elliptic curve. □

Remark 5.3. Theorem 5.2 implies that every oriented generalized elliptic curve $\mathcal{E}$ over any Nash manifold X can be defined by *global* equations. That is to say, there are global Nash functions f_1 and f_2 on X such that $\mathcal{E}$ is isomorphic to the oriented generalized elliptic curve $\mathcal{F}$ in the projective plane $\mathbb{P}^2_X$ over X, defined by the equation $y^2 = x^3 + f_1x + f_2$. Indeed, f_1 and f_2 are the coordinate functions of the morphism $f\colon X \to S^1$ whose existence is asserted in Theorem 5.2, and $\mathcal{F}$ is simply $f^\star\mathcal{W}^1$.

6 Coarse moduli of real generalized elliptic curves

Before we prove our main result on the moduli problem $\mathcal{M}$ of generalized elliptic curves on the category Nash of Nash manifolds, we need some preparation. Let $\mathcal{E}$ be a generalized elliptic curve over a Nash manifold X. Then, the sheaf of automorphisms $\mathrm{Aut}(\mathcal{E}/X)$ of $\mathcal{E}$ over X is canonically isomorphic to the constant sheaf $\mathbb{Z}/2\mathbb{Z}$ over X. In particular, one can twist a generalized elliptic curve over X by an element of the first cohomology group $H^1(X, \mathbb{Z}/2\mathbb{Z})$. Thus, we have an action of $H^1(X, \mathbb{Z}/2\mathbb{Z})$ on the set $\mathcal{M}(X)$. This action is easily seen to be fixed point-free.

The best one can prove for the moduli problem $\mathcal{M}$ of generalized elliptic curves is the following result.

Theorem 6.1. *Let X be a Nash manifold and let $\mathcal{E}$ be any generalized elliptic curve over X. Then there is a unique morphism f from X into S^1 and a unique cohomology class $\gamma \in H^1(X, \mathbb{Z}/2\mathbb{Z})$ such that*

$$\gamma \cdot f^\star \mathcal{W}^1 \cong \mathcal{E}.$$

In particular, one has an isomorphism of functors

$$\mathcal{M} \cong H^1(\cdot, \mathbb{Z}/2\mathbb{Z}) \times \mathcal{M}'.$$

Proof. There is a unique cohomology class $\gamma \in H^1(X, \mathbb{Z}/2\mathbb{Z})$ such that $\gamma^{-1} \cdot \mathcal{E}$ is orientable. Then, by Theorem 5.2, there is a unique morphism $f \colon X \to S^1$ such that $f^\star \mathcal{W}^1 \cong \gamma^{-1} \cdot \mathcal{E}$. □

Remark 6.2. It follows from Theorem 6.1 that the moduli problem $\mathcal{M}$ on Nash has no fine moduli since the presheaf $\mathcal{M}$ on the site Nash, being isomorphic to $H^1(\cdot, \mathbb{Z}/2\mathbb{Z}) \times \mathcal{M}'$, is not a sheaf.

Remark 6.3. Theorem 6.1 and Remark 5.3 imply that every generalized elliptic curve $\mathcal{E}$ over any Nash manifold X is a $H^1(X, \mathbb{Z}/2\mathbb{Z})$-twist of a generalized elliptic curve over X defined by global equations.

Since every generalized elliptic curve over a Nash manifold is locally orientable, Theorem 5.2 implies the existence of a morphism of functors

$$\varphi \colon \mathcal{M} \longrightarrow h_{S^1}.$$

In fact, if X is a Nash manifold and $\mathcal{E}$ is a generalized elliptic curve over X then $\varphi_X(\mathcal{E})$ is the morphism of Nash manifolds from X into S^1 determined by $\varphi_X(\mathcal{E})(p) = (a, b)$, where (a, b) is the unique point of S^1 such that $\mathcal{W}_{(a,b)}$ is isomorphic to the fiber $\mathcal{E}_p$ of $\mathcal{E}$ at p.

Theorem 6.4. *The Nash manifold S^1, or more precisely the pair (S^1, φ), is a coarse moduli space for real generalized elliptic curves. In particular, the moduli problem $\mathcal{M}$ has coarse Nash moduli.*

Proof. The only thing to prove is that any natural transformation $\psi\colon \mathcal{M} \to h_M$, for some Nash manifold M, factorizes through φ. Since h_M is a sheaf on the site Nash, ψ factorizes through the sheafification of $\mathcal{M}$. But this sheafification is nothing but the transformation φ, by Theorems 6.1 and 5.2. $\square$

Corollary 6.5. *The coarse moduli space of real elliptic curves is the Nash manifold $S^1 \backslash \{\Delta = 0\}$.*

Remark 6.6. Although the coarse moduli space of elliptic curves happens to be a real algebraic variety (in the sens of [9]), real elliptic curves do not have real algebraic coarse moduli. Indeed, suppose M is a structure of a real algebraic variety on the set of isomorphism classes of real elliptic curves having the following property. For any real algebraic variety X and for all elliptic curves $\mathcal{E}$ over X, let f be the map from X into M defined by letting $f(p)$ be the isomorphism class of the fiber $\mathcal{E}_p$. Then, f is a morphism of real algebraic varieties.

We show that such a structure M does not exists by taking $X = \mathbb{R}^{2\star}$ and $\mathcal{E} = \mathcal{W}$. Then, according to Proposition 2.3, the map $f\colon X \to M$ factorizes through the quotient of $\mathbb{R}^{2\star}$ by $\mathbb{R}^\star$ *in the category of real algebraic varieties.* This quotient is easily seen to be the map

$$q\colon \mathbb{R}^{2\star} \longrightarrow \mathbb{P}^1(\mathbb{R})$$

defined by $q(a,b) = (a^3 : b^2)$. Then, $q(0,1) = q(0,-1)$, so that $f(0,1) = f(0,-1)$, i.e., the real elliptic curves $\mathcal{W}_{(0,1)}$ and $\mathcal{W}_{(0,-1)}$ are isomorphic. Contradiction, by Proposition 2.3. (In fact, the map q is, up to a linear transformation, the complex j-invariant of the curve $\mathcal{W}_{(a,b)}$, cf. Section 7).

Remark 6.7. A particular consequence of Corollary 6.5 is that the set of isomorphism classes of true real elliptic curves has a natural structure of a semialgebraic space (cf. [10] for the notion of semialgebraic spaces). Indeed, $\varphi_\star$ is a bijection from the set $\mathcal{M}(\star)$ of isomorphism classes of real generalized elliptic curves onto S^1. Then, the restriction to $S^1\backslash\{\Delta = 0\}$ of $\varphi_\star^{-1}$ is a bijection

$$w\colon S^1\backslash\{\Delta = 0\} \longrightarrow \mathrm{Ell}$$

onto the set Ell of isomorphism classes of real elliptic curves. Since w is a bijection, the set Ell gets a structure of a semialgebraic space by transport of structure. This structure on Ell is natural in the sense that it comes from a moduli problem.

The set Ell of isomorphism classes of real elliptic curves was already given a structure of a semialgebraic space [2], Theorem 6.1. Strictly speaking, that structure is just a particular semialgebraic structure. It is not *the* semialgebraic structure on Ell coming from moduli, i.e., the one induced by w. To see this, we need to recall its construction. Let $\mathcal{A}^1_{\mathbb{R}}$ be the semialgebraic subset of the upper half plane $\mathbb{H}$, consisting of those complex numbers whose real part is equal to 0 or to $\frac{1}{2}$. Let

$$g\colon \mathcal{A}^1_{\mathbb{R}} \longrightarrow \mathrm{Ell}$$

be the map that associates to $\tau \in \mathcal{A}^1_{\mathbb{R}}$ the isomorphism class of the real elliptic curve $y^2 = 4x^3 - g_2x - g_3$, where

$$g_2 = 60 \cdot \sum_{(m,n)\in\mathbb{Z}^{2\star}} \frac{1}{(m+n\tau)^4} \quad \text{and} \quad g_3 = 140 \cdot \sum_{(m,n)\in\mathbb{Z}^{2\star}} \frac{1}{(m+n\tau)^6}.$$

Then g is a bijection, hence Ell gets a structure of a semialgebraic space by transport of structure via g.

Now, I claim that the semialgebraic structure on the set Ell obtained via g does not coincide with the semialgebraic structure on Ell induced by w. Indeed, the map

$$w^{-1} \circ g\colon \mathcal{A}^1_{\mathbb{R}} \longrightarrow S^1\backslash\{\Delta = 0\}$$

is not a semialgebraic map. This can be seen as follows. Let

$$k\colon S^1\backslash\{\Delta = 0\} \to \mathbb{R}$$

be the semialgebraic map defined by $k(a,b) = 2^8 3^3 a^3/\Delta$. If $w^{-1} \circ g$ had been semialgebraic then $k \circ w^{-1} \circ g$ would have been semialgebraic too. But $k \circ w^{-1} \circ g\colon \mathcal{A}^1_{\mathbb{R}} \to \mathbb{R}$ is nothing but the restriction to $\mathcal{A}^1_{\mathbb{R}}$ of the modular j-function $j\colon \mathbb{H} \to \mathbb{C}$. Therefore, the map $k \circ w^{-1} \circ g$ cannot be semialgebraic. Hence, $w^{-1} \circ g$ is not semialgebraic.

7 The real j-invariant

One can reformulate Theorem 6.4 as follows. Let $\sigma\colon S^1 \to \mathbb{P}^1(\mathbb{R})$ be the stereographic projection defined by $\sigma(x,y) = x/(1+y)$. Then $\mathbb{P}^1(\mathbb{R})$, together with the morphism

$$\psi = h_\sigma \circ \varphi\colon \mathcal{M} \longrightarrow h_{\mathbb{P}^1(R)},$$

is a coarse moduli space for the moduli problem $\mathcal{M}$. We define the *real j-invariant* to be the map

$$j_{\mathbb{R}} = \psi_\star\colon \mathcal{M}(\star) \longrightarrow \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\},$$

that is, $j_{\mathbb{R}}(E)$ is the unique element λ of $\mathbb{R} \cup \{\infty\}$ such that $\mathcal{W}_{\sigma^{-1}(\lambda)} \cong E$. Then, one has the following consequence of Theorem 6.4.

Corollary 7.1. *Let E and F be real generalized elliptic curves. Then,*

$$E \cong F \Longleftrightarrow j_{\mathbb{R}}(E) = j_{\mathbb{R}}(F). \quad \square$$

We present a formula for the real j-invariant of the real generalized elliptic curve $\mathcal{W}_{(a,b)}$ given by the Weierstrass equation $y^2 = x^3 + ax + b$. Let $(a,b) \in \mathbb{R}^{2\star}$. Then by definition, the real j-invariant $j_{\mathbb{R}}(\mathcal{W}_{(a,b)})$ of $\mathcal{W}_{(a,b)}$ is $\sigma \circ \pi(a,b) \in \mathbb{P}^1(\mathbb{R})$. Recall that $\pi\colon \mathbb{R}^{2\star} \to S^1$ is the morphism defined by $\pi(a,b) = \rho\cdot(a,b)$, where $\rho = \rho(a,b)$ is the unique root in $\mathbb{R}^+$ of Equation (1). Put

$$c = \frac{27b^4 - 2a^6}{54b^6} \quad \text{and} \quad d = \sqrt{\frac{27b^4 - 4a^6}{108b^8}}.$$

Then,

$$\rho(a,b) = \sqrt[4]{\frac{9b^4\sqrt[3]{c+d}^2 - 3a^2b^2\sqrt[3]{c+d} + a^4}{9b^4\sqrt[3]{c+d}}}.$$

Hence, the real j-invariant of the real generalized elliptic curve $\mathcal{W}_{(a,b)}$ is equal to

$$j_{\mathbb{R}}(E_{a,b}) = \frac{\rho(a,b)^4 a}{1+\rho(a,b)^6 b}.$$

This answers a question of Silhol [3], Remark 5.4(ii).

Next we study the relation between the real j-invariant and the usual j-invariant of a real generalized elliptic curve [11], §3.1. We will call the usual j-invariant of a generalized elliptic curve E over $\mathbb{R}$ the *complex j-invariant* and we will denote it by $j_{\mathbb{C}}(E)$.

Recall that

$$j_{\mathbb{C}}(\mathcal{W}_{(a,b)}) = 2^8 3^3 \frac{a^3}{\Delta}.$$

Since $\sigma^{-1}\colon \mathbb{P}^1(\mathbb{R}) \to S^1$ is given by

$$\sigma^{-1}(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right),$$

we have $j_{\mathbb{C}} = j_{\mathbb{C}}(E_{\sigma^{-1}(j_{\mathbb{R}})}) = f(j_{\mathbb{R}})$, where

$$f(t) = \frac{2^{11}3^3t^3}{3^3t^6 - 3^3t^4 + 2^5t^3 - 3^3t^2 + 3^3}.$$

Clearly, the morphism $f\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ is ramified at $t = 0, \infty$, its ramification indices being equal to 3 at these points. Since $f(1/t) = f(t)$, the morphism f is ramified at $t = \pm 1$, its ramification indices being equal to 2 at these points. One can check that f is unramified, i.e., étale, outside the set $\{-1, 0, 1, \infty\}$ (see Figure 2).

Remark 7.2. One might think that the coarse moduli space S^1 of generalized elliptic curves could have been obtained by considering a suitable modular curve over $\mathbb{R}$. This is however not the case. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ be a congruence subgroup defining a modular curve $X(\Gamma)$ over $\mathbb{R}$. Suppose that $X(\Gamma)(\mathbb{R})$ is

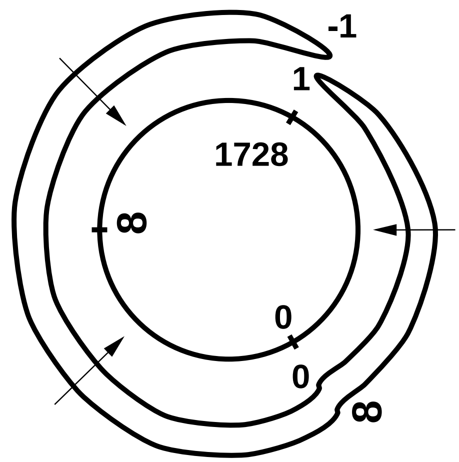

FIG. 2: The morphism $f\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ having the property that $j_{\mathbb{C}} = f(j_{\mathbb{R}})$. This map is 2-to-1. It is étale outside $\{-1, 0, 1, \infty\}$. Its ramification indices at the points $-1, 0, 1, \infty$ are $2, 3, 2, 3$, respectively. Special values of f: $f(-1) = f(1) = 2^6 3^3 = 1728$ and $f(0) = f(\infty) = 0$.

a coarse moduli space of generalized real elliptic curves. Then, there would have been an isomorphism $f\colon X(\Gamma)(\mathbb{R}) \to S^1$ such that the diagram

$$\begin{array}{ccc} X(\Gamma)(\mathbb{R}) & \xrightarrow{\;f\;} & S^1 \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1(\mathbb{R}) & \end{array}$$

commutes where the skew arrows are induced by the complex j-invariant. Since this complex j-invariant map from S^1 into $\mathbb{P}^1(R)$ is unramified above ∞, the group Γ contains the isotropy groups of the cusps ∞ and 0, in particular, Γ contains the elements

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

But since

$$A^{-1}B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } A^{-1}BA^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

are generators of $\mathrm{PSL}_2(\mathbb{Z})$, one necessarily has $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The map $X(\Gamma)(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ would then be an isomorphism. Contradiction.

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