# The moduli space of anisotropic Gaussian curves

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**Abstract**

Let $X$ be a real hyperelliptic curve. Its opposite curve $X^-$ is the curve obtained from $X$ by twisting the real structure on $X$ by the hyperelliptic involution. The curve $X$ is said to be Gaussian if $X^-$ is isomorphic to $X$. In an earlier paper, we have studied Gaussian curves having real points [4]. In the present paper we study Gaussian curves without real points, i.e. anisotropic Gaussian curves. We prove that the moduli space of such curves is a reducible connected real analytic subset of the moduli space of all anisotropic hyperelliptic curves, and determine its irreducible components.

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## 1 Introduction

Let $X$ be a real hyperelliptic curve. Its opposite curve is the curve obtained from $X$ by twisting the real structure on $X$ by the hyperelliptic involution. The opposite curve of $X$ is denoted by $X^-$. The curve $X$ is said to be Gaussian if $X^-$ is isomorphic to $X$ [4]. The terminology is inspired by the fact that such a curve carries a complex automorphism $\alpha$ whose complex conjugate is equal to $[-1]\circ\alpha$, where $[-1]$ denotes the hyperelliptic involution. As explained in the introduction of [4], Gaussian curves arise naturally when one studies the real Schottky problem [6].

In an earlier paper, we have studied Gaussian curves having real points [4]. Their study relies heavily on the fact that they are ramified double coverings of the ordinary real projective line. In the present paper we study Gaussian curves without real points. Such curves are ramified double coverings of the anisotropic conic given by the equation $x^2+y^2+z^2=0$, in the real projective plane (Lemma 3.1). Their study, therefore, needs entirely different methods.

Now, in another paper, we studied general ramified double coverings of the anisotropic conic, i.e., anisotropic hyperelliptic curves, or anisotropic curves, for short [5]. We developed a method to study anisotropic curves through so-called bordered line arrangements. This method will allow us here to study anisotropic Gaussian curves. More precisely, we will derive a characterization of anisotropic Gaussian curves in terms of bordered line arrangements (Theorem 3.2). This characterization leads to a parametrization of the moduli space of anisotropic Gaussian curves by a space of bordered line arrangements.

More explicitly, we obtain the following result. Denote by $\mathcal{G}_g$ the moduli space of anisotropic Gaussian curves of genus $g$, where $g$ is a natural integer $\geq 2$. Denote by $\mathcal{H}_g$ the moduli space of all anisotropic curves of genus $g$. Recall that $\mathcal{H}_g$ has a natural structure of a semianalytic variety [4]. The variety $\mathcal{H}_g$ is connected irreducible and of dimension $2g-1$.

**Theorem 1.1.** *Let $g$ be a natural integer satisfying $g \geq 2$. The subset $\mathcal{G}_g$ of $\mathcal{H}_g$ is a connected real analytic subset of the semianalytic variety $\mathcal{H}_g$. Furthermore, the following statements hold.*

1. *If $g+1$ is a power of 2 then the number of irreducible components of $\mathcal{G}_g$ is equal to $\frac{1}{2}(g+1)$. Moreover, the dimension of each irreducible component is equal to $g-1$.*
2. *If $g$ is odd and $g+1$ is not a power of 2 then the number of irreducible components of $\mathcal{G}_g$ is equal to $\frac{1}{2}(g+1)+\frac{1}{2}(h+1)$, where $h$ is the greatest odd divisor of $g+1$. Moreover, the dimension of each irreducible component is either equal to $g-1$ or to $h-1$.*
3. *If $g$ is even then $\mathcal{G}_g$ is empty.*

In fact, we have a much more precise statement (Theorem 6.3), but since it is rather technical, we postpone it to Section 6.

The paper is organized as follows. After recalling some facts about anisotropic curves and bordered line arrangements, we give a characterization of Gaussian curves, in Sections 2 and 3. Sections 4 and 5 constitute the core of the paper. There, we study the moduli space of bordered line arrangements associated to anisotropic Gaussian curves. In Section 6 we derive our main results on anisotropic Gaussian curves. Finally, in Section 7, we apply our results, and describe all Gaussian anisotropic curves of genus 3.

## 2 Anisotropic real curves

A smooth projective real algebraic curve $X$ of genus $g \geq 2$ is said to be *hyperelliptic* if the image of the canonical morphism $k\colon X \to \mathbb{P}^{g-1}$ is a rational curve. Now, there are, up to isomorphism, exactly 2 rational curves: the real projective line $\mathbb{P}^1$ and the anisotropic conic $\mathbb{S}^1$. The latter curve is the real algebraic curve defined by the equation $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2$. A real hyperelliptic curve $X$ is said to be *isotropic* if $k(X)$ is isomorphic to $\mathbb{P}^1$, and *anisotropic* if $k(X)$ is isomorphic to $\mathbb{S}^1$. It is clear that an anisotropic curve has no real points, and well known that such a curve has odd genus [3, Proposition 6.1].

In order to study anisotropic curves, it is useful to have at one's disposal the notion of bordered line arrangements. A *real line arrangement* is a reduced real algebraic curve in $\mathbb{P}^2$ all of whose irreducible components are real projective lines. More explicitly, a real line arrangement in $\mathbb{P}^2$ is the union of finitely many distinct real projective lines in $\mathbb{P}^2$. A *bordered line arrangement* is a pair $(A, O)$, where $A$ is a real line arrangement and $O$ is a closed subset of $\mathbb{P}^2(\mathbb{R})$ whose boundary $\partial O$ is equal to $A(\mathbb{R})$. The *degree* of a bordered line arrangement $(A, O)$ is the degree of $A$, i.e. the number of lines that $A$ contains. It is not difficult to see that the degree of a bordered line arrangement is even [5, Proposition 5.1].

Let $p$ be a nonzero homogeneous real polynomial in $x, y, z$ such that $p = \prod_{i=1}^{d} L_i$, where $L_1, \ldots, L_d$ are real linear forms in $x, y, z$ and $L_i$ is not a multiple of $L_j$, whenever $i \neq j$. If $d$ is even then $p$ defines a bordered line arrangement of degree $d$. Indeed, let $A$ be the vanishing set of $p$ and let $O$ be the set of real points of $\mathbb{P}^2$ where $p$ is nonnegative. Since $d$ is even, the latter subset is a well-defined subset of $\mathbb{P}^2(\mathbb{R})$. It is clear that $(A, O)$ is a bordered line arrangement. In fact, it is not hard to see that each bordered line arrangement arises in this way. Moreover, two polynomials $p$ and $q$ as above define the same bordered line arrangement if and only if there is a positive real number $\lambda$ such that $p = \lambda q$.

It will be convenient to have a notation for the set of the homogeneous polynomials $p$ as above. Let $\mathbb{R}[x, y, z]_d^1$ denote the set of all nonzero homogeneous real polynomials $p$ of degree $d$ in $x, y, z$ such that $p = \prod_{i=1}^{d} L_i$, where $L_1, \ldots, L_d$ are real linear forms in $x, y, z$, and $L_i$ is not a multiple of $L_j$, whenever $i \neq j$.

Let $g$ be an odd natural integer $\geq 2$. A homogeneous polynomial $p \in \mathbb{R}[x, y, z]_{g+1}^1$ not only defines a bordered line arrangement, but also an anisotropic curve. Indeed, let $X$ be the normalization of the curve in $\mathbb{P}^3$ defined by the equations

$$x^2 + y^2 + z^2 = 0 \quad \text{and} \quad w^2 z^{d-2} = p(x, y, z).$$

Then $X$ is an anisotropic curve of genus $g$ [3, Proposition 6.2]. In fact, for any anisotropic curve $X$ of genus $g$ there is a homogeneous polynomial $p \in \mathbb{R}[x, y, z]^1_{g+1}$ such that the anisotropic curve associated to $p$ is isomorphic to $X$. Moreover, two homogeneous polynomials $p, q \in \mathbb{R}[x, y, z]^1_{g+1}$ give rise to isomorphic anisotropic curves if and only if there are $\beta \in \mathrm{SO}_3(\mathbb{R})$ and $\lambda \in \mathbb{R}^+$ such that $\beta^\star p = \lambda q$ [5, Corollary 7.3]. Here $\beta^\star p$ denotes the natural action of the special orthogonal group $\mathrm{SO}_3(\mathbb{R})$ on $\mathbb{R}[x, y, z]^1_{g+1}$.

Now, if we put the two constructions together, one has a map from the set of bordered line arrangements into the set of isomorphism classes of anisotropic curves. Indeed, let $(A, O)$ be a bordered line arrangement of degree $g + 1$. Then, there is a $p \in \mathbb{R}[x, y, z]^1_{g+1}$ such that $A$ is the vanishing set of $p$, and $O$ is the set of real points of $\mathbb{P}^2$ on which $p$ is nonnegative. The anisotropic curve $X$ *associated* to $(A, O)$ is the anisotropic curve $X$ associated to $p$ above. By what has been said above, any anisotropic curve is isomorphic to a curve associated to a bordered line arrangement.

Recall that an *isometry* of $\mathbb{P}^2$ is an element of the projective special orthogonal group $\mathrm{PSO}_3(\mathbb{R})$, which is the image, in $\mathrm{PGL}_3(\mathbb{R})$, of the special orthogonal group $\mathrm{SO}_3(\mathbb{R})$. The group $\mathrm{PSO}_3(\mathbb{R})$ acts on the set of bordered line arrangement if one defines $\beta \cdot (A, O) = (\beta(A), \beta(O))$.

The following statement has been proved in [5, Corollary 7.2], and will be useful in the sequel of the paper.

**Theorem 2.1.** *Let $(A, O)$ and $(A', O')$, be bordered line arrangements in $\mathbb{P}^2$. Let $X$ and $X'$ be the anisotropic curves associated to $(A, O)$ and $(A', O')$, respectively. Then the following conditions are equivalent.*

1. *The anisotropic curves $X$ and $X'$ are isomorphic.*
2. *There is a $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that $\beta \cdot (A, O) = (A', O')$.* □

## 3 Anisotropic Gaussian curves

Let $X$ be a real hyperelliptic curve. Let $[-1]$ be the hyperelliptic involution on $X$. Let $G = \{1, \sigma\}$ be the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, acting naturally on the complexification $X_\mathbb{C} = X \times_\mathbb{R} \mathbb{C}$ of $X$. Let $\mathrm{Aut}_\mathbb{R}(X_\mathbb{C})$ be the group of $\mathbb{R}$-automorphism of the scheme $X_\mathbb{C}$. Then, the action of $G$ induces a morphism of groups $\varphi \colon G \to \mathrm{Aut}_\mathbb{R}(X_\mathbb{C})$.

One can twist the $\varphi$-action of $G$ on $X_\mathbb{C}$ by defining a morphism $\psi \colon G \to \mathrm{Aut}_\mathbb{R}(X_\mathbb{C})$ by $\psi(\sigma) = [-1] \circ \varphi(\sigma)$. The morphism $\psi$ defines another action of $G$ on $X_{\mathbb{C}}$. The quotient of $X_{\mathbb{C}}$ by the $\psi$-action of $G$ is a real algebraic curve $X^-$, which is said to be obtained from $X$ by *twisting* the real structure. The curve $X^-$ is called the *opposite* curve of $X$.

The real curves $X$ and $X^-$ are not isomorphic, in general. When they are isomorphic, $X$ is said to be a *Gaussian* curve.

**Lemma 3.1.** *Let $X$ be a Gaussian curve without real points. Then, $X$ is an anisotropic curve. In particular, the genus of $X$ is an odd natural integer.*

*Proof.* Let $X$ be a Gaussian curve without real points. In order to show that $X$ is anisotropic, suppose, to the contrary, that $X$ is isotropic. Then the image $k(X)$ of the canonical morphism $k\colon X \to \mathbb{P}^{g-1}$ is isomorphic to the real projective line $\mathbb{P}^1$. Identify $k(X)$ with $\mathbb{P}^1$, and consider $k$ as a morphism from $X$ into $\mathbb{P}^1$. Since $k$ is a morphism of degree 2, there is a real polynomial $p \in \mathbb{R}[x]$ such that $X$ is the smooth projective model of the affine plane curve defined by the equation $y^2 = p(x)$. Since $X$ has no real points, the polynomial $p$ is strictly negative on $\mathbb{R}$. The curve $X^-$ obtained by twisting the real structure on $X$, as defined above, is the smooth projective model of the affine plane curve defined by the equation $y^2 = -p(x)$. In particular, $X^-$ does have real points. Hence, $X^-$ is not isomorphic to $X$. Contradiction, since $X$ is Gaussian. It follows that $X$ is anisotropic. In particular, the genus of $X$ is an odd integer [3, Proposition 6.1]. □

For the statement of the main result of this section, we need to introduce some further terminology. Let $(A, O)$ be a bordered line arrangement. The *opposite* bordered line arrangement is $(A, O^-)$, where $O^-$ is the closure of the complement $\mathbb{P}^2(\mathbb{R}) \setminus O$ of $O$ in $\mathbb{P}^2(\mathbb{R})$. Let $X$ be the anisotropic curve associated to $(A, O)$. It follows from the definition of the anisotropic curve associated to a bordered line arrangement (see Section 2), that the anisotropic curve associated to the opposite bordered line arrangement $(A, O^-)$ is the opposite curve $X^-$.

Let $\beta \in \mathrm{PSO}_3(\mathbb{R})$ be a nontrivial isometry. The *center* of $\beta$ is the unique point $C_\beta \in \mathbb{P}^2(\mathbb{R})$ that is fixed by $\beta$ and that is not contained in any line of fixed points of $\beta$. The *line at infinity* of $\beta$ is the unique real projective line $L_\beta$ of $\mathbb{P}^2$ which is stable under the action of $\beta$ and which does not pass through $C_\beta$.

More explicitly, there is an orthonormal basis $v_1, v_2, v_3$ of $\mathbb{R}^3$ with respect to which the matrix of $\beta$ is equal to

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for some real number $\theta$. Then, the center $C_\beta$ of $\beta$ is the point $[v_3]$ of $\mathbb{P}^2(\mathbb{R})$. The line at infinity $L_\beta$ of $\beta$ is the real projective line in $\mathbb{P}^2$ passing through $[v_1]$ and $[v_2]$.

Note that $\beta$ is necessarily of finite order in $\mathrm{PSO}_3(\mathbb{R})$ if there is a line arrangement $A$ of degree $\geq 2$ such that $\beta(A) = A$. We will use this fact tacitly throughout the rest of the paper.

**Theorem 3.2.** *Let $g$ be an odd integer $\geq 2$. Let $(A, O)$ be a bordered line arrangement of degree $g+1$, and let $X$ be the associated anisotropic curve of genus $g$. Then the following conditions are equivalent.*

1. *$X$ is a Gaussian curve.*

2. *There is an $\alpha \in \mathrm{Aut}(X_\mathbb{C})$ such that $\overline{\alpha} = [-1] \circ \alpha$, where $\overline{\alpha}$ is the complex conjugate automorphism of $\alpha$, and $[-1]$ is the hyperelliptic involution on the complexification $X_\mathbb{C}$ of $X$.*

3. *There is a $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that*
   - *(a) $\beta(A) = A$, and*
   - *(b) $\beta(O) = O^-$.*

4. *There is a $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that*
   - *(a) $\beta(A) = A$,*
   - *(b) $n$ divides $2\ell$, where $n$ is the order of $\beta$ and $\ell$ is the number of lines of $A$ through $C_\beta$, and*
   - *(c) $\frac{2\ell}{n}$ is odd.*

5. *There is a $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that*
   - *(a) $\beta(A) = A$,*
   - *(b) the order $n$ of $\beta$ is even, and*
   - *(c) either $L_\beta$ is a line of $A$, or $n$ does not divide $g+1$.*

*Proof.* $1 \Leftrightarrow 2$: This has already been shown for isotropic Gaussian curves [4]. The proof for anisotropic Gaussian curve is similar.

$1 \Leftrightarrow 3$: Since $X$ is associated to the bordered line arrangement $(A, O)$, the curve $X^-$ is associated to the bordered line arrangement $(A, O^-)$, as observed above. By Theorem 2.1, $X$ is Gaussian if and only if there is $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that $\beta(A) = A$ and $\beta(O) = O^-$.

$3 \Leftrightarrow 4$: Suppose that there is a $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that $\beta(A) = A$. We have to show that $\beta(O) = O^-$ if and only if $n$ divides $2\ell$ with odd quotient.

Since $\mathrm{SO}_3(\mathbb{R}) = \mathrm{PSO}_3(\mathbb{R})$, we consider $\beta$ as an element of $\mathrm{SO}_3(\mathbb{R})$, whenever it is convenient. Let $p \in \mathbb{R}[x,y,z]^1_{g+1}$ be such that $A$ is equal to the vanishing set of $p$ and such that $O$ is the set of real points of $\mathbb{P}^2$ where $p$ is nonnegative. It is clear that $\beta(O) = O^-$ if and only if $\beta^\star p = -p$.

Observe that the order $n$ of $\beta$ is even, if $\beta(O) = O^-$, or if $n$ divides $2\ell$ with odd quotient. Since we want to show the equivalence of the last two conditions, we may as well suppose that $n$ is even. Observe also that $\beta^{n/2}$ acts trivially on the set of real projective lines through $C_\beta$, and that an orbit of a real projective line through $C_\beta$ under the action of the subgroup generated by $\beta$ contains exactly $n/2$ elements. It follows that $n/2$ divides $\ell$, i.e., $n$ divides $2\ell$.

Since the number of lines of $A$ passing through $C_\beta$ is equal to $\ell$, there are nonzero real linear forms $L_1, \ldots, L_\ell$ in $x, y, z$ such that

$$p = p' \cdot \prod_{i=1}^{\ell} L_i,$$

where $p' \in \mathbb{R}[x,y,z]^1_{g+1-\ell}$ does not vanish at $C_\beta$. Since $\beta(A) = A$, one may assume, after renumbering if necessary, that

$$\prod_{i=1}^{\ell} L_i = \prod_{i=1}^{\ell/(n/2)} \left( \prod_{j=0}^{n/2-1} (\beta^j)^\star L_i \right)$$

It is clear that $\beta^\star p' = p'$. Since $(\beta^{n/2})^\star L_i = -L_i$, it follows that $\beta^\star p = -p$ if and only if $\ell/(n/2)$ is odd. This proves that $\beta(O) = O^-$ if and only if $n$ divides $2\ell$ with odd quotient.

$4 \Leftrightarrow 5$: Let $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that $\beta(A) = A$. Let $n$ be the order of $\beta$. Since both conditions 4 and 5 imply that $n$ is even, we may assume that $n$ is even. The action of the subgroup $\langle\beta\rangle$ generated by $\beta$ on the set of real projective lines in $\mathbb{P}^2$ has 3 types of orbits: the orbit $\{L_\beta\}$ of $L_\beta$, the orbit of a line through $C_\beta$ and the orbit of a line different from the preceding ones. The latter orbit has cardinality $n$. The orbit of a line through $C_\beta$ has cardinality $\frac{n}{2}$. When one restricts the action of $\langle\beta\rangle$ to the set of lines contained in $A$, one gets the orbit formula

$$g + 1 = \varepsilon + k\tfrac{n}{2} + mn,$$

where $k$ is the number of orbits contained in $A$ of lines through $C_\beta$, $m$ is the number of orbits contained in $A$ of lines different from $L_\beta$ and not passing through $C_\beta$, and $\varepsilon = 1$ or $0$ according to whether or not $L_\beta$ belongs to $A$.

Since the number of lines of $A$ through $C_\beta$ is equal to $\ell$, one has $\ell = k\frac{n}{2}$. In particular, $n$ divides $2\ell$.

In order to show the equivalence between 4 and 5, we need to show that $k = 2\ell/n$ is odd if and only if $\varepsilon = 1$ or $n$ does not divide $g+1$. If $k$ is odd then $n$ does not divide $g+1-\varepsilon$, as follows from the above equation. Hence, $\varepsilon = 1$ or $n$ does not divide $g+1$. Conversely, if $\varepsilon = 1$ or $n$ does not divide $g+1$, then $n$ does not divide $g+1-\varepsilon$, since $n$ is even and $g$ is odd. It follows from the above equation that $k$ is odd. □

*Remark* 3.3. Condition 2 has been added for completeness. It is the condition that suggested the terminology "Gaussian curve" in [4].

As a corollary of Theorem 3.2, one gets that a Gaussian anisotropic curve is determined, up to isomorphism, by the associated line arrangement $A$, or equivalently, by the ramification locus in $\mathbb{S}^1$ of the associated ramified double covering of $\mathbb{S}^1$.

**Corollary 3.4.** *Let $(A, O)$ and $(A', O')$ be bordered line arrangements of degree $\geq 3$. Let $X$ and $X'$ be the anisotropic curves associated to $(A, O)$ and $(A', O')$, respectively. Let $B$ and $B'$ be the ramification loci in $\mathbb{S}^1$ of the double coverings $X \to \mathbb{S}^1$ and $X' \to \mathbb{S}^1$, respectively. If $X$ is Gaussian then the following conditions are equivalent.*

1. *The curves $X$ and $X'$ are isomorphic.*
2. *There is a $\beta \in \mathrm{Aut}(\mathbb{S}^1)$ such that $\beta(B) = B'$.*
3. *There is a $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that $\beta(A) = A'$.*

*Proof.* 1 ⇒ 2: Let $f\colon X \to X'$ be an isomorphism. Then, $f$ induces an automorphism $\beta$ of $\mathbb{S}^1$ such that $\beta(B) = B'$.

2 ⇒ 3: Let $\beta$ be an automorphism of $\mathbb{S}^1$ such that $\beta(B) = B'$. Since $\mathbb{S}^1$ is a rational normal curve in $\mathbb{P}^2$, the automorphism group of $\mathbb{S}^1$ is equal to $\mathrm{PSO}_3(\mathbb{R})$. It follows that $\beta \in \mathrm{PSO}_3(\mathbb{R})$, and that $\beta(A) = A'$.

3 ⇒ 1: Let $\beta \in \mathrm{PSO}_3(\mathbb{R})$ such that $\beta(A) = A'$. One has either $\beta(O) = O'$ or $\beta(O^-) = O'$. If $\beta(O) = O'$ then $X$ and $X'$ are isomorphic by Theorem 2.1. If $\beta(O^-) = O'$ then there is a $\gamma \in \mathrm{PSO}_3(\mathbb{R})$ such that $\gamma(A) = A$ and $\gamma(O) = O^-$, by Theorem 3.2. It follows that $\beta\circ\gamma(A) = A$ and $\beta\circ\gamma(O) = O'$. One concludes again by Theorem 2.1 that $X$ and $X'$ are isomorphic. □

## 4 The space of Gaussian bordered line arrangements

Let $d$ be a natural integer. Let $\tilde{\mathcal{A}}_d$ denote the set of bordered line arrangements of degree $d$. The elements of $\tilde{\mathcal{A}}_d$ are pairs $(A, O)$, where $A$ is a real line arrangement in $\mathbb{P}^2$, and $O$ is a closed subset of $\mathbb{P}^2(\mathbb{R})$ whose boundary is equal to $A(\mathbb{R})$. Recall from [5] that $\tilde{\mathcal{A}}_d$ is nonempty if and only if $d$ is even and nonzero. Moreover, if $d$ is even and nonzero then $\tilde{\mathcal{A}}_d$ is a connected real analytic manifold of dimension $2d$.

Let $\tilde{\mathcal{B}}_d$ denote the set of all triples $(A, O, \beta)$, where $(A, O)$ is a bordered real line arrangement in $\mathbb{P}^2$ of degree $d$ and $\beta$ is an isometry of $\mathbb{P}^2$ such that $\beta(A) = A$ and $\beta(O) = O^-$. Bordered line arrangements $(A, O)$ for which such an isometry exists give rise to anisotropic Gaussian curves, as follows from Theorem 3.2. The object of this section is to study $\tilde{\mathcal{B}}_d$ and its connected components.

Let $n$ be a natural integer, and let $\tilde{\mathcal{B}}_d^n$ be the subset of $\tilde{\mathcal{B}}_d$ of all triples $(A, O, \beta)$ such that the order of $\beta$ is equal to $n$. In order to study $\tilde{\mathcal{B}}_d^n$, we need to recall some classical facts about elements of $\mathrm{PSO}_3(\mathbb{R})$ that are of finite order.

Let $\mathrm{PSO}_3(\mathbb{R})[n]$ denote the subset of $\mathrm{PSO}_3(\mathbb{R})$ of all elements of order $n$. It is clear that $\mathrm{PSO}_3(\mathbb{R})[n]$ is a compact real analytic subvariety of $\mathrm{PSO}_3(\mathbb{R})$. The group $\mathrm{PSO}_3(\mathbb{R})$ acts on $\mathrm{PSO}_3(\mathbb{R})[n]$ by conjugation. The orbits for this action are easily seen to be open. Moreover, they are connected and closed since $\mathrm{PSO}_3(\mathbb{R})$ is connected and compact. Therefore, the action is transitive on each connected component of $\mathrm{PSO}_3(\mathbb{R})[n]$. It follows that $\mathrm{PSO}_3(\mathbb{R})[n]$ is a smooth closed real analytic subvariety of $\mathrm{PSO}_3(\mathbb{R})$, i.e., it is a closed real analytic submanifold of $\mathrm{PSO}_3(\mathbb{R})$.

Suppose that $n \geq 2$, and let

$$C\colon \mathrm{PSO}_3(\mathbb{R})[n] \longrightarrow \mathbb{P}^2(\mathbb{R})$$

be the map that associates to $\beta$ its center $C_\beta$. It is clear that $C$ is a surjective real analytic map. It is also clear that its fibers are finite and of cardinality $\varphi(n)$, where $\varphi$ denotes Euler's totient function. If $\gamma \in \mathrm{PSO}_3(\mathbb{R})$ and $\beta \in \mathrm{PSO}_3(\mathbb{R})[n]$, then the center of $\gamma\beta\gamma^{-1}$ is $\gamma(C_\beta)$. This means that the map $C$ is equivariant with respect to the actions of $\mathrm{PSO}_3(\mathbb{R})$ on $\mathrm{PSO}_3(\mathbb{R})[n]$ and $\mathbb{P}^2(\mathbb{R})$. It follows that $C$ is a local real analytic isomorphism. Since $\varphi(2) = 1$, the map $C$ is a global real analytic isomorphism if $n = 2$. If $n \geq 3$, an element $\beta$ of $\mathrm{PSO}_3(\mathbb{R})$ of order $n$ determines an orientation of the tangent space of $\mathbb{P}^2(\mathbb{R})$ at $C_\beta$. It follows that $C$ factorizes through the orientation double covering $S^2 \to \mathbb{P}^2(\mathbb{R})$. Since the sphere $S^2$ is simply connected, the number of connected components of $\mathrm{PSO}_3(\mathbb{R})[n]$ is equal to $\frac{1}{2}\varphi(n)$, and each connected component of $\mathrm{PSO}_3(\mathbb{R})[n]$ is real analytically isomorphic to $S^2$.

Now, the subset $\tilde{\mathcal{B}}_d^n$ of $\tilde{\mathcal{B}}_d$ can be identified with the subset of $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$ of all pairs $((A, O), \beta)$ such that $\beta(A) = A$ and $\beta(O) = O^-$. The latter subset is clearly a closed real analytic subvariety of $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$. We show that it is smooth:

**Proposition 4.1.** *The subset* $\tilde{\mathcal{B}}_d^n$ *of* $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$ *is a closed real analytic submanifold of* $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$.

*Proof.* Let $\beta \in \mathrm{PSO}_3(\mathbb{R})[n]$. Let

$$f\colon \mathrm{PSO}_3(\mathbb{R}) \longrightarrow \mathrm{PSO}_3(\mathbb{R})[n]$$

be the map defined by $f(\gamma) = \gamma\beta\gamma^{-1}$. Since $f$ is a real analytic submersion, there are a locally closed real analytic submanifold $U$ of $\mathrm{PSO}_3(\mathbb{R})$ containing the identity, and an open neighborhood $V$ of $\beta$ in $\mathrm{PSO}_3(\mathbb{R})[n]$ such that the restriction of $f$ to $U$ is a real analytic isomorphism. Define a map

$$F\colon \tilde{\mathcal{A}}_d \times U \longrightarrow \tilde{\mathcal{A}}_d \times V$$

by $F((A, O), \gamma) = ((\gamma A, \gamma O), f(\gamma))$. Then $F$ is a real analytic isomorphism, and $F^{-1}(\tilde{\mathcal{B}}_d^n)$ is equal to the subset $M$ of $\tilde{\mathcal{A}}_d \times U$ of all elements $((A, O), \gamma)$ such that $\beta(A) = A$ and $\beta(O) = O^-$. To put it otherwise, $M$ is equal to $\tilde{\mathcal{A}}_d^\beta \times U$, where $\tilde{\mathcal{A}}_d^\beta$ is the set of fixed points of $\beta$ on $\tilde{\mathcal{A}}_d$. Since $\tilde{\mathcal{A}}_d$ is a real analytic manifold, and $\beta$ acts as a real analytic automorphism of finite order on $\tilde{\mathcal{A}}_d$, the subset $\tilde{\mathcal{A}}_d^\beta$ is a smooth closed real analytic subvariety of $\tilde{\mathcal{A}}_d$. It follows that $\tilde{\mathcal{B}}_d^n$ is a smooth closed analytic subvariety of $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$, i.e., it is a closed real analytic submanifold. □

Let $\ell$ be a natural integer, and let $\tilde{\mathcal{B}}_d^{n,\ell}$ be the subset of $\tilde{\mathcal{B}}_d^n$ of all triples $(A, O, \beta)$ such that $A$ contains exactly $\ell$ lines that pass through the center $C_\beta$ of $\beta$.

**Proposition 4.2.** *Let* $d, n, \ell$ *be natural integers. The subset* $\tilde{\mathcal{B}}_d^{n,\ell}$ *is nonempty if and only if*

$$\begin{cases} d \text{ and } n \text{ are even and nonzero, and } \ell \le d, \\ n \text{ divides } 2\ell \text{ with odd quotient, and} \\ n \text{ divides } d-\ell \text{ or } d-\ell-1. \end{cases} \tag{1}$$

*Proof.* Suppose that $\tilde{\mathcal{B}}_d^{n,\ell}$ is nonempty. Let $(A, O, \beta)$ be an element of $\tilde{\mathcal{B}}_d^{n,\ell}$. As observed earlier, $d$ is even and nonzero. Of course, $n \neq 0$ and $\ell \le d$. By Theorem 3.2, $n$ is even, and divides $2\ell$ with odd quotient. Moreover, since $\beta$ acts freely on the set of real projective lines of $\mathbb{P}^2$ that do not pass through $C_\beta$ and that are different from $L_\beta$, the integer $n$ divides $d-\ell$ or $d-\ell-1$.

Conversely, suppose that $d, n, \ell$ satisfy condition 1. Choose any $\beta \in \mathrm{PSO}_3(\mathbb{R})$ of order $n$. Let $k$ be the quotient of $\ell$ by $n/2$. Choose $k$ generic real projective lines $L_1, \ldots, L_k$ through $C_\beta$. Let $m$ be the quotient of $d-\ell$ by $n$, if $n$ divides $d-\ell$, and the quotient of $d-\ell-1$, if $n$ divides $d-\ell-1$. Choose $m$

generic real projective lines $L'_1, \dots, L'_m$ in $\mathbb{P}^2$. Let $A$ be the union of the orbits, under the action of the subgroup generated by $\beta$, of $L_1, \dots, L_k, L'_1, \dots, L'_m$ if $n$ divides $d-\ell$, and the union of the orbits of $L_1, \dots, L_k, L'_1, \dots, L'_m, L_\beta$, if $n$ divides $d-\ell-1$. Then $A$ is a real line arrangement of degree $d$. Since $d$ is even, there is a closed subset $O$ of $\mathbb{P}^2(\mathbb{R})$ such that $\partial O = A(\mathbb{R})$. By Theorem 3.2, one has $\beta(A) = A$ and $\beta(O) = O^-$. It follows that $(A, O, \beta)$ is an element of $\tilde{\mathcal{B}}_d^{n,\ell}$. □

Let $d, n, \ell$ be natural integers. We will say that the triplet $(d, n, \ell)$ is *admissible* if $d, n, \ell$ satisfy condition (1). By the preceding proposition, $\tilde{\mathcal{B}}_d^{n,\ell}$ is nonempty if and only if $(d, n, \ell)$ is admissible.

Recall that $\varphi(n)$ denotes Euler's totient function, i.e., $\varphi(n)$ is the number of integers $k$ with $1 \le k \le n$ that are relatively prime with $n$. To put it otherwise, $\varphi(n)$ is equal to the cardinality of the group of generators $(\mathbb{Z}/n\mathbb{Z})^\star$ of $\mathbb{Z}/n\mathbb{Z}$.

**Proposition 4.3.** *Suppose that $(d, n, \ell)$ is admissible. The subset $\tilde{\mathcal{B}}_d^{n,\ell}$ is an open and closed subset of $\tilde{\mathcal{B}}_d^n$. In particular, $\tilde{\mathcal{B}}_d^{n,\ell}$ is a closed real analytic submanifold of $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$. Moreover, the action of $(\mathbb{Z}/n\mathbb{Z})^\star$ on $\tilde{\mathcal{B}}_d^{n,\ell}$ defined by $\overline{k} \cdot (A, O, \beta) = (A, O, \beta^k)$ induces a free and transitive action of $(\mathbb{Z}/n\mathbb{Z})^\star/\{\pm 1\}$ on the set of connected components of $\tilde{\mathcal{B}}_d^{n,\ell}$. In particular, the number of connected components of $\tilde{\mathcal{B}}_d^{n,\ell}$ is equal to $\frac{1}{2}\varphi(n)$, except when $n = 2$, in which case $\tilde{\mathcal{B}}_d^{n,\ell}$ has 1 connected component.*

*Proof.* Let $\tilde{\mathcal{B}}_d^{n,\le\ell}$ be the subset of $\tilde{\mathcal{B}}_d^n$ of all triples $(A, O, \beta)$ such that the number of lines of $A$ through $C_\beta$ is less than or equal to $\ell$. It is clear that $\tilde{\mathcal{B}}_d^{n,\le\ell}$ is an open subset of $\tilde{\mathcal{B}}_d^n$. Let us show that it is also closed.

Suppose that $(A_k, O_k, \beta_k)$ is a sequence in $\tilde{\mathcal{B}}_d^{n,\le\ell}$ that converges to an element $(A, O, \beta)$ of $\tilde{\mathcal{B}}_d^n$. We prove that the number of lines of $A$ through $C_\beta$ is less than or equal to $\ell$. Suppose to the contrary that this number is greater than $\ell$. Then there is a line $L$ in $A$ through $C_\beta$ that is the limit of a sequence of lines $L_k$ of $A_k$ that do not pass through the center of $\beta_k$. Let $L'_k$ be the image of $L_k$ by $\beta_k^{n/2}$. Then $L'_k$ is a line of $A_k$ distinct from $L_k$. The limit of $L'_k$ is equal to $\beta^{n/2}(L)$. Now, since $L$ passes through the center of $\beta^{n/2}$, and since $\beta^{n/2}$ is of order 2, the line $\beta^{n/2}(L)$ is equal to $L$. It follows that $L$ is a line of $A$ of multiplicity 2. Contradiction, since $A$ is a reduced real line arrangement. This proves that $\tilde{\mathcal{B}}_d^{n,\le\ell}$ is an open and closed subset of $\tilde{\mathcal{B}}_d^n$. It follows that

$$\tilde{\mathcal{B}}_d^{n,\ell} = \tilde{\mathcal{B}}_d^{n,\le\ell} \setminus \tilde{\mathcal{B}}_d^{n,\le\ell-1}$$

is an open and closed subset of $\tilde{\mathcal{B}}_d^n$.

For a given isometry $\gamma$ of order $n$, the subset of $\tilde{\mathcal{B}}_d^{n,\ell}$ consisting of all elements $(A, O, \beta)$ such that $\beta = \gamma$ is easily seen to be connected. Therefore, in order to prove the last statement of the lemma, it suffices to show the following. The action of $(\mathbb{Z}/n\mathbb{Z})^\star$ on $\mathrm{PSO}_3(\mathbb{R})[n]$ defined by $\overline{k} \cdot \beta = \beta^k$ induces a free and transitive action of $(\mathbb{Z}/n\mathbb{Z})^\star/\{\pm 1\}$ on the set of connected components of $\mathrm{PSO}_3(\mathbb{R})[n]$. Indeed, as we have seen above, the map $C$ from $\mathrm{PSO}_3(\mathbb{R})[n]$ into $\mathbb{P}^2(\mathbb{R})$ is a topological covering map of degree $\varphi(n)$. In fact, it is the quotient map of for the action of $(\mathbb{Z}/n\mathbb{Z})^\star$ on $\mathrm{PSO}_3(\mathbb{R})[n]$. In particular, it is an isomorphism if $n = 2$, which shows that $\tilde{\mathcal{B}}_d^{n,\ell}$ is connected if $n = 2$. Now, assume that $n \geq 3$. Let $\lambda\colon [0,1] \to \mathbb{P}^2(\mathbb{R})$ be a nontrivial loop in $\mathbb{P}^2(\mathbb{R})$, and let $\tilde{\lambda}$ be a lift to $\mathrm{PSO}_3(\mathbb{R})[n]$. Then, $\tilde{\lambda}(1) = \tilde{\lambda}(0)^{-1}$. Since $n \geq 3$, each connected component of $\mathrm{PSO}_3(\mathbb{R})[n]$ is a nontrivial covering of degree 2 of $\mathbb{P}^2(\mathbb{R})$, and the group of covering automorphisms is the subgroup $\{\pm 1\}$ of $(\mathbb{Z}/n\mathbb{Z})^\star$. It follows that the induced action of $(\mathbb{Z}/n\mathbb{Z})^\star/\{\pm 1\}$ on the set of connected components of $\mathrm{PSO}_3(\mathbb{R})[n]$ is free and transitive. □

## 5 Moduli of Gaussian bordered line arrangements

Let $\mathcal{A}_d$ be the set of isometry classes of bordered real line arrangements of degree $d$. More precisely, $\mathcal{A}_d$ is the quotient of $\tilde{\mathcal{A}}_d$ by the action of $\mathrm{PSO}_3(\mathbb{R})$. Recall from [5] that $\mathcal{A}_d$ has a natural structure of a connected semianalytic variety of dimension $2d$, if $d$ is even. Denote by $\mathcal{B}_d$ the subset of $\mathcal{A}_d$ of isometry classes of *Gaussian* bordered line arrangements. More precisely,

$$\mathcal{B}_d = \{(A, O) \in \mathcal{A}_d \,|\, \exists \beta \in \mathrm{PSO}_3(\mathbb{R})\colon \beta(A) = A \text{ and } \beta(O) = O^-\}.$$

**Proposition 5.1.** *Let $d$ be an even nonzero natural integer. The subset $\mathcal{B}_d$ of $\mathcal{A}_d$ of Gaussian bordered line arrangements of degree $d$ is a closed real analytic subset of $\mathcal{A}_d$. In particular, $\mathcal{B}_d$ has a natural structure of a semianalytic variety.*

*Proof.* Define an involution $\iota$ on $\mathcal{A}_d$ by $\iota(A, O) = (A, O^-)$. The involution $\iota$ is easily seen to be real analytic. Since $\mathcal{B}_d$ is equal to the set of fixed points of $\iota$ on $\mathcal{A}_d$, the subset $\mathcal{B}_d$ is a closed real analytic subset of $\mathcal{A}_d$. □

In this section we study $\mathcal{B}_d$ and determine its irreducible components. Define a forgetful map

$$\psi\colon \tilde{\mathcal{B}}_d \longrightarrow \mathcal{B}_d$$

by $\psi(A, O, \beta) = (A, O)$.

**Proposition 5.2.** *Let $d$ be an even nonzero natural integer. The map $\psi$ is a real analytic surjection.*

*Proof.* The map $\psi$ is clearly real analytic. The map $\psi$ is surjective by definition of $\mathcal{B}_d$. □

Let $(d, n, \ell)$ be an admissible triple. Define a subset $\mathcal{B}_d^{n,\ell}$ of $\mathcal{B}_d$ by

$$\mathcal{B}_d^{n,\ell} = \psi(\tilde{\mathcal{B}}_d^{n,\ell}).$$

Explicitly, $\mathcal{B}_d^{n,\ell}$ is the set of isometry classes of all Gaussian bordered line arrangements $(A, O)$ of degree $d$ such that there is an isometry $\beta$ of $\mathbb{P}^2$ with the following properties:

1. $\beta(A) = A$ and $\beta(O) = O^-$,
2. the order of $\beta$ is equal to $n$, and
3. the number of lines of $A$ passing through $C_\beta$ is equal to $\ell$.

**Proposition 5.3.** *Let $(d, n, \ell)$ be an admissible triple. The subset $\mathcal{B}_d^{n,\ell}$ is an irreducible real analytic subset of $\mathcal{B}_d$. Moreover,*

$$\dim(\mathcal{B}_d^{n,\ell}) = \begin{cases} \frac{2d}{n} - 1 & \text{if } \ell \text{ is even,} \\ \frac{2d-2}{n} - 1 & \text{if } \ell \text{ is odd,} \end{cases}$$

*Proof.* By Proposition 4.3, $\tilde{\mathcal{B}}_d^{n,\ell}$ is a closed real analytic submanifold of $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$. Let

$$p \colon \tilde{\mathcal{B}}_d^{n,\ell} \longrightarrow \tilde{\mathcal{A}}_d$$

be the restriction to $\tilde{\mathcal{B}}_d^{n,\ell}$ of the projection of $\tilde{\mathcal{A}}_d \times \mathrm{PSO}_3(\mathbb{R})[n]$ onto the first factor. It is clear that $p$ is a proper immersion of $\tilde{\mathcal{B}}_d^{n,\ell}$ into $\tilde{\mathcal{A}}_d$. It follows that the image $M$ of $p$ is a real analytic submanifold of $\tilde{\mathcal{A}}_d$. Since $p$ is constant on the orbits in $\tilde{\mathcal{B}}_d^{n,\ell}$ for the action of $(\mathbb{Z}/n\mathbb{Z})^\star$, the map $p$ factors through the quotient $\tilde{\mathcal{B}}_d^{n,\ell}/(\mathbb{Z}/n\mathbb{Z})^\star$. The latter quotient is a connected real analytic manifold by Proposition 4.3. It follows that the image $M$ of $p$ is a connected closed real analytic submanifold of $\tilde{\mathcal{A}}_d$. Since $M$ is stable for the action of $\mathrm{PSO}_3(\mathbb{R})$ on $\tilde{\mathcal{A}}_d$, its image in the quotient $\mathcal{A}_d$ of $\tilde{\mathcal{A}}_d$ is an irreducible closed real analytic subvariety. The latter image is nothing but the subset $\mathcal{B}_d^{n,\ell}$.

In order to determine the dimension of $\mathcal{B}_d^{n,\ell}$, let us first determine the dimension of $\tilde{\mathcal{B}}_d^{n,\ell}$. An element of $\tilde{\mathcal{B}}_d^{n,\ell}$ is a triple $(A, O, \beta)$ where $\beta$ is an isometry of $\mathbb{P}^2$ of order 2 with $\beta(A) = A$ and $\beta(O) = O^-$, such that the number of lines of $A$ passing through $C_\beta$ is equal to $\ell$. Now, the set $\mathrm{PSO}_3(\mathbb{R})[n]$ is of dimension 2. For a given $\beta \in \mathrm{PSO}_3(\mathbb{R})[n]$, the number of lines through $C_\beta$ that one can choose more or less arbitrary in order to get a line arrangement stable for $\beta$, is equal to $\ell/(n/2)$. If $\ell$ is odd then the line $L_\beta$ is in $A$. The number of remaining lines that one choose more or less arbitrary in $\mathbb{P}^2$, is equal to $(d-\ell-1)/n$, if $\ell$ is odd. It follows that

$$\dim(\tilde{\mathcal{B}}_d^{n,\ell}) = 2 + \tfrac{\ell}{n/2} + 2\tfrac{d-\ell-1}{n} = \tfrac{2d-2}{n} + 2,$$

if $\ell$ is odd. Therefore, $\dim(\mathcal{B}_d^{n,\ell}) = \frac{2d-2}{n} - 1$ if $\ell$ is odd. The case $\ell$ is even is similar, except that $L_\beta$ does not belong to such line arrangements. □

Although all subsets $\mathcal{B}_d^{n,\ell}$ of $\mathcal{B}_d$ are irreducible, they are are not all irreducible components of $\mathcal{B}_d$, as follows from the following statement. Recall that $\mathrm{ord}_2(n)$ denotes the 2-valuation of an integer $n$.

**Proposition 5.4.** *Let $(d,n,\ell)$ be an admissible triple. Let $i = \mathrm{ord}_2(n)$. Then,*

$$\mathcal{B}_d^{n,\ell} \subseteq \mathcal{B}_d^{2^i,\ell}.$$

*Proof.* Let $(A,O)$ be a Gaussian bordered line arrangement in $\mathcal{B}_d^{n,\ell}$. This means that there is an isometry $\beta$ of order $n$ of $\mathbb{P}^2$ such that $\beta(A) = A$ and $\beta(O) = O^-$. Moreover, the number of lines belonging to $A$ that pass through $C_\beta$ is equal to $\ell$. Since $i = \mathrm{ord}_2(n)$, there is an odd natural integer $k$ such that $n = 2^i k$. Now, the isometry $\beta^k$ satisfies $\beta^k(A) = A$ and $\beta^k(O) = O^-$ since $k$ is odd. Also, $\beta^k \neq \mathrm{id}$ since $n$ is even. Therefore, $\beta^k$ has a center, and this center is obviously equal to $C_\beta$. It follows that $(A,O)$ is an element of $\mathcal{B}_d^{2^i,\ell}$. □

Let $d$ be an even nonzero natural integer. We will determine the set of pairs $(n,\ell)$ of natural integers such that $(d,n,\ell)$ is admissible and such that $n$ is a power of 2.

Let $i = \mathrm{ord}_2(d)$, and let $e$ be the natural integer such that $d = 2^i(2e+1)$. Define subsets $I_d^{\mathrm{I}}$ and $I_d^{\mathrm{II}}$ of $\mathbb{N}\times\mathbb{N}$ by

$$I_d^{\mathrm{I}} = \{(n,\ell) \,|\, n = 2,\ \ell = 2k+1,\ k \in \mathbb{N},\ k \leq \tfrac{d}{2} - 1\}$$

and

$$I_d^{\mathrm{II}} = \{(n,\ell) \,|\, n = 2^{i+1},\ \ell = 2^i(2k+1),\ k \in \mathbb{N},\ k \leq e\}.$$

Note that the sets $I_d^{\mathrm{I}}$ and $I_d^{\mathrm{II}}$ are disjoint.

**Lemma 5.5.** *Let $d$ be an even nonzero natural integer. The set of pairs $(n,\ell)$ of natural integers such that $n$ is a power of 2 and such that $(d,n,\ell)$ is admissible, is equal to $I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}$.*

*Proof.* Suppose that $(n,\ell) \in I_d^{\mathrm{I}}$. Then $n = 2^{i+1}$ and $\ell = 2^i(2k+1)$, where $i = \mathrm{ord}_2(d)$ and $0 \le k \le e$ with $d = 2^i(2e+1)$. It follows that $n$ is even and nonzero, that $\ell \le d$, that $n$ divides $2\ell$ and that the quotient $2\ell/n$ is odd. Moreover, $n$ divides $d-\ell$. Therefore, $(d,n,\ell)$ is admissible.

Suppose that $(n,\ell) \in I_d^{\mathrm{II}}$. Then $n = 2$ and $\ell = 2k+1$, where $0 \le k \le \frac{d}{2}-1$. It follows that $n$ is even and nonzero, that $\ell \le d$, that $n$ divides $2\ell$ and that the quotient $2\ell/n$ is odd. Moreover, $n$ divides $d-\ell-1$. Therefore, $(d,n,\ell)$ is admissible.

Conversely, suppose that $(d,n,\ell)$ is admissible, and that $n$ is a power of 2. Since $n$ is an even power of 2, there is a natural integer $j$ such that $n = 2^{j+1}$. Since $n$ divides $2\ell$ and the quotient is odd, $\ell = 2^j(2k+1)$, for some natural integer $k$.

Now, $n$ divides $d-\ell$ or $n$ divides $d-\ell-1$. If $n$ divides $d-\ell$, then $\mathrm{ord}_2(d) = \mathrm{ord}_2(\ell)$, i.e., $i = j$. Moreover, since $\ell \le d$, one has $k \le e$. Therefore, $(n,\ell) \in I_d^{\mathrm{II}}$ if $n$ divides $d-\ell$. If $n$ divides $d-\ell-1$, then $\ell$ is odd, as $n$ and $d$ are even. Hence, $j=0$. Moreover, since $\ell \le d$, one has $k \le \frac{d}{2}-1$. Therefore, $(n,\ell) \in I_d^{\mathrm{I}}$ if $n$ divides $d-\ell-1$. It follows that $(n,\ell) \in I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}$. □

**Lemma 5.6.** *Let $(n,\ell),(n',\ell') \in I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}$ with $(n,\ell) \neq (n',\ell')$. Then*

$$\mathcal{B}_d^{n,\ell} \subseteq \mathcal{B}_d^{n',\ell'}$$

*if and only if*

1. *$d = 2$, or*
2. *$d$ is a power of 2, $(n,\ell) = (2d,d)$ and $(n',\ell') = (2,1)$.*

*Proof.* Let us treat the case $d = 2$ separately. If $d = 2$ then $I_d^{\mathrm{I}} = \{(2,1)\}$ and $I_d^{\mathrm{II}} = \{(4,2)\}$. It is easy to see that $\mathcal{B}_d^{2,1} = \mathcal{B}_d^{4,2}$ if $d = 2$. Indeed, the isometry class of the bordered line arrangement defined by the inequality $xy \ge 0$ is the only element of both $\mathcal{B}_d^{2,1}$ and $\mathcal{B}_d^{4,2}$. Therefore, we may assume for the rest of the proof that $d \ge 4$.

If $d$ is a power of 2, $(n,\ell) = (2d,d)$ and $(n',\ell') = (2,1)$ then $\mathcal{B}_d^{n,\ell}$ is contained in $\mathcal{B}_d^{n',\ell'}$. Indeed, the subset $\mathcal{B}_d^n\ell$ of $\mathcal{B}_d$ consists of one element only. Let $\beta$ be an isometry of $\mathbb{P}^2$ of order $2d$. Let $L$ be a real projective line through $C_\beta$. Let $A$ be the real line arrangement in $\mathbb{P}^2$ of degree $d$ containing $L$ and its $d-1$ successive images by $\beta$. Choose any closed subset $O$ of $\mathbb{P}^2(\mathbb{R})$ such that $\partial O = A(\mathbb{R})$. Then, the isometry class of $(A,O)$ is the unique element of $\mathcal{B}_d^{2d,d}$. Let $\beta'$ be the reflection of $\mathbb{P}^2$ in $L$. Then, $\beta'(A) = A$ and $\beta'(O) = O^-$. Also, the center $C_{\beta'}$ of $\beta'$ lies on $\beta^{d/2}(L)$, the line of $A$ perpendicular to $L$. This shows that the isometry class of $(A,O)$ belongs also to $\mathcal{B}_d^{2,1}$. It follows that $\mathcal{B}_d^{2d,d}$ is contained in $\mathcal{B}_d^{2,1}$, when $d$ is a power of 2.

Let $i = \mathrm{ord}_2(d)$, as above. In order to prove the lemma, we prove, conversely, the following 4 statements:

1. if $n = 2^{i+1}$ and $n' = 2^{i+1}$ then $\mathcal{B}_d^{n,\ell} \cap \mathcal{B}_d^{n',\ell'} = \emptyset$,
2. if $n = 2$ and $n' = 2^{i+1}$ then $\dim(\mathcal{B}_d^{n,\ell}) > \dim(\mathcal{B}_d^{n',\ell'})$,
3. if $n = 2^{i+1}$ and $n' = 2$ then $\dim(\mathcal{B}_d^{n,\ell} \cap \mathcal{B}_d^{n',\ell'}) = \dim(\mathcal{B}_d^{n,\ell})$ implies that $d$ is a power of 2, $(n, \ell) = (2d, d)$ and $(n', \ell') = (2, 1)$, and
4. if $n = 2$ and $n' = 2$ then $\dim(\mathcal{B}_d^{n,\ell} \cap \mathcal{B}_d^{n',\ell'}) < \dim(\mathcal{B}_d^{n,\ell})$.

These 4 statements together suffice to prove the lemma. Indeed, if $\mathcal{B}_d^{n,\ell}$ is contained in $\mathcal{B}_d^{n',\ell'}$, where $(n, \ell), (n', \ell') \in I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}$ with $(n, \ell) \neq (n', \ell')$, then the conclusions of statements 1, 2 and 4 cannot hold. Therefore, $n = 2^{i+1}$ and $n' = 2$. Since the inclusion also implies that $\dim(\mathcal{B}_d^{n,\ell} \cap \mathcal{B}_d^{n',\ell'}) = \dim(\mathcal{B}_d^{n,\ell})$, it follows from statement 4 that $d$ is a power of 2, $(n, \ell) = (2d, d)$ and $(n', \ell') = (2, 1)$, as was to be proved.

In order to prove statement 1 above, suppose that $n = n' = 2^{i+1}$. We show that the two subvarieties $\mathcal{B}_d^{n,\ell}$ and $\mathcal{B}_d^{n',\ell'}$ are disjoint, if $\ell \neq \ell'$. Suppose, to the contrary, that $(A, O)$ is a bordered real line arrangement whose isometry class belongs to the intersection of the two varieties. Then, there are isometries $\beta$ and $\beta'$ of $\mathbb{P}^2$, both of order $n$, such that $\beta(A) = A$ and $\beta'(A) = A$, and such that the number of lines of $A$ through $C_\beta$ is equal to $\ell$, and the number of lines of $A$ through $C_{\beta'}$ is equal to $\ell'$. Since $\ell \neq \ell'$, the isometries $\beta$ and $\beta'$ have different centers in $\mathbb{P}^2(\mathbb{R})$. It follows from the classification of finite subgroups of $\mathrm{PSO}_3(\mathbb{R})$ that the subgroup of $\mathrm{PSO}_3(\mathbb{R})$ generated by $\beta$ and $\beta'$ contains an element $\gamma$ such that $\gamma(C_\beta) = C_{\beta'}$. Since $\gamma$ belongs to the subgroup generated by $\beta$ and $\beta'$, one has $\gamma(A) = A$. It follows that the number of lines of $A$ through $C_\beta$ is equal to the number of lines of $A$ through $C_{\beta'}$, i.e., $\ell = \ell'$. Contradiction, since, by hypothesis, $\ell \neq \ell'$. Therefore, the intersection of $\mathcal{B}_d^{n,\ell}$ and $\mathcal{B}_d^{n',\ell'}$ is empty. This proves statement 1 above.

In order to prove statement 2, suppose that $n = 2$ and $n' = 2^{i+1}$. We show that the dimension of $\mathcal{B}_d^{n,\ell}$ is strictly greater than the dimension of $\mathcal{B}_d^{n',\ell'}$. Indeed, since $n = 2$, $\ell$ is odd and the dimension of $\mathcal{B}_d^{n,\ell}$ is equal to $(2d-2)/n - 1 = d - 2$, by Proposition 5.3. Since $n' = 2^{i+1}$, $\ell'$ is even and the dimension of $\mathcal{B}_d^{n',\ell'}$ is equal to $2d/n' - 1 = d/2^i - 1$, by that same proposition. Since $d \geq 4$, one has

$$\dim(\mathcal{B}_d^{n,\ell}) = d - 2 > \tfrac{d}{2} - 1 \geq \tfrac{d}{2^i} - 1 = \dim(\mathcal{B}_d^{n',\ell'}).$$

This proves statement 2.

In order to prove statement 3, suppose that $n = 2^{i+1}$ and $n' = 2$. We first determine the dimension of the intersection of the two varieties $\mathcal{B}_d^{n,\ell}$ and

$\mathcal{B}_d^{n',\ell'}$. Let $(A, O)$ be a bordered line arrangement in the intersection of the two varieties. Then there is an isometry $\beta$ of order $n$ and an isometry $\beta'$ of order $n'$ such that $\beta(A) = A$, $\beta'(A) = A$ and such that exactly $\ell$ lines of $A$ pass through $C_\beta$, and exactly $\ell'$ lines of $A$ pass through $C_{\beta'}$. Since $2\ell/n$ and $2\ell'/n'$ are odd, $\ell \neq \ell'$, and the points $C_\beta$ and $C_{\beta'}$ of $\mathbb{P}^2(\mathbb{R})$ are distinct. It follows from the classification of finite subgroups of $\mathrm{PSO}_3(\mathbb{R})$ that $\beta$ and $\beta'$ generate a subgroup isomorphic to the dihedral group $D_n$. We identify $D_n$ with this subgroup.

The action of $D_n$ on the set of lines of $A$ may have orbits of cardinality $2n$, $n$, $\frac{n}{2}$ and 1. Let $a$, $a'$, $a''$ and $a'''$ be the number of orbits of cardinality $2n$, $n$, $\frac{n}{2}$ and 1, respectively. Since $A$ is an arrangement of degree $d$,

$$2na + na' + \tfrac{n}{2}a'' + a''' = d.$$

A line of $A$ is contained in an orbit of cardinality $n$ or $\frac{n}{2}$ if and only if it passes through $C_\beta$. Therefore

$$na' + \tfrac{n}{2}a'' = \ell.$$

A line of $A$ is contained in an orbit of cardinality 1 if and only if it goes through the centers of all elements of $D_n$ of order 2. Therefore, $a''' = 0$ or 1. Since $\ell$ and $d$ are even, $d - \ell = 2na + a'''$ is even. It follows that $a''' = 0$.

The lines of $A$ that are in an orbit of cardinality $\frac{n}{2}$ are lines that pass through $C_\beta$ and the center of an element of order 2 of $D_n$. The set of centers of elements of $D_n$ of order 2 contains exactly 2 orbits for the action of $D_n$. It follows that $a'' = 0$, 1 or 2. Since $2\ell/n$ is odd, one has $a'' = 1$.

The lines of $A$ that are in an orbit of cardinality $\frac{n}{2}$ or 1 are rigid with respect to $D_n$. Indeed, a line of $A$ contained in such an orbit is a line that goes through at least 2 centers of elements of $D_n$. The number of moduli of a line of $A$ contained in an orbit of cardinality $n$ is equal to 1. The number of moduli of a line of $A$ contained in an orbit of cardinality $2n$ is equal to 2. It follows that

$$\dim(\mathcal{B}_d^{n,\ell} \cap \mathcal{B}_d^{n',\ell'}) = 2a + a'.$$

Now, we are ready to prove statement 3. Suppose that $\dim(\mathcal{B}_d^{n,\ell} \cap \mathcal{B}_d^{n',\ell'})$ is equal to $\dim(\mathcal{B}_d^{n,\ell})$. Since $\ell$ is even, one has

$$\begin{aligned} \tfrac{2d}{n} - 1 = 2a + a' = \tfrac{1}{n}(2na + na') = \tfrac{1}{n}(d - \tfrac{n}{2}a'' - a''') = \\ = \tfrac{1}{n}(d - \tfrac{n}{2}) = \tfrac{d}{n} - \tfrac{1}{2} = \tfrac{1}{2}(\tfrac{2d}{n} - 1). \end{aligned}$$

It follows that $n = 2d$. In particular, $d$ is a power of 2. Since $n$ divides $2\ell$ with odd quotient, one also obtains that $\ell = d$. Therefore, the arrangement $A$

is an arrangement of degree $d$ of $d$ lines passing through one and the same point. It follows that $\ell' = 1$. This proves statement 3.

The proof of statement 4 is quite similar to the proof of statement 3: suppose that $n = n' = 2$. We determine the dimension of the intersection of the two varieties $\mathcal{B}_d^{n,\ell}$ and $\mathcal{B}_d^{n',\ell'}$, if $\ell \neq \ell'$. Let $(A, O)$ be a bordered line arrangement in the intersection of the two varieties. Then there are two isometries $\beta$ and $\beta'$ of order 2 such that $\beta(A) = A$ and $\beta'(A) = A$, and such that exactly $\ell$ lines of $A$ pass through $C_\beta$, and exactly $\ell'$ lines of $A$ pass through $C_{\beta'}$. Since $\ell \neq \ell'$, the points $C_\beta$ and $C_{\beta'}$ of $\mathbb{P}^2(\mathbb{R})$ are distinct. It follows from the classification of finite subgroups of $\mathrm{PSO}_3(\mathbb{R})$ that $\beta$ and $\beta'$ generate a subgroup isomorphic to the dihedral group $D_2$. We identify $D_2$ with this subgroup.

The action of $D_2$ on the set of lines of $A$ may have orbits of cardinality 4, 2 and 1. Let $a$, $a'$ and $a''$ be the number of orbits of cardinality 4, 2 and 1, respectively. Since $A$ is an arrangement of degree $d$,

$$4a + 2a' + a'' = d.$$

A line of $A$ is contained in an orbit of cardinality 4 if and only if it does not pass through the center of any nontrivial element of $D_2$. A line of $A$ is contained in an orbit of cardinality 2 if and only if it passes through the center of exactly 1 nontrivial element of $D_2$. A line of $A$ is contained in an orbit of cardinality 1 if and only if it passes through the centers of exactly 2 nontrivial elements of $D_2$. Since $D_2$ contains 3 nontrivial elements, one has $a'' = 0$, 1, 2 or 3. Since $d$ is even, one cannot have $a'' = 1$ or $a'' = 3$. Therefore, $a'' = 0$ 0r 2. Since the number of lines of $A$ through $C_\beta$ is odd, one has $a'' \neq 0$. Hence, $a'' = 2$. It follows that

$$\dim(\mathcal{B}_d^{n,\ell} \cap \mathcal{B}_d^{n',\ell'}) = 2a + a' = \tfrac{1}{2}(d - a'') = \tfrac{1}{2}d - 1 < d - 2 = \dim(\mathcal{B}_d^{n,\ell}),$$

where the last equality is a consequence of Proposition 5.3. This proves statement 4. □

**Theorem 5.7.** *Let $d$ be an even nonzero natural integer. Then the set of irreducible components of $\mathcal{B}_d$ is*

$$\{\mathcal{B}_d^{n,\ell} \mid (n, \ell) \in I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}\},$$

*except when $d$ is a power of 2, i.e., when $e = 0$, in which case the set of irreducible components of $\mathcal{B}_d$ is*

$$\{\mathcal{B}_d^{n,\ell} \mid (n, \ell) \in I_d^{\mathrm{I}}\}.$$

*Moreover, $\mathcal{B}_d$ is connected.*

*Proof.* By Proposition 5.4 and Lemma 5.5,

$$\mathcal{B}_d = \bigcup_{(n,\ell)\in I_d^{\mathrm{I}}\cup I_d^{\mathrm{II}}} \mathcal{B}_d^{n,\ell}.$$

By Proposition 5.3, the real analytic subsets $\mathcal{B}_d^{n,\ell}$ are irreducible subsets of $\mathcal{B}_d$. Suppose that $d$ is not a power of 2. Then, by Lemma 5.6, $\mathcal{B}_d^{n,\ell}$ is not contained in $\mathcal{B}_d^{n',\ell'}$ for all $(n,\ell),(n',\ell') \in I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}$ with $(n,\ell) \neq (n',\ell')$. It follows that the irreducible components of $\mathcal{B}_d$ are the subsets $\mathcal{B}_d^{n,\ell}$, where $(n,\ell)$ runs through $I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}$, if $d$ is not a power of 2.

If $d$ is a power of 2, then $I_d^{\mathrm{II}} = \{(2d,d)\}$, and

$$\mathcal{B}_d = \bigcup_{(n,\ell)\in I_d^{\mathrm{I}}} \mathcal{B}_d^{n,\ell},$$

by Lemma 5.6. It follows from the same Lemma 5.6 that the irreducible components of $\mathcal{B}_d$ are the subsets $\mathcal{B}_d^{n,\ell}$, where $(n,\ell)$ runs through $I_d^{\mathrm{I}}$, if $d$ is a power of 2.

Let us show the connectedness of $\mathcal{B}_d$. As we have seen in the proof of Lemma 5.6, the intersection of $\mathcal{B}_d^{n,\ell}$ and $\mathcal{B}_d^{n',\ell'}$ is non empty if $n = 2$ or $2^i$, and $n' = 2$ ,where $(n,\ell),(n',\ell') \in I_d^{\mathrm{I}} \cup I_d^{\mathrm{II}}$. It follows from what has been said above that $\mathcal{B}_d$ is connected. □

## 6 Moduli of anisotropic Gaussian curves

Let $g \geq 2$ be a natural integer. Let $\mathcal{H}_g$ denote the moduli space of anisotropic real hyperelliptic curves of genus $g$. Recall that $\mathcal{H}_g$ is nonempty if and only if $g$ is odd and $g \geq 2$ [3, Proposition 6.1], and that $\mathcal{H}_g$ has a natural structure of a semianalytic variety [5]. Let $\kappa\colon \mathcal{H}_g \to \mathcal{H}_g$ be the involution defined by $\kappa(X) = X^-$. Its is easy to see that $\kappa$ is a real analytic involution on $\mathcal{H}_g$. The set of fixed points $\mathcal{G}_g$ of $\kappa$ is the set of isomorphism classes of anisotropic real hyperelliptic curves that are Gaussian. Since $\kappa$ is a real analytic involution on a semianalytic variety, $\mathcal{G}_g$ is a real analytic subset of $\mathcal{H}_g$. In particular, $\mathcal{G}_g$ has a natural structure of a semianalytic variety.

Let

$$\rho\colon \mathcal{B}_{g+1} \longrightarrow \mathcal{G}_g$$

be the map that maps an element $(A,O)$ to the Gaussian anisotropic curve associated to $(A,O)$.

**Theorem 6.1.** *The map $\rho$ is a real analytic isomorphism.*

*Proof.* The map $\rho$ extends to a map, again denoted by $\rho$, from $\mathcal{A}_{g+1}$ into $\mathcal{H}_g$ defined similarly: if $(A, O)$ is a bordered real line arrangement of degree $g+1$ then $\rho(A, O)$ is the anisotropic real hyperelliptic curve associated to $(A, O)$. It is proved in [5] that $\rho$ is a real analytic isomorphism. Since $\kappa \circ \rho = \rho \circ \iota$, the restriction of $\rho$ to set of fixed points of $\iota$ on $\mathcal{A}_{g+1}$ is an isomorphism onto the set of fixed points of $\kappa$ on $\mathcal{H}_g$. The former set of fixed points is equal to $\mathcal{B}_{g+1}$, the latter set of fixed points is equal to $\mathcal{G}_g$. □

Let $n$ and $\ell$ be natural integers. Define the subset $\mathcal{G}_g^{n,\ell}$ of $\mathcal{G}_g$ by

$$\mathcal{G}_g^{n,\ell} = \rho(\mathcal{B}_{g+1}^{n,\ell}).$$

Since $\rho$ is a bijection, the subset $\mathcal{G}_g^{n,\ell}$ of $\mathcal{G}_g$ is nonempty if and only if the triple $(g+1, n, \ell)$ is admissible, i.e., if and only if

$$\begin{cases} g \text{ is odd and } g \geq 2,\ n \text{ is even and nonzero, and } \ell \leq g+1, \\ n \text{ divides } 2\ell \text{ with odd quotient, and} \\ n \text{ divides } g+1-\ell \text{ or } g-\ell. \end{cases} \tag{2}$$

**Theorem 6.2.** *Let $g$, $n$ and $\ell$ be natural integers satisfying condition (2). Then the subset $\mathcal{G}_g^{n,\ell}$ is an irreducible real analytic subset of $\mathcal{G}_g$. Moreover,*

$$\dim(\mathcal{G}_g^{n,\ell}) = \begin{cases} \frac{2g+2}{n} - 1 & \text{if } \ell \text{ is even,} \\ \frac{2g}{n} - 1 & \text{if } \ell \text{ is odd,} \end{cases}$$

*Proof.* The statement is a direct consequence of Proposition 5.3 and Theorem 6.1. □

**Theorem 6.3.** *Let $g$ be an odd natural integer with $g \geq 2$. Then the irreducible components of $\mathcal{G}_g$ are the real analytic subvarieties $\mathcal{G}_g^{n,\ell}$, where $(n, \ell)$ runs through $I_{g+1}^{\mathrm{I}} \cup I_{g+1}^{\mathrm{II}}$, if $g+1$ is not a power of 2. If $g+1$ is a power of 2, then the irreducible components of $\mathcal{G}_g$ are the real analytic subvarieties $\mathcal{G}_g^{n,\ell}$, where $(n, \ell)$ runs through $I_{g+1}^{\mathrm{I}}$. In particular, $\mathcal{G}_g$ is not irreducible.*

*Proof.* The statement is a direct consequence of Theorem 5.7 and Theorem 6.1, except for the last assertion. In order to see that $\mathcal{G}_g$ is not irreducible, it suffices to note that $I_{g+1}^{\mathrm{I}}$ contains at least 2 elements. □

*Proof of Theorem 1.1.* As observed above, $\mathcal{G}_g$ is a real analytic subset of $\mathcal{H}_g$. It follows from Theorem 5.7 that $\mathcal{G}_g$ is connected, if $g$ is odd. It is well known that $\mathcal{H}_g$ is empty if $g$ is even [3, Proposition 6.1]. Therefore, $\mathcal{G}_g$ is empty and connected, if $g$ is even. If $g$ is odd then $\mathcal{G}_g$ is nonempty and reducible, as will follow from the fact, proved below, that the number of irreducible components of $\mathcal{G}_g$ is nonzero.

If $g+1$ is a power of 2, then the number of irreducible components of $\mathcal{G}_g$ is equal to the cardinality of the set $I^{\mathrm{I}}_{g+1}$, by Theorem 6.3. This cardinality is equal to $\frac{1}{2}(g+1)$. Therefore, the number of irreducible components of $\mathcal{G}_g$ is equal to $\frac{1}{2}(g+1)$, if $g+1$ is a power of 2. Moreover, all irreducible components of $\mathcal{G}_g$ are of dimension $g-1$, by Theorem 6.2 and Theorem 6.3, if $g+1$ is a power of 2.

Similarly, if $g$ is odd and $g+1$ is not a power of 2 then the number of irreducible components of $\mathcal{G}_g$ is equal to the cardinality of the set $I^{\mathrm{I}}_{g+1} \cup I^{\mathrm{II}}_{g+1}$, by Theorem 6.3. This cardinality is equal to $\frac{1}{2}(g+1)+\frac{1}{2}(h+1)$, where $h$ is the greatest odd divisor of $g+1$. Therefore, the number of irreducible components of $\mathcal{G}_g$ is equal to $\frac{1}{2}(g+h+2)$, if $g$ is odd and $g+1$ is not a power of 2. Moreover, one of such an irreducible component is of dimension $g-1$ or $h-1$, by Theorem 6.2 and Theorem 6.3. $\square$

## 7 Examples

In the present section we apply our results to anisotropic Gaussian curves of genus 3. To simplify some statements, we introduce an additional notation.

Let $(A, O)$ be a bordered line arrangement and let $X$ be the associated anisotropic curve. Let $\mathrm{Isom}(A)$ be the group of all isometries $\beta$ of $\mathbb{P}^2$ such that $\beta(A) = A$. We denote by $\mathrm{Isom}^+(A)$ the subgroup of $\mathrm{Isom}(A)$ of isometries that satisfy $\beta(O) = O$, and by $\mathrm{Isom}^-(A)$ its complement in $\mathrm{Isom}(A)$. Since every isometry $\beta \in \mathrm{Isom}(A)$ is induced by a pair of automorphisms $\alpha$ and $\alpha \circ [-1]$ on the curve $X_{\mathbb{C}}$, it is clear that one can view $\mathrm{Isom}(A)$ as a subgroup of the *reduced* automorphism group $\overline{\mathrm{Aut}}(X_{\mathbb{C}}) = \mathrm{Aut}(X_{\mathbb{C}})/\langle[-1]\rangle$.

Note that the above considerations highlight a natural link between the study of anisotropic Gaussian curves and two others kind of problems: the study of automorphisms groups of anisotropic curves (see [2]) and the study of anisotropic real structures on complex hyperelliptic curves (see [1]).

**Example 7.1.** Let $X$ be the anisotropic curve of genus 3 defined by the homogeneous polynomial

$$p(x, y, z) = yz(y^2 - 3x^2)$$

and let $(A, O)$ be the bordered line arrangement defined by $p$. It is easy to see that the groups $\mathrm{Isom}^+(A)$ and $\mathrm{Isom}(A)$ are isomorphic to the dihedral groups $\mathrm{D}_3$ and $\mathrm{D}_6$, respectively. Moreover $X_{\mathbb{C}}$ is isomorphic to the hyperelliptic complex curve defined by the affine plane equation $v^2 = u(u^6-1)$. It is known that, for such a curve, the reduced automorphisms group $\overline{\mathrm{Aut}}(X_{\mathbb{C}})$ is isomorphic to the dihedral group $D_6$. Therefore, one has:

$$\mathrm{Isom}^+(A) \subsetneq \mathrm{Isom}(A) = \overline{\mathrm{Aut}}(X_{\mathbb{C}}).$$

We describe the 6 elements of the set $\mathrm{Isom}^-(A)$.

There is exactly 1 isometry $\beta \in \mathrm{Isom}^-(A)$ of order $n = 2$ such that the number of lines $\ell$ of $A$ through $C_\beta$ is equal to 3. It is the rotation $\beta$ of center $(0,0,1)$, line at infinity $z = 0$, and angle $\pi$. Note that $\beta$ is induced by the automorphism $\alpha\colon X_{\mathbb{C}} \to X_{\mathbb{C}}$ defined by $\alpha(w,x,y,z) = (iw,x,y,-z)$. The corresponding isomorphism $\gamma\colon X \to X^-$ is given by $\gamma(w,x,y,z) = (w,x,y,-z)$.

There are exactly 3 isometries in $\mathrm{Isom}^-(A)$ with $n = 2$ and $\ell = 1$. These are the rotations of angle $\pi$, of centers $(0,1,0)$, $(\sqrt{3},1,0)$ and $(\sqrt{3},-1,0)$, and lines at infinity $y = 0$, $x\sqrt{3} + y = 0$ and $x\sqrt{3} - y = 0$, respectively.

There are exactly 2 isometries in $\mathrm{Isom}^-(A)$ with $n = 6$ and $\ell = 3$. These are the rotations of center $(0,0,1)$, line at infinity $z = 0$, and angles $\pm\frac{\pi}{3}$.

In terms of the moduli spaces $\mathcal{G}_g^{n,\ell}$, this means that

$$X \in \mathcal{G}_3^{2,1} \cap \mathcal{G}_3^{2,3} \cap \mathcal{G}_3^{6,3}.$$

We give a more precise description of the entire moduli space $\mathcal{G}_3$ of all anisotropic Gaussian curves of genus 3, in the next example.

**Example 7.2.** By Theorem 6.3 and Theorem 6.2, the moduli space $\mathcal{G}_3$ has two irreducible components, each of dimension 2:

$$\mathcal{G}_3 = \mathcal{G}_3^{2,1} \cup \mathcal{G}_3^{2,3}.$$

The component $\mathcal{G}_3^{2,1}$ is equal to the set of isomorphism classes of anisotropic Gaussian curves $X$ defined by a homogeneous polynomial $p$ of the form:

$$p(x,y,z) = yz(ax + y + bz)(ax - y + bz) \qquad (a,b \in \mathbb{R}, (a,b) \neq (0,0)).$$

For such a curve, an isomorphism $\gamma$ from $X$ into $X^-$ is given by $\gamma(w,x,y,z) = (w,x,-y,z)$.

The component $\mathcal{G}_3^{2,3}$ is equal to the set of isomorphism classes of anisotropic Gaussian curves $Y$ defined by a homogeneous polynomial $p$ of the form:

$$p(x,y,z) = yz(x + cy)(x + dy) \qquad (c,d \in \mathbb{R}, c \neq d).$$

For such a curve, an isomorphism from $Y$ into $Y^-$ is given by $\gamma(w,x,y,z) = (w,x,y,-z)$.

The intersection $\mathcal{G}_3^{2,1} \cap \mathcal{G}_3^{2,3}$ is not empty. More precisely, by the proof of Lemma 5.6, it is a subspace of $\mathcal{G}_3$ of dimension 1. In fact, if, in the above equation of the curve $X$, one takes $b = 0$ (or, which amounts to the same, if, in the equation of the curve $Y$, one take $d = -c$), one obtains a one-dimensional family of anisotropic Gaussian curves, that are both in $\mathcal{G}_3^{2,1}$ and $\mathcal{G}_3^{2,3}$.

For completeness, let us also describe the other nonempty irreducible moduli spaces $\mathcal{G}_3^{n,\ell}$. By condition (2) of Section 6, $(n, \ell) = (6, 3)$ or $(8, 4)$. By Theorem 6.2, the moduli spaces $\mathcal{G}_3^{6,3}$ and $\mathcal{G}_3^{8,4}$ are of dimension 0.

The moduli space $\mathcal{G}_3^{6,3}$ consists of the curve described in Example 7.1. Therefore, one has a strict inclusion

$$\mathcal{G}_3^{6,3} \subsetneq \mathcal{G}_3^{2,1} \cap \mathcal{G}_3^{2,3}.$$

The moduli space $\mathcal{G}_3^{8,4}$ consists of the anisotropic curve defined by the homogeneous polynomial $p(x, y, z) = yz(y + z)(-y + z)$. In fact, if $(A, O)$ is the bordered line arrangement defined by $p$, then an isometry $\beta \in \operatorname{Isom}^-(A)$, of order 8 and class 4, is the rotation of center $(1, 0, 0)$, line at infinity $x = 0$ and angle $\frac{\pi}{4}$. Note that one can obtain this curve by taking $a = 0$ and $b = 1$ in the equation of the curve $X \in \mathcal{G}_3^{2,1}$ above. Therefore, one has the strict inclusion

$$\mathcal{G}_3^{8,4} \subsetneq \mathcal{G}_3^{2,1}.$$

As a final remark, the intersection $\mathcal{G}_3^{2,1} \cap \mathcal{G}_3^{2,3}$, as well as the moduli spaces $\mathcal{G}_3^{6,3}$ and $\mathcal{G}_3^{8,4}$, lie in the boundary of the entire moduli space $\mathcal{H}_3$ of all anisotropic curves of genus 3. But the general anisotropic Gaussian curve of the moduli space $\mathcal{G}_3$ does not lie in the boundary of $\mathcal{H}_3$.

The following is an example of an anisotropic curves which is not a Gaussian curve.

**Example 7.3.** Let $X$ be the anisotropic real curve of genus 3 defined by the homogeneous polynomial

$$p(x, y, z) = (x + y + z)(x - y + z)(-x + y + z)(-x - y + z).$$

Let $(A, O)$ be the bordered line arrangement in $\mathbb{P}^2$ defined by $p$.

Proceeding as in the Example 7.1, one easily sees that the group $\operatorname{Isom}^+(A)$ is isomorphic to the symmetric group $S_4$. Since $\operatorname{Isom}^+(A)$ is a subgroup of $\overline{\operatorname{Aut}}(X_{\mathbb{C}})$ and since there are no hyperelliptic complex curves of genus 3 having a reduced automorphism group of order greater than 24, we deduce that $X_{\mathbb{C}}$ is isomorphic to the hyperelliptic complex curve defined by the affine plane equation $v^2 = u^8 + 14u^4 + 1$. In fact, it is known to be the only hyperelliptic complex curve of genus 3 for which $\overline{\mathrm{Aut}}(X_{\mathbb{C}}) \simeq S_4$.

Therefore, we have

$$\mathrm{Isom}^+(A) = \mathrm{Isom}(A) = \overline{\mathrm{Aut}}(X_{\mathbb{C}}).$$

In particular, $\mathrm{Isom}^-(A)$ is empty, that is, $X$ is not a Gaussian curve.

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