

# ALGEBRAIC DIFFEOMORPHISMS OF REAL RATIONAL SURFACES AND WEIGHTED BLOW-UP SINGULARITIES

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ABSTRACT. Let  $X$  be a real rational surface having only weighted blow-up singularities. Denote by  $\text{Diff}_{\text{alg}}(X)$  the group of algebraic automorphisms or algebraic diffeomorphisms of  $X$  into itself. Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ . Denote by  $X^{\mathbf{e}}$  the set of  $\ell$ -tuples  $(P_1, \dots, P_\ell)$  of distinct nonsingular infinitely near points of  $X$  of orders  $(e_1, \dots, e_\ell)$ . We show that the group  $\text{Diff}_{\text{alg}}(X)$  acts transitively on  $X^{\mathbf{e}}$ . This statement generalizes earlier work where the case of the trivial partition  $\mathbf{e} = [1, \dots, 1]$  was treated under the supplementary condition that  $X$  is nonsingular.

As an application we classify rational real surfaces having only weighted blow-up singularities.

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## 1. INTRODUCTION

Let  $X$  be a nonsingular compact connected real algebraic surface, i.e.  $X$  is a nonsingular compact connected real algebraic subset of some  $\mathbb{R}^m$  of dimension 2. Recall that  $X$  is *rational* if the field of rational functions  $\mathbb{R}(X)$  of  $X$  is a purely transcendental field extension of  $\mathbb{R}$  of transcendence degree 2. More geometrically,  $X$  is rational if there are nonempty Zariski open subsets  $U$  and  $V$  of  $\mathbb{R}^2$  and  $X$ , respectively, such that there is an isomorphism of real algebraic varieties—in the sense of [BCR98]—between  $U$  and  $V$ . Loosely speaking,  $X$  is rational if a nonempty Zariski open subset of  $X$  admits a rational parametrization by a nonempty Zariski open subset of  $\mathbb{R}^2$ . A typical example of a rational compact connected real algebraic surface is the unit sphere  $S^2$  in  $\mathbb{R}^3$ . A rational parametrization in that case is the inverse stereographic projection.

It has recently been shown that any rational nonsingular compact connected real algebraic surface is isomorphic either to the real algebraic torus  $S^1 \times S^1$ , or to a real algebraic surface obtained from the real algebraic sphere  $S^2$  by blowing up a finite number of points [BH07, HM08].

In the sequel, it will be convenient to identify the real algebraic surface  $X$  with the affine scheme  $\text{Spec } \mathcal{R}(X)$ , where  $\mathcal{R}(X)$  denotes the  $\mathbb{R}$ -algebra of all algebraic—also called regular—functions on  $X$  [BCR98]. A real-valued function  $f$  on  $X$  is *algebraic* if there are real polynomials  $p$  and  $q$  in  $x_1, \dots, x_m$

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such that  $q$  does not vanish on  $X$  and such that  $f = p/q$  on  $X$ . The algebra  $\mathcal{R}(X)$  is the localization of the coordinate ring  $\mathbb{R}[x_1, \dots, x_m]/I$  with respect to the multiplicative system of all polynomials that do not vanish on  $X$ , where  $I$  denotes the vanishing ideal of  $X$ . It is the subring of  $\mathbb{R}(X)$  of all rational functions on  $X$  that do not have any poles on  $X$ .

Thanks to the above convention, we can define an *infinitely near point* of  $X$  to be a closed subscheme  $P$  of  $X$  that is isomorphic to  $\text{Spec } \mathbb{R}[x]/(x^e)$ , for some nonzero natural integer  $e$ . We call  $e$  the *length* or the *order* of  $P$ . An infinitely near point of  $X$  is also called an *infinitesimal arc* on  $X$ . An infinitely near point of length 1 is an ordinary point of  $X$ . An infinitely near point of length 2 is a pair  $(P, L)$ , where  $P$  is a point of  $X$  and  $L$  is 1-dimensional subspace of the tangent space  $T_P X$  of  $X$  at  $P$ . Equivalently, an infinitely near point  $P$  of  $X$  of length 2 is a point of the exceptional divisor of the real algebraic surface obtained by blowing up  $X$  in an ordinary point. By induction, an infinitely near point of  $X$  of length  $e$  is an infinitely near point of length  $e-1$  on the exceptional divisor  $E$  of a blow-up of  $X$  (cf. [Mu, p. 171]).

Let  $P$  be an infinitely near point of  $X$ . The reduced scheme  $P_{\text{red}}$  is an ordinary point of  $X$ . Let  $P$  and  $Q$  be infinitely near points of  $X$ . We say that  $P$  and  $Q$  are *distinct* if the points  $P_{\text{red}}$  and  $Q_{\text{red}}$  of  $X$  are different.

Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ , where  $\ell$  is some natural integer. Denote by  $X^{\mathbf{e}}$  the set of  $\ell$ -tuples  $(P_1, \dots, P_\ell)$  of distinct infinitely near points  $P_1, \dots, P_\ell$  of  $X$  of orders  $e_1, \dots, e_\ell$ , respectively.

Recall that an *algebraic diffeomorphism* or *algebraic automorphism* of  $X$  is a bijective map  $f$  from  $X$  into itself such that all coordinate functions of  $f$  and  $f^{-1}$  are algebraic functions on  $X$  [HM08]. Denote by  $\text{Diff}_{\text{alg}}(X)$  the group of algebraic diffeomorphisms of  $X$  into itself. Equivalently,  $\text{Diff}_{\text{alg}}(X)$  is the group of  $\mathbb{R}$ -algebra automorphisms of  $\mathcal{R}(X)$ . One has a natural action of  $\text{Diff}_{\text{alg}}(X)$  on  $X^{\mathbf{e}}$ . One of the main results of the paper is the following.

**Theorem 1.1.** *Let  $X$  be a nonsingular rational compact connected real algebraic surface. Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ , for some natural integer  $\ell$ . Then the group  $\text{Diff}_{\text{alg}}(X)$  acts transitively on  $X^{\mathbf{e}}$ .*

Roughly speaking, Theorem 1.1 states that the group  $\text{Diff}_{\text{alg}}(X)$  acts  $\ell$ -transitively on infinitely near points of  $X$ , for any  $\ell$ . The statement generalizes earlier work where  $n$ -transitivity was proved for ordinary points only, i.e., in case of the trivial partition  $\mathbf{e} = [1, \dots, 1]$  (cf. [HM08]).

The statement of Theorem 1.1 motivates the following question.

**Question 1.2.** *Let  $X$  be a nonsingular rational compact connected real algebraic surface. Is the subset  $\text{Diff}_{\text{alg}}(X)$  of algebraic diffeomorphisms of  $X$  dense in the set  $\text{Diff}(X)$  of all diffeomorphisms of  $X$ ? Equivalently, can any diffeomorphism of  $X$  be approximated by algebraic diffeomorphisms?*

The problem of approximating smooth maps between real algebraic varieties by algebraic maps has been studied by numerous authors [BK87a, BK87b, BKS97, Ku99, JK03, JM04, Ma06].

It should be noted that Theorem 1.1 does not seem to follow from the known  $n$ -transitivity of  $\text{Diff}_{\text{alg}}(X)$  on ordinary points. The difficulty is that if two  $n$ -tuples  $P$  and  $Q$  of ordinary distinct points of  $X$  tend to two  $\ell$ -tuples of distinct infinitely near points with lengths  $e_1, \dots, e_\ell$ , then the algebraic diffeomorphisms mapping  $P$  to  $Q$  do not necessarily have a limit in  $\text{Diff}_{\text{alg}}(X)$ .

Our proof of Theorem 1.1 goes as follows. First we show that the statement of Theorem 1.1 is valid for the real algebraic surfaces  $S^2$  and  $S^1 \times S^1$  by explicit construction of algebraic diffeomorphisms (see Theorems 3.1 and 2.1). This step does use the earlier work mentioned above. Then, we use the fore-mentioned fact that an arbitrary nonsingular rational compact connected real algebraic surface is either algebraically diffeomorphic to  $S^1 \times S^1$ , or to a real algebraic surface obtained from  $S^2$  by blowing up a finite number of distinct ordinary points [HM08, Theorem 4.3], and conclude by induction.

In order to give an application of Theorem 1.1, we need to recall the following. Let  $X$  be a nonsingular rational compact connected real algebraic surface, and let  $P$  be an infinitely near point of  $X$ . The *blow-up* of  $X$  at  $P$  is the blow-up  $B_P(X)$  of  $X$  at the sheaf of ideals defined by the closed subscheme  $P$ . Explicitly, if  $P$  is defined by the ideal  $(x^e, y)$  on the real affine plane  $\mathbb{R}^2$ , then the blow-up of  $\mathbb{R}^2$  at  $P$  is the real algebraic subvariety of  $\mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$  defined by the equation  $vx^e - uy = 0$ , where  $(u : v)$  are homogeneous coordinates on the real projective line  $\mathbb{P}^1(\mathbb{R})$ . The blow-up  $B_P(X)$  is also called a *weighted blow-up*, for obvious reasons. If  $e = 1$ , the blow-up  $B_P(X)$  is the ordinary blow-up of  $X$  at  $P$ . If  $e \geq 2$  then the blow-up  $B_P(X)$  has a singular point. A local equation of the singularity is  $x^e = uy$  in  $\mathbb{R}^3$ . This is often called a singularity of type  $A_{e-1}^-$  (see e.g. [Ko00, Definition 2.1]).

We note that weighted blow-ups recently turned out to have interesting applications in real algebraic geometry (see [CM08a, CM08b, Ko99, Ko00]).

We will say that a real surface  $X$  has *weighted blow-up singularities* if  $X$  is obtained from a nonsingular real algebraic surface  $Y$  by blowing up a finite number of nonsingular infinitely near points. Here, we say that an infinitely near point  $P$  of a surface is *nonsingular* if  $P_{\text{red}}$  is a nonsingular point.

The following statement implies that any rational compact connected real algebraic surface having weighted blow-up singularities, is obtained from  $S^1 \times S^1$  or  $S^2$  by blowing-up a finite number of distinct infinitely near points of  $S^1 \times S^1$  or  $S^2$ , respectively.

**Theorem 1.3.** *Let  $X$  be a rational compact connected real algebraic surface having weighted blow-up singularities. Then*

- *there are distinct infinitely near points  $P_1, \dots, P_\ell$  on  $S^1 \times S^1$  such that  $X$  is isomorphic to the real algebraic surface obtained from  $S^1 \times S^1$  by blowing up  $P_1, \dots, P_\ell$ , or*
- *there are distinct infinitely near points  $P_1, \dots, P_\ell$  on  $S^2$  such that  $X$  is isomorphic to the real algebraic surface obtained from  $S^2$  by blowing up  $P_1, \dots, P_\ell$ .*

Let  $X$  be a rational compact connected real algebraic surface having weighted blow-up singularities. Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ , where  $\ell$  is some natural integer. Denote

by  $X^e$  the set of  $\ell$ -tuples  $(P_1, \dots, P_\ell)$  of distinct nonsingular infinitely near points  $P_1, \dots, P_\ell$  of  $X$  of orders  $e_1, \dots, e_\ell$ , respectively.

Denote again by  $\text{Diff}_{\text{alg}}(X)$  the group of algebraic diffeomorphisms of the possibly singular real algebraic surface  $X$ . Note that the definition of algebraic diffeomorphism above makes perfectly sense for singular varieties. Alternatively, one can define an algebraic diffeomorphism of  $X$  to be an automorphism of  $\text{Spec } \mathcal{R}(X)$ . Anyway, one has a natural action of  $\text{Diff}_{\text{alg}}(X)$  on  $X^e$ .

We have the following generalization of Theorem 1.1 above.

**Theorem 1.4.** *Let  $X$  be a rational compact connected real algebraic surface having weighted blow-up singularities. Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ , for some natural integer  $\ell$ . Then the group  $\text{Diff}_{\text{alg}}(X)$  acts transitively on  $X^e$ .*

As an application of Theorem 1.1 and Theorem 1.3, we prove the following statement.

**Theorem 1.5.** *Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ , for some natural integer  $\ell$ . Let  $X$  and  $Y$  be two rational compact connected real algebraic surfaces having weighted blow-up singularities. Assume that each of the surfaces  $X$  and  $Y$  has exactly one singularity of type  $A_{e_i}^-$  for each  $i = 1, \dots, \ell$ . Then  $X$  and  $Y$  are isomorphic as real algebraic surfaces if and only if they are homeomorphic as singular topological surfaces.*

Theorem 1.5 generalizes to certain singular rational real algebraic surfaces an earlier result for nonsingular ones ([BH07, Theorem 1.2] and [HM08, Theorem 1.5]). We will show by an example that the statement of Theorem 1.5 does not hold for the slightly more general class of rational compact connected real algebraic surfaces that have singularities of type  $A^-$  (see Example 7.1).

## 2. INFINITELY NEAR POINTS ON THE TORUS

The object of this section is to prove Theorem 1.1 in the case of the real algebraic torus:

**Theorem 2.1.** *Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ , for some natural integer  $\ell$ . The group  $\text{Diff}_{\text{alg}}(S^1 \times S^1)$  acts transitively on  $(S^1 \times S^1)^e$ .*

The above statement is a generalization of the following statement, that we recall for future reference.

**Theorem 2.2** ([BH07, Theorem 1.3]). *Let  $n$  be a natural integer. The group  $\text{Diff}_{\text{alg}}(S^1 \times S^1)$  acts  $n$ -transitively on  $S^1 \times S^1$ .  $\square$*

For the proof of Theorem 2.1, we need several lemmas. It will turn out to be convenient to replace  $S^1$  by the isomorphic real projective line  $\mathbb{P}^1(\mathbb{R})$ .

**Lemma 2.3.** *Let  $p, q \in \mathbb{R}[x]$  be real polynomials in  $x$  of the same degree. Suppose that  $q$  does not have any real roots. Define*

$$\varphi: \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \longrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$$

by

$$\varphi(x, y) = \left( x, y + \frac{p}{q} \right).$$

Then  $\varphi$  is an algebraic diffeomorphism of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  into itself.

*Proof.* It suffices to prove that  $\varphi$  is an algebraic map. We write  $\varphi$  in bi-homogeneous coordinates:

$$\varphi([x_0 : x_1], [y_0 : y_1]) = ([x_0 : x_1], [\bar{q}(x_0, x_1)y_0 + \bar{p}(x_0, x_1)y_1 : \bar{q}(x_0, x_1)y_1]),$$

where  $\bar{p}$  and  $\bar{q}$  are the homogenizations of  $p$  and  $q$ , respectively. Since  $q$  has no real zeros, the homogeneous polynomial  $\bar{q}$  does not vanish on  $\mathbb{P}^1(\mathbb{R})$ . Therefore, if

$$\begin{aligned} \bar{q}(x_0, x_1)y_0 + \bar{p}(x_0, x_1)y_1 &= 0, \text{ and} \\ \bar{q}(x_0, x_1)y_1 &= 0 \end{aligned}$$

then  $y_1 = 0$  and  $y_0 = 0$ . It follows that  $\varphi$  is a well defined algebraic map from  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  into itself.  $\square$

**Definition 2.4.** Let  $P$  be an infinitely near point of a nonsingular rational compact connected real algebraic surface  $X$ . If  $R$  is an ordinary point of  $X$ , then  $P$  is said to be infinitely near to  $R$  if  $P_{\text{red}} = R$ .

**Definition 2.5.** Let  $P$  be an infinitely near point of  $S^1 \times S^1$ . We say that  $P$  is vertical if  $P$  is tangent to a vertical fiber  $\{x\} \times S^1$ , for some  $x \in S^1$ , i.e. if the scheme-theoretic intersection  $P \cdot (\{x\} \times S^1)$  is not reduced.

**Lemma 2.6.** Let  $P_1, \dots, P_\ell$  be distinct infinitely near points of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ . Then there is an algebraic diffeomorphism  $\varphi$  of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  such that

- $\varphi(P_i)$  is infinitely near to the point  $(i, 0)$  of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ , and
- $\varphi(P_i)$  is not vertical,

for all  $i$ .

*Proof.* By Theorem 2.2, we may assume that  $P_1, \dots, P_\ell$  are infinitely near to the points  $(1, 0), \dots, (\ell, 0)$  of the real algebraic torus  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ , respectively.

Let  $v_i = (a_i, b_i)$  be a tangent vector to  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  at  $P_i$  that is tangent to  $P_i$ . This means the following. If  $P_i$  is an ordinary point then  $v_i = 0$ . If  $P_i$  is not an ordinary point then  $v_i \neq 0$  and the 0-dimensional subscheme of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  of length 2 defined by  $v_i$  is contained in the closed subscheme  $P$  of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ .

Let  $p, q \in \mathbb{R}[y]$  be real polynomials in  $y$  of the same degree such that

- $q$  does not have any real roots,
- $p(0) = 0, q(0) = 1, q'(0) = 0$  and
- $a_i + b_i p'(0) \neq 0$  whenever  $v_i \neq 0$ .

Define  $\varphi: \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}) \longrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  by

$$\varphi(x, y) = \left( x + \frac{p}{q}, y \right).$$

According to Lemma 2.3—exchanging  $x$  and  $y$ —the map  $\varphi$  is an algebraic diffeomorphism of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ . Since  $p(0) = 0$ , one has  $\varphi((i, 0)) = (i, 0)$ . It follows that  $\varphi(P_i)$  is also infinitely near to  $(i, 0)$ .

The Jacobian of  $\varphi$  at  $(i, 0)$  is equal to

$$D_{(i,0)}\varphi = \begin{pmatrix} 1 & \frac{p'(0)q(0)-p(0)q'(0)}{q(0)^2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p'(0) \\ 0 & 1 \end{pmatrix}$$

By construction,  $(D_{(i,0)}\varphi)v_i$  has first coordinate non zero whenever  $v_i \neq 0$ . Therefore,  $\varphi(P_i)$  is not vertical, for all  $i$ .  $\square$

*Proof of Theorem 2.1.* Let  $P_1, \dots, P_\ell$  be distinct infinitely near points of the real algebraic torus  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  of orders  $e_1, \dots, e_\ell$ , respectively. Let  $Q_i$  be the infinitely near point of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  defined by the ideal  $((x - i)^{e_i}, y)$  in  $\mathcal{R}(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R}))$ . It suffices to show that there is an algebraic diffeomorphism  $\varphi$  of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  such that  $\varphi(Q_i) = P_i$  for all  $i$ .

By Lemma 2.6, we may assume that the infinitely near point  $P_i$  is infinitely near to  $(i, 0)$  and that  $\varphi(P_i)$  is not vertical. It follows that  $P_i$  is defined by an ideal of the form

$$((x - i)^{e_i}, y - f_i),$$

where  $f_i \in \mathbb{R}[x]$ .

Let  $p, q \in \mathbb{R}[x]$  be of the same degree such that

- $q$  does not have any real roots,
- $p = f_i q$  modulo  $(x - i)^{e_i}$  for all  $i$ .

Such polynomials abound by the Chinese Remainder Theorem.

By Lemma 2.3, the polynomials  $p$  and  $q$  give rise to an algebraic diffeomorphism  $\varphi$  of  $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$  defined by

$$\varphi(x, y) = \left( x, y + \frac{p}{q} \right).$$

In order to show that  $\varphi(Q_i) = P_i$  for all  $i$ , we compute

$$(\varphi^{-1})^*((x - i)^{e_i}) = (x - i)^{e_i}$$

and

$$(\varphi^{-1})^*(y) = y - \frac{p}{q} = y - f_i$$

modulo  $(x - i)^{e_i}$ . Indeed,  $q$  is invertible modulo  $(x - i)^{e_i}$ , and  $p = f_i q$  modulo  $(x - i)^{e_i}$ , by construction of  $p$  and  $q$ . It follows that  $\varphi(Q_i) = P_i$ .  $\square$

### 3. INFINITELY NEAR POINTS ON THE UNIT SPHERE

The object of this section is to prove Theorem 1.1 in the case of the real algebraic sphere  $S^2$ :

**Theorem 3.1.** *Let  $n$  be a natural integer and let  $\mathbf{e} = [e_1, \dots, e_\ell]$  be a partition of  $n$ , for some natural integer  $\ell$ . The group  $\text{Diff}_{\text{alg}}(S^2)$  acts transitively on  $(S^2)^{\mathbf{e}}$ .*

The above statement is a generalization of the following statement, that we recall for future reference.

**Theorem 3.2** ([HM08, Theorem 2.3]). *Let  $n$  be a natural integer. The group  $\text{Diff}_{\text{alg}}(S^2)$  acts  $n$ -transitively on  $S^2$ .*  $\square$

For the proof of Theorem 3.1, we need several lemmas.

**Lemma 3.3** ([HM08, Lemma 2.1]). *Let  $p, q, r \in \mathbb{R}[x]$  be such that*

- $r$  does not have any roots in the interval  $[-1, 1]$ , and
- $p^2 + q^2 = r^2$ .

Define  $\varphi: S^2 \rightarrow S^2$  by

$$\varphi(x, y, z) = \left( x, \frac{yp - zq}{r}, \frac{yq + zp}{r} \right).$$

Then  $\varphi$  is an algebraic diffeomorphism of  $S^2$ . □

**Definition 3.4.** *Let  $P$  be infinitely near to a point of the equator  $\{z = 0\}$  of  $S^2$ . We say that  $P$  is vertical if  $P$  is tangent to the great circle of  $S^2$  passing through the North pole.*

As for the torus above, we need some standard points on  $S^2$ . Let

$$R_i = (x_i, y_i, z_i) = \left( \cos\left(\frac{i\pi}{2\ell+1}\right), \sin\left(\frac{i\pi}{2\ell+1}\right), 0 \right)$$

for  $i = 1, \dots, \ell$ . Note that  $x_i \neq 0$  and  $y_i \neq 0$  for all  $i$ .

**Lemma 3.5.** *Let  $P_1, \dots, P_\ell$  be distinct infinitely near points of  $S^2$ . Then there is an algebraic diffeomorphism  $\varphi$  of  $S^2$  such that*

- $\varphi(P_i)$  is infinitely near to  $R_i$ , and
- $\varphi(P_i)$  is not vertical,

for all  $i$ .

*Proof.* By Theorem 3.2, we may assume that  $P_i$  is infinitely near to the point  $R_i$ , for all  $i$ . Let  $v_i = (a_i, b_i, c_i)$  be a tangent vector to  $S^2$  at  $P_i$  that is tangent to  $P_i$ . Let  $p, q, r \in \mathbb{R}[x]$  be such that

- $r$  does not vanish on  $[-1, 1]$ ,
- $p^2 + q^2 = r^2$ ,
- $p(0) = 1, q(0) = 0, r(0) = 1, p'(0) = 0, r'(0) = 0$ , and
- $a_i - c_i y_i q'(0) \neq 0$  or  $b_i + c_i x_i q'(0) \neq 0$ , whenever  $v_i \neq 0$ .

Such polynomials abound. Take, for example,

$$p(z) = (1 + z^2)^2 - (\lambda z)^2, \quad q(z) = 2(1 + z^2)\lambda z, \quad r(z) = (1 + z^2)^2 + (\lambda z)^2,$$

where  $\lambda$  is any real number such that  $a_i - 2\lambda y_i c_i \neq 0$  or  $b_i + 2\lambda x_i c_i \neq 0$  whenever  $v_i \neq 0$ .

Define  $\varphi: S^2 \rightarrow S^2$  by

$$\varphi(x, y, z) = \left( \frac{xp(z) - yq(z)}{r(z)}, \frac{xq(z) + yp(z)}{r(z)}, z \right).$$

According to Lemma 3.3—permuting the roles of  $x, y, z$ —the map  $\varphi$  is an algebraic diffeomorphism of  $S^2$ . Since  $p(0) = 1, q(0) = 0$  and  $r(0) = 1$ , the infinitely near point  $\varphi(P_i)$  is again infinitely near to  $R_i$ , for all  $i$ .

The Jacobian of  $\varphi$  at  $R_i$  is equal to

$$D_{R_i}\varphi = \begin{pmatrix} \frac{p(0)}{r(0)} & \frac{-q(0)}{r(0)} & \frac{x_i p'(0)r(0) - y_i q'(0)r(0) - x_i p(0)r'(0) + y_i q(0)r'(0)}{r(0)^2} \\ \frac{q(0)}{r(0)} & \frac{p(0)}{r(0)} & \frac{x_i q'(0)r(0) + y_i p'(0)r(0) - x_i q(0)r'(0) - y_i p(0)r'(0)}{r(0)^2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -y_i q'(0) \\ 0 & 1 & x_i q'(0) \\ 0 & 0 & 1 \end{pmatrix}$$

By construction,  $(D_{R_i}\varphi)v_i$  has first or second coordinate non zero whenever  $v_i \neq 0$ . Therefore,  $\varphi(P_i)$  is not vertical, for all  $i$ .  $\square$

**Lemma 3.6.** *Let  $e$  be a nonzero natural integer, and let  $i \in \{1, \dots, \ell\}$ . Let  $f, g, h \in \mathbb{R}[x]/(x - x_i)^e$  be such that*

$$(3.7) \quad x^2 + f^2 = 1, \quad \text{and} \quad x^2 + g^2 + h^2 = 1$$

in  $\mathbb{R}[x]/(x - x_i)^e$ . Assume, moreover, that  $f(x_i) = g(x_i) = y_i$ . Then there is  $a \in \mathbb{R}[x]/(x - x_i)^e$  such that

$$(3.8) \quad (1 - a^2)f = (1 + a^2)g, \quad \text{and} \quad 2af = (1 + a^2)h$$

in  $\mathbb{R}[x]/(x - x_i)^e$ . Moreover, there is such an element  $a$  such that  $1 + a^2$  is invertible in  $\mathbb{R}[x]/(x - x_i)^e$ .

*Proof.* If  $h = 0$  then  $f = g$ , and one can take  $a = 0$ . Therefore, we may assume that  $h \neq 0$ . Let  $d$  be the valuation of  $h$ , i.e.,  $h = (x - x_i)^d h'$ , where  $h' \in \mathbb{R}[x]/(x - x_i)^e$  is invertible. Since  $f(x_i) = g(x_i)$ , one has  $h(x_i) = 0$ , i.e.,  $h$  is not invertible in  $\mathbb{R}[x]/(x - x_i)^e$  and  $d \neq 0$ . By Hensel's Lemma, there are lifts  $\bar{f}, \bar{g}, \bar{h}$  in  $\mathbb{R}[x]/(x - x_i)^{e+2d}$  of  $f, g, h$ , respectively, satisfying the equations (3.7) in the ring  $\mathbb{R}[x]/(x - x_i)^{e+2d}$ . Note that  $\bar{f} + \bar{g}$  is invertible in  $\mathbb{R}[x]/(x - x_i)^{e+2d}$ , and that  $\bar{h}$  has valuation  $d$ .

In order to simplify notation, we denote again by  $f, g, h$  the elements  $\bar{f}, \bar{g}, \bar{h}$ , respectively. Let  $k \in \mathbb{R}[x]/(x - x_i)^{e+2d}$  be the inverse of  $f + g$ . Let  $a = hk$ . We verify that equations (3.8) hold and that  $1 + a^2$  is invertible in  $\mathbb{R}[x]/(x - x_i)^e$ .

The element  $1 + a^2$  is clearly invertible in  $\mathbb{R}[x]/(x - x_i)^{e+2d}$  since  $h$  is not invertible.

Since

$$(f - g)(f + g) = f^2 - g^2 = (1 - x^2) - (1 - x^2 - h^2) = h^2,$$

one has

$$f - g = h^2 k = h^2 k^2 (f + g) = a^2 (f + g).$$

It follows that

$$(1 - a^2)f = (1 + a^2)g$$

in  $\mathbb{R}[x]/(x - x_i)^{e+2d}$ , and therefore also in  $\mathbb{R}[x]/(x - x_i)^e$ .

In order to prove that the other equation of (3.8) holds as well, observe that

$$\begin{aligned} (f - g)^2 h - 2f(f - g)h + h^3 &= (f - g)h(f - g - 2f) + h^3 = \\ &= -(f - g)h(f + g) + h^3 = -(f^2 - g^2)h + h^3 = 0 \end{aligned}$$



by what we have seen above. Substituting  $f - g = ah$ , one obtains

$$0 = a^2h^3 - 2afh^2 + h^3 = h^2(a^2h - 2af + h)$$

in  $\mathbb{R}[x]/(x - x_i)^{e+2d}$ . Since the valuation of  $h$  is equal to  $d$ , one deduces that  $a^2h - 2af + h = 0$  in  $\mathbb{R}[x]/(x - x_i)^e$ . Hence,  $2af = (1 + a^2)h$ , as was to be proved.  $\square$

*Proof of Theorem 3.1.* Let  $P_1, \dots, P_\ell$  be distinct infinitely near points of  $S^2$  of orders  $e_1, \dots, e_\ell$ , respectively. Let  $Q_i$  be the infinitely near point of  $S^2$  defined by the ideal

$$((x - x_i)^{e_i}, y - f_i, z)$$

in  $\mathbb{R}[x, y, z]$ , where  $f_i$  is the Taylor polynomial in  $x - x_i$  of  $\sqrt{1 - x^2}$  at  $x_i$  of order  $e_i - 1$ . Note that  $Q_i$  is infinitely near to  $R_i$  for all  $i$ . We show that there is an algebraic diffeomorphism  $\varphi$  of  $S^2$  such that  $\varphi(Q_i) = P_i$  for all  $i$ .

By Lemma 3.5, we may assume that  $P_1, \dots, P_\ell$  are infinitely near to the points  $R_1, \dots, R_\ell$  of  $S^2$ , respectively, and that they are not vertical. It follows that  $P_i$  is defined by an ideal of the form

$$((x - x_i)^{e_i}, y - g_i, z - h_i)$$

where  $g_i, h_i \in \mathbb{R}[x]$  are of degree  $< e_i$ . Moreover, since  $P_i$  is an infinitely near point of  $S^2$ , we have

$$x^2 + g_i^2 + h_i^2 = 1 \pmod{(x - x_i)^{e_i}}.$$

By Lemma 3.6, there is  $a_i \in \mathbb{R}[x]/(x - x_i)^{e_i}$  such that

$$\begin{aligned} (1 - a_i^2)f_i &= (1 + a_i^2)g_i, \quad \text{and} \\ 2a_i f_i &= (1 + a_i^2)h_i \end{aligned}$$

in  $\mathbb{R}[x]/(x - x_i)^{e_i}$ , and, moreover,  $1 + a_i^2$  is invertible.

By the Chinese Remainder Theorem, there is a polynomial  $a \in \mathbb{R}[x]$  such that  $a = a_i \pmod{(x - x_i)^{e_i}}$ , for all  $i$ . Then

$$\begin{aligned} (1 - a^2)f_i &= (1 + a^2)g_i \pmod{(x - x_i)^{e_i}} \\ 2af_i &= (1 + a^2)h_i \pmod{(x - x_i)^{e_i}}, \end{aligned}$$

and  $1 + a^2$  is invertible in  $\mathbb{R}[x]/(x - x_i)^{e_i}$ , for all  $i$ .

Put

$$p = 1 - a^2, \quad q = 2a, \quad r = 1 + a^2.$$

Then

$$\begin{aligned} pf_i &= rg_i \pmod{(x - x_i)^{e_i}} \\ qf_i &= rh_i \pmod{(x - x_i)^{e_i}}, \end{aligned}$$

and  $r$  is invertible in  $\mathbb{R}[x]/(x - x_i)^{e_i}$ , for all  $i$ . Moreover,

- $r$  does not have any roots in the interval  $[-1, 1]$ , and
- $p^2 + q^2 = r^2$ .

By Lemma 3.3, the polynomials  $p, q, r$  give rise to an algebraic diffeomorphism  $\varphi$  of  $S^2$  defined by

$$\varphi(x, y, z) = \left( x, \frac{yp - zq}{r}, \frac{yq + zp}{r} \right).$$

In order to show that  $\varphi(Q_i) = P_i$  for all  $i$ , we compute

$$\begin{aligned} (\varphi^{-1})^*((x - x_i)^{e_i}) &= (x - x_i)^{e_i} \\ u_i &= (\varphi^{-1})^*(y - f_i) = \frac{yp + zq}{r} - f_i \\ v_i &= (\varphi^{-1})^*(z) = \frac{-yq + zp}{r}, \end{aligned}$$

so that  $\varphi(Q_i)$  is the infinitely near point of  $S^2$  defined by the ideal  $((x - x_i)^{e_i}, u_i, v_i)$ . We have

$$\frac{p}{r}u_i - \frac{q}{r}v_i = y - \frac{p}{r}f_i = y - g_i \pmod{(x - x_i)^{e_i}}$$

and

$$\frac{q}{r}u_i + \frac{p}{r}v_i = z - \frac{q}{r}f_i = z - h_i \pmod{(x - x_i)^{e_i}}.$$

It follows that  $\varphi(Q_i) = P_i$ .  $\square$

#### 4. ALGEBRAIC DIFFEOMORPHISMS OF NONSINGULAR RATIONAL SURFACES

The proof of Theorem 1.1 is now similar to the proof of [HM08, Theorem 1.4]. We include it for convenience of the reader.

*Proof of Theorem 1.1.* Let  $X$  be a nonsingular real rational surface. As mentioned before,  $X$  is algebraically diffeomorphic to  $S^1 \times S^1$  or to the blow-up of  $S^2$  at a finite number of distinct points  $R_1, \dots, R_m$ . If  $X$  is algebraically diffeomorphic to  $S^1 \times S^1$  then  $\text{Diff}_{\text{alg}}(X)$  acts transitively on  $X^{\mathbf{e}}$  by Theorem 2.1. Therefore, we may assume that  $X$  is algebraically diffeomorphic to the blow-up  $B_{R_1, \dots, R_m}(S^2)$  of  $S^2$  at  $R_1, \dots, R_m$ . We will show that  $\text{Diff}_{\text{alg}}(X)$  acts transitively on  $X^{\mathbf{e}}$  for all partitions  $\mathbf{e}$  of any natural integer  $n$ , by induction on  $m$ .

If  $m = 0$ , then  $X$  is algebraically diffeomorphic to  $S^2$  and  $\text{Diff}_{\text{alg}}(X)$  acts transitively on  $X^{\mathbf{e}}$ , for all  $\mathbf{e}$ , by Theorem 3.1. Let  $m > 0$ , and let  $X$  be  $B_{R_1, \dots, R_m}(S^2)$ . Let  $(P_1, \dots, P_n)$  and  $(Q_1, \dots, Q_n)$  be two  $n$ -tuples in  $X^{\mathbf{e}}$ . We want to show that there is an algebraic diffeomorphism  $\varphi$  of  $X$  such that  $\varphi(P_j) = Q_j$  for all  $j$ .

We identify  $\mathbb{P}^2(\mathbb{R})$  with  $B_{R_m}(S^2)$  via an algebraic diffeomorphism. We may consider  $R_1, \dots, R_{m-1}$  as points of  $\mathbb{P}^2(\mathbb{R})$  and the surface  $X$  is the surface  $B_{R_1, \dots, R_{m-1}}(\mathbb{P}^2(\mathbb{R}))$ . Let  $\pi: X \rightarrow \mathbb{P}^2(\mathbb{R})$  be the blowing-up morphism. Let  $L$  be a line in  $\mathbb{P}^2(\mathbb{R})$  that does not contain any of the points  $R_k, \pi(P_j)_{\text{red}}, \pi(Q_j)_{\text{red}}$ . The inverse image  $\tilde{L}$  of  $L$  in  $X$  is a contractible real algebraic curve in  $X$ , see [HM08, Theorem 3.2].

Let  $Y$  be the nonsingular rational compact connected real algebraic surface obtained from  $X$  by contracting  $\tilde{L}$  to a point. Let  $\rho: X \rightarrow Y$  be the morphism that contracts  $\tilde{L}$  to a point  $P$ . Let  $\sigma: \mathbb{P}^2(\mathbb{R}) \rightarrow S^2$  be the morphism that contracts the line  $L$  of  $\mathbb{P}^2(\mathbb{R})$  to a point. Then  $\pi$  induces a

morphism  $\tau: Y \rightarrow S^2$ , i.e., one has the following diagram:

$$\begin{array}{ccc}
 X & & \\
 \downarrow \pi & \searrow \rho & \\
 \mathbb{P}^2(\mathbb{R}) & & Y \\
 & \searrow \sigma & \downarrow \tau \\
 & & S^2
 \end{array}$$

The morphism  $\tau$  is the blow-up of  $S^2$  at the points  $R_1, \dots, R_{m-1}$ . Since the real algebraic curve  $\tilde{L}$  does not contain any of the points  $(P_j)_{\text{red}}$  or  $(Q_j)_{\text{red}}$  of  $X$ , the infinitely near points  $\rho(P_1), \dots, \rho(P_n)$  are distinct, and the same holds for the points  $\rho(Q_1), \dots, \rho(Q_n)$ . Moreover,  $P \neq \sigma(P_j)_{\text{red}}$  and  $P \neq \sigma(Q_j)_{\text{red}}$  for all  $j$ . By the induction hypothesis, the group  $\text{Diff}_{\text{alg}}(Y)$  acts transitively on  $Y^{\mathbf{d}}$ , where  $\mathbf{d}$  is the partition  $[e_1, \dots, e_\ell, 1]$  of  $n+1$ . Therefore, there is an algebraic diffeomorphism  $\psi$  of  $Y$  such that  $\psi(\rho(P_j)) = \rho(Q_j)$  and  $\psi(P) = P$ . Since  $X$  is the blow-up of  $Y$  at  $P$ , the map  $\psi$  induces an algebraic diffeomorphism  $\varphi$  of  $X$  with the required property.  $\square$

## 5. RATIONAL SURFACES WITH $A^-$ SINGULARITIES

The object of this section is to prove Theorem 1.3 that asserts that a real rational surface with weighted blow-up singularities is isomorphic to a real algebraic surface obtained from  $S^2$  or  $S^1 \times S^1$  by blowing up a finite number of distinct infinitely near points. We need the following lemmas.

**Lemma 5.1.** *Let  $X$  be a nonsingular rational real algebraic Klein bottle. Let  $S$  be a finite subset of  $X$ . Then there is an algebraic map  $f: X \rightarrow S^2$  such that*

- (1)  $f$  is the blow-up of  $S^2$  at 2 distinct real points  $Q_1, Q_2$ , and
- (2)  $Q_i \notin f(S)$ , for  $i = 1, 2$ .  $\square$

*Proof.* Since all nonsingular rational compact connected real algebraic Klein bottles are algebraically diffeomorphic (see [Ma06, Theorem 1.3], [BH07, Theorem 1.2] or [HM08, Theorem 1.5]), we may assume that  $K$  is obtained from  $S^2$  by blowing up the points  $(0, 0, \pm 1)$ . Let  $\beta: K \rightarrow S^2$  be the blow-up map. There are disjoint euclidean circles  $C_1$  and  $C_2$  on  $S^2$  with the following properties:

- $(0, 0, 1) \in C_1$ ,  $(0, 0, -1) \notin C_1$ ,
- $(0, 0, 1) \notin C_2$ ,  $(0, 0, -1) \in C_2$ , and
- the strict transforms  $\tilde{C}_1$  and  $\tilde{C}_2$  of  $C_1$  and  $C_2$  in  $K$  do not contain any of the points belonging to  $S$ .

The real algebraic curves  $\tilde{C}_1$  and  $\tilde{C}_2$  are contractible in  $K$ , according to [HM08, Theorem 3.2]. The nonsingular real algebraic surface obtained from  $K$  by

contracting  $\tilde{C}_1$  and  $\tilde{C}_2$  is homeomorphic to  $S^2$ , hence algebraically diffeomorphic to  $S^2$ . The algebraic map  $f: K \rightarrow S^2$  contracting  $\tilde{C}_1$  and  $\tilde{C}_2$  has the required properties.  $\square$

**Lemma 5.2.** *Let  $P$  be an infinitely near point of  $S^1 \times S^1$ , and let  $C$  be a real algebraic curve in  $S^1 \times S^1$  such that there is a nonsingular projective complexification  $\mathcal{X}$  of  $S^1 \times S^1$  having the following properties:*

- (1) *the Zariski closure  $\mathcal{C}$  of  $C$  in  $\mathcal{X}$  is nonsingular and rational,*
- (2) *the self-intersection of  $\mathcal{C}$  in  $\mathcal{X}$  is even and non-negative,*
- (3)  *$P_{\text{red}} \in \mathcal{C}$ , and*
- (4)  *$\mathcal{C}$  is not tangent to  $P$ , i.e., the scheme-theoretic intersection  $P \cdot \mathcal{C}$  is of length 1.*

*Then, there is an algebraic map*

$$f: B_P(S^1 \times S^1) \rightarrow Z$$

*that is a blow-up at an infinitely near point  $Q$  whose exceptional curve  $f^{-1}(Q_{\text{red}})$  is equal to the strict transform of  $C$  in  $B_P(S^1 \times S^1)$ , where  $Z$  is either the real algebraic torus  $S^1 \times S^1$ , or the rational real algebraic Klein bottle  $K$ .*

*Proof.* Let  $Y$  be the blow-up of  $S^1 \times S^1$  at  $P$ . Let  $\beta: \mathcal{Y} \rightarrow \mathcal{X}$  be the blow-up of  $\mathcal{X}$  at  $P$ . It is clear that  $\mathcal{Y}$  is a nonsingular projective complexification of  $Y$ .

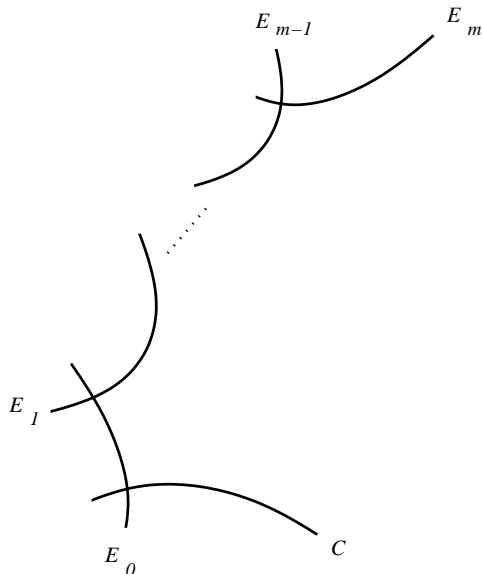
Let  $m + 1$  be the length of the infinitely near point  $P$ , where  $m \geq 0$ . Let  $\rho: \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  be the minimal resolution of  $\mathcal{Y}$ . If  $P$  is a point of length 1, then  $\rho = \text{id}$ , of course. The morphism  $\beta \circ \rho$  is a repeated blow-up of  $\mathcal{X}$ . More precisely, there is a sequence of morphisms of algebraic varieties over  $\mathbb{R}$

$$\tilde{\mathcal{Y}} = \mathcal{X}_{m+1} \xrightarrow{f_m} \mathcal{X}_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_0} \mathcal{X}_0 = \mathcal{X},$$

with the following properties. Each morphism  $f_i$  is an ordinary blow-up of  $\mathcal{X}_i$  at a nonsingular ordinary real point  $P_i$  of  $\mathcal{X}_i$ , for all  $i$ . One has  $P_0 = P_{\text{red}}$ , i.e.,  $f_0$  is the blow-up of  $\mathcal{X}$  at  $P_{\text{red}}$ . Moreover, denoting by  $\mathcal{E}_i$  the exceptional curve of  $f_i$  in  $\mathcal{X}_{i+1}$ , the center of blow-up  $P_{i+1}$  belongs to  $\mathcal{E}_i$  but does not belong to the strict transform of any of the curves  $\mathcal{E}_j$  in  $\mathcal{X}_{i+1}$  for all  $j < i$ .

Denote again by  $\mathcal{E}_i$  and  $\mathcal{C}$  the strict transforms of  $\mathcal{E}_i$  and  $\mathcal{C}$  in  $\tilde{\mathcal{Y}}$ , respectively. The curves  $\mathcal{E}_0, \dots, \mathcal{E}_{m-1}$  have self-intersection  $-2$ , the curve  $\mathcal{E}_m$  has self-intersection  $-1$ . Since  $\mathcal{C}$  is not tangent to  $P$ , the curve  $\mathcal{C}$  in  $\tilde{\mathcal{Y}}$  has odd self-intersection  $\geq -1$ . The curves  $\mathcal{C}, \mathcal{E}_0, \dots, \mathcal{E}_m$  form a chain of curves over  $\mathbb{R}$  in  $\mathcal{Y}$ , intersecting in real points only (See Figure 1). The morphism  $\rho: \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  is the contraction of the curves  $\mathcal{E}_0, \dots, \mathcal{E}_{m-1}$ . The morphism  $\beta: \mathcal{Y} \rightarrow \mathcal{X}$  is the contraction of  $\rho(\mathcal{E}_m)$ , i.e.,  $\beta^{-1}(P_{\text{red}}) = \rho(\mathcal{E}_m)$ .

Let  $k$  be the self-intersection of  $\mathcal{C}$  in  $\tilde{\mathcal{Y}}$ . Since  $k \geq -1$  and  $k \equiv -1 \pmod{2}$ , the integer  $k + 1$  is even and non-negative. Let  $R_1, \dots, R_{k+1}$  be pairwise complex conjugate of  $\mathcal{C}$ . Denote by  $\tilde{\mathcal{Y}}'$  the blow-up of  $\tilde{\mathcal{Y}}$  in  $R_1, \dots, R_{k+1}$ . The algebraic variety  $\tilde{\mathcal{Y}}'$  is again defined over  $\mathbb{R}$ . The strict transform of  $\mathcal{C}$  in  $\tilde{\mathcal{Y}}'$  is a nonsingular rational curve of self-intersection  $-1$ . Denote again by  $\mathcal{E}_i$  the strict transform of  $\mathcal{E}_i$  in  $\tilde{\mathcal{Y}}'$ . The self-intersection of  $\mathcal{E}_i$  is equal to  $-2$ , if  $i \neq m$ , the self-intersection of  $\mathcal{E}_m$  is equal to  $-1$ .

FIGURE 1. The chain of exceptional curves in  $\tilde{\mathcal{Y}}$ .

Let  $\mathcal{Y}'$  be the algebraic surface defined over  $\mathbb{R}$  obtained from  $\tilde{\mathcal{Y}}'$  by contracting the union of the curves  $\mathcal{E}_0 \dots, \mathcal{E}_{m-1}$  to a point, and let  $\rho': \tilde{\mathcal{Y}}' \rightarrow \mathcal{Y}'$  be the contracting morphism. Let  $\mathcal{X}'$  be the algebraic surface defined over  $\mathbb{R}$  obtained from  $\mathcal{Y}'$  by contracting  $\rho'(C)$  to a point, and let  $\beta': \mathcal{Y}' \rightarrow \mathcal{X}'$  be the contracting morphism. Since  $\beta' \circ \rho'$  is a repeated blow-down of  $-1$ -curves, the algebraic surface  $\mathcal{X}'$  is nonsingular. Moreover, the morphism  $\beta'$  is a blow-up of  $\mathcal{X}'$  at a nonsingular infinitely near point  $Q$  of  $\mathcal{X}'$ . Denote again by  $C$  the curve  $\rho'(C)$  in  $\mathcal{Y}'$ . The curve  $C$  in  $\mathcal{Y}'$  is the exceptional curve of  $\beta'$ .

Now take the associated real algebraic varieties, denoted by the corresponding roman characters. Since the points  $R_1, \dots, R_{k+1}$  are non real, one has  $\tilde{\mathcal{Y}}'(\mathbb{R}) = \tilde{\mathcal{Y}}(\mathbb{R})$ , i.e.,  $\tilde{Y}' = \tilde{Y}$ , the minimal resolution of  $Y$ . It follows that  $Y' = Y$ , and that the induced algebraic map  $b: Y' \rightarrow X'$  is the blow-up of the infinitely near point  $Q$  of the nonsingular compact connected real algebraic surface  $X'$ . The exceptional curve of  $b$  is equal to the strict transform of  $C$  in  $Y$ .

The only thing that is left to prove is the fact that the real algebraic surface  $X'$  is isomorphic to  $S^1 \times S^1$  or to the rational real algebraic Klein bottle  $K$ . In order to establish this, observe that  $\tilde{Y}$ , as an  $(m+1)$ -fold blow-up of  $S^1 \times S^1$ , is homeomorphic to the connected sum of  $S^1 \times S^1$  and  $m+1$  copies of  $\mathbb{P}^2(\mathbb{R})$ . Since  $Y'$  also is homeomorphic to the connected sum of  $X'$  and  $m+1$  copies of  $\mathbb{P}^2(\mathbb{R})$ , it follows that  $X'$  is homeomorphic to a torus or a Klein bottle. By [Ma06, Theorem 1.3], or [BH07, Theorem 1.2], or [HM08, Theorem 1.5],  $X'$  is algebraically diffeomorphic to  $S^1 \times S^1$  or the rational real algebraic Klein bottle  $K$ .  $\square$

A similar, but easier, argument applies and proves the following lemma.

**Lemma 5.3.** *Let  $P$  be an infinitely near point of  $S^2$ , and let  $C$  be a real algebraic curve in  $S^2$  such that there is a nonsingular projective complexification  $\mathcal{X}$  of  $S^2$  having the following properties:*

- (1) the Zariski closure  $\mathcal{C}$  of  $C$  in  $\mathcal{X}$  is nonsingular and rational,
- (2) the self-intersection of  $\mathcal{C}$  in  $\mathcal{X}$  is even and non-negative,
- (3)  $P_{\text{red}} \in \mathcal{C}$ , and
- (4)  $\mathcal{C}$  is not tangent to  $P$ , i.e., the scheme-theoretic intersection  $P \cdot \mathcal{C}$  is of length 1.

Then, there is an algebraic map

$$f: B_P(S^2) \rightarrow S^2$$

that is the blow-up of  $S^2$  at an infinitely near point  $Q$ . Moreover, the exceptional curve  $f^{-1}(Q)$  is equal to the strict transform of  $C$  in  $B_P(S^2)$ .  $\square$

*Proof of Theorem 1.3.* There is a nonsingular rational compact connected real algebraic surface  $Y$  such that  $X$  is isomorphic to the real algebraic surface obtained from  $Y$  by repeatedly blowing up a nonsingular infinitely near point. Since  $Y$  is a nonsingular rational compact connected real algebraic surface,  $Y$  is obtained either from  $S^2$  or from  $S^1 \times S^1$ , by repeatedly blowing up an ordinary point (cf. [BH07, Theorem 3.1] or [HM08, Theorem 4.1]). Hence, there is a sequence of algebraic maps

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 = Z,$$

where  $Z = S^2$  or  $Z = S^1 \times S^1$ , and each map  $f_i$  is a blow-up at a nonsingular infinitely near point  $Q_i$  of  $X_{i-1}$ , possibly of length 1, for  $i = 1, \dots, n$ . Denote by  $E_i$  the exceptional divisor  $f_i^{-1}((Q_i)_{\text{red}})$  of  $f_i$  in  $X_i$ .

Let  $\mathcal{F}$  be the set of the infinitely near points  $Q_i$ . Define a partial ordering on  $\mathcal{F}$  by  $Q_i \leq Q_j$  if the composition  $f_i \circ \cdots \circ f_{j-1}$  maps  $(Q_j)_{\text{red}}$  to  $(Q_i)_{\text{red}}$ . It is clear that  $\mathcal{F}$  is a forest, i.e., a disjoint union of trees.

Let  $s$  be the number of edges in the forest  $\mathcal{F}$ . We show the statement of the theorem by induction on  $s$ . The statement is clear for  $s = 0$ . Suppose, therefore, that  $s \neq 0$ . We may assume that  $Q_1$  is the root of a tree of  $\mathcal{F}$  of nonzero height.

Let  $C$  be a real algebraic curve in  $Z$  satisfying the conditions of Lemma 5.2 if  $Z = S^1 \times S^1$ , and of Lemma 5.3 if  $Z = S^2$ , with  $P = (Q_1)_{\text{red}}$ . Such curves abound: one can take a bi-degree  $(1, 1)$  in  $S^1 \times S^1$ , or a Euclidean circle in  $S^2$ , respectively. Moreover, we may assume that the strict transform of  $C$  in  $X_i$  does not contain  $(Q_{i+1})_{\text{red}}$ , for all  $i \geq 1$ . Applying Lemma 5.2 and Lemma 5.3, respectively, one obtains a sequence

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f'_1} X'_0 = Z',$$

where  $f'_1$  contracts the strict transform of  $C$  in  $X_1$  to a point  $Q'_1$ . The real algebraic surface  $Z'$  is either the real algebraic sphere  $S^2$ , or the real algebraic torus  $S^1 \times S^1$ , or the rational real algebraic Klein bottle  $K$ . By construction, the number of edges in the forest  $\mathcal{F}'$  associated to the latter sequence of blow-ups is equal to  $s - 1$ . Therefore, if  $Z' = S^2$  or  $Z' = S^1 \times S^1$ , we are done. If  $Z'$  is the real algebraic Klein bottle  $K$ , then, according to Lemma 5.1, there is a sequence of blow-ups

$$Z' = X'_0 \xrightarrow{f_0} X'_{-1} \xrightarrow{f_{-1}} X_{-2} = S^2$$

at ordinary points such that the images of the centers  $Q'_1, Q_2, \dots, Q_n$  in  $X_{-1}$  and  $X_{-2}$  are distinct from the centers of the blow-ups  $f_0$  and  $f_{-1}$ . We conclude also in this case by the induction hypothesis.  $\square$

A close inspection of the above proof reveals that the following slightly more technical statement holds.

**Theorem 5.4.** *Let  $X$  be a rational compact connected real algebraic surface whose singularities are weighted blow-ups, and let  $S \subseteq X$  be a finite subset of nonsingular points of  $X$ . Then there is an algebraic map  $f: X \rightarrow S^2$  or  $f: X \rightarrow S^1 \times S^1$  with the following properties:*

- (1) *there are distinct infinitely near points  $P_1, \dots, P_\ell$  on  $S^2$  or  $S^1 \times S^1$ , respectively, such that  $f$  is the blow-up at  $P_1, \dots, P_\ell$ , and*
- (2)  *$(P_i)_{\text{red}} \notin f(S)$ , for all  $i$ .*  $\square$

## 6. INFINITELY NEAR POINTS ON A SINGULAR RATIONAL SURFACE

The object of this section is to prove Theorem 1.4.

*Proof of Theorem 1.4.* Let  $(P_1, \dots, P_\ell)$  and  $(Q_1, \dots, Q_\ell)$  be two elements of  $X^e$ . We prove that there is an algebraic automorphism  $\varphi$  of  $X$  such that  $\varphi(P_i) = Q_i$ , as infinitely near points.

Let  $S$  be the set of ordinary points  $(P_1)_{\text{red}}, \dots, (P_\ell)_{\text{red}}, (Q_1)_{\text{red}}, \dots, (Q_\ell)_{\text{red}}$  of  $X$ . Since  $S$  is a finite set of nonsingular points of  $X$ , there is, by Theorem 5.4, an algebraic map  $f: X \rightarrow S^2$  or  $f: X \rightarrow S^1 \times S^1$  with the following properties:

- there are distinct infinitely near points  $R_1, \dots, R_n$  on  $S^2$  such that  $f$  is the blow-up at  $R_1, \dots, R_n$ , and
- $(R_i)_{\text{red}} \notin f(S)$ , for all  $i$ .

Since  $f$  is an isomorphism at a neighborhood of  $S$ , the image  $f(P_i)$  is an infinitely near point of  $S^2$  of length  $e_i$ , and the same holds for  $f(Q_i)$ , for all  $i$ .

By Theorems 2.1 and 3.1, there is an algebraic diffeomorphism  $\psi$  of  $S^2$  or  $S^1 \times S^1$ , respectively, such that  $\psi(P_i) = Q_i$  for  $i = 1, \dots, \ell$ , and  $\psi(R_i) = R_i$  for  $i = 1, \dots, n$ . Then,  $\psi$  induces an algebraic diffeomorphism  $\varphi$  of  $X$  with the required property.  $\square$

## 7. ISOMORPHIC RATIONAL REAL ALGEBRAIC SURFACES

*Proof of Theorem 1.5.* Let  $X$  and  $Y$  be rational compact connected real algebraic surfaces having weighted blow-up singularities, such that each of the surfaces  $X$  and  $Y$  has exactly one singularity of type  $A_{e_i}^-$ , for each  $i = 1, \dots, \ell$ .

If  $X$  and  $Y$  are isomorphic, then, of course, the singular topological surfaces  $X$  and  $Y$  are homeomorphic.

Conversely, suppose that  $X$  and  $Y$  are homeomorphic. By Theorem 1.3, there are nonsingular real rational surfaces  $X'$  and  $Y'$ , and  $\ell$ -tuples  $(P_1, \dots, P_\ell) \in (X')^e$  and  $(Q_1, \dots, Q_\ell) \in (Y')^e$  such that  $X$  is the blow-up of  $X'$  at  $P_1, \dots, P_\ell$  and  $Y$  is the blow-up of  $Y'$  at  $Q_1, \dots, Q_\ell$ . Since  $X$  and  $Y$  are homeomorphic,  $X'$  and  $Y'$  are homeomorphic. It follows that  $X'$  and  $Y'$  are algebraically diffeomorphic. By Theorem 1.1, there is an

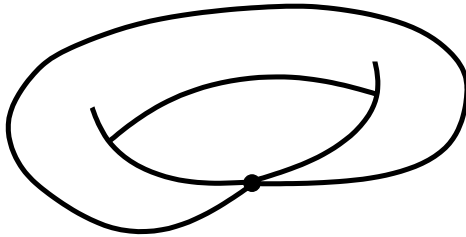


FIGURE 2. The once-pinched torus.

algebraic diffeomorphism  $\varphi: X \rightarrow Y$  such that  $\varphi(P_i) = Q_i$  for  $i = 1, \dots, \ell$ . The diffeomorphism  $\varphi$  induces an algebraic isomorphism between  $X$  and  $Y$ .  $\square$

The following example shows that the statement of Theorem 1.5 does not hold for the slightly more general class of rational compact connected real algebraic surfaces that have singularities of type  $A^-$ .

*Example 7.1.* Let  $X$  be the real algebraic surface obtained from the real algebraic torus  $S^1 \times S^1$  by contracting a fiber  $S^1 \times \{\star\}$  to a point. Then  $X$  is a rational compact connected real algebraic surface having only one singular point. Its singularity is of type  $A_1^-$ .

Let  $P$  be a point of  $\mathbb{P}^2(\mathbb{R})$ . The real algebraic surface  $K$  obtained from  $\mathbb{P}^2(\mathbb{R})$  by blowing up  $P$  is a real algebraic Klein bottle. Let  $Y$  be the real algebraic surface obtained from the real algebraic Klein bottle  $K$  by contracting to a point the strict transform of a real projective line in  $\mathbb{P}^2(\mathbb{R})$  that passes through  $P$ . Then  $Y$  is a rational compact connected real algebraic surface having only one singular point. Its singularity is of type  $A_1^-$ .

It is clear that  $X$  and  $Y$  are homeomorphic singular surfaces. Indeed, they are both rational real algebraic models of the once-pinched torus (Figure 2). However, they are non isomorphic as real algebraic surfaces. Indeed, if they were isomorphic, their minimal resolutions  $S^1 \times S^1$  and  $K$  were isomorphic, which is absurd.

Note that  $Y$  is a rational real algebraic surface having weighted blow-up singularities, whereas  $X$  is not.

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