On the enumerative geometry of real algebraic curves having many real branches

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Fulton (1984): Study the number of real solutions of enumerative problems in algebraic geometry

Example 1. Determine the number of real conics, tangent to 5 given smooth real conics. All of the 3264 complex conics tangent to them can be real (Fulton and Ronga-Tognoli-Vust, 1997)

Example 2. Given positive integers $l_1, \dots, l_s$ with $l_1 + \cdots + l_s = 2n - 2$, there are real linear subspaces $L_1, \dots, L_s$ of $\mathbb{P}^n$ with $\dim L_i = n - 1 - l_i$ such that all complex lines meeting all subspaces L_i are real (Sottile, 1997)

Let $C \subseteq \mathbb{P}^2$ be a smooth real algebraic curve, $\deg(C) = c$, $g(C) = \frac{1}{2}(c-1)(c-2)$.

$C(\mathbb{R})$ is a smooth 1-dimensional submanifold of $\mathbb{P}^2(\mathbb{R})$. Therefore, $C(\mathbb{R})$ is a finite disjoint union of smooth circles in $\mathbb{P}^2(\mathbb{R})$. In particular, $\#\pi_0(C(\mathbb{R})) < \infty$.

Harnack's Inequality (1876):

$$\#\pi_0(C(\mathbb{R})) \leq g + 1.$$

C is an M-*curve* if $\#\pi_0(C(\mathbb{R})) = g + 1$.
C is an $(M-1)$-*curve* if $\#\pi_0(C(\mathbb{R})) = g$.

We say that C *has many real branches* if C is an M or $(M-1)$-curve and $g \geq 1$.

Let $C \subseteq \mathbb{P}^2$ be a smooth real algebraic curve having many real branches.

Theorem 1. *Let* $\mathbf{e}$ *be a partition of* $c(c-1)$ *of length* g. *Let* ν *be the number of real plane curves of degree* $c-1$ *having tangency* $\mathbf{e}$ *to* g *real branches of* C. *Then,* ν *is finite. Moreover,* $\nu \neq 0$ *if and only if* $\mathbf{e}$ *is an even partition. In that case,*

$$\nu = \begin{cases} \dfrac{g!}{m_1! \cdots m_r!} \cdot \displaystyle\prod_{i=1}^{g} \mathbf{e}_i & \text{if } C \text{ is an } (M-1), \\ \dfrac{(g+1)!}{m_1! \cdots m_r!} \cdot \displaystyle\prod_{i=1}^{g} \mathbf{e}_i & \text{if } C \text{ is an } M\text{-curve}, \end{cases}$$

where $m_1, \ldots, m_r$ *are the multiplicities of* $\mathbf{e}$.

Example 3. $c = 4$, $g = 3$, $\#\pi_0(C(\mathbb{R})) \geq 3$. The even partitions of $4 \cdot 3 = 12$ of length 3 are:

$$(8,2,2),\ (6,4,2) \text{ and } (4,4,4).$$

Let $\mathbf{e}$ be one of them. Let ν be the number of real cubics tangent to 3 real branches of C with orders of tangency $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

If $\#\pi_0(C(\mathbb{R})) = 3$ then

$$\nu = \begin{cases} 96 & \text{if } \mathbf{e} = (8,2,2), \\ 288 & \text{if } \mathbf{e} = (6,4,2), \\ 64 & \text{if } \mathbf{e} = (4,4,4). \end{cases}$$

If $\#\pi_0(C(\mathbb{R})) = 4$ then

$$\nu = \begin{cases} 384 & \text{if } \mathbf{e} = (8,2,2), \\ 1152 & \text{if } \mathbf{e} = (6,4,2), \\ 256 & \text{if } \mathbf{e} = (4,4,4). \end{cases}$$

Theorem 1 is a consequence of the following statement:

Theorem 2. *Let* $B_1, \ldots, B_g$ *be mutually distinct real branches of* C *and put*

$$B = \prod_{i=1}^{g} B_i.$$

Let $e_1, \ldots, e_g$ *be nonzero natural integers, and let*

$$\varphi\colon B \longrightarrow \mathrm{Pic}(C)$$

be the map defined by

$$\varphi(P) = \mathrm{cl}(\sum_{i=1}^{g} e_i P_i),$$

where cl *denotes the divisor class. Then,* φ *is a topological covering of its image of degree* $\prod_{i=1}^{g} e_i$.

Proof. Let $P \in B, v \in T_P B$. Suppose that $T\varphi(v) = 0$. Each (P_i, v_i) determines a morphism

$$f_i \colon T \longrightarrow C' = C \times_{\mathsf{Spec}(\mathbb{R})} T,$$

where $T = \mathsf{Spec}(\mathbb{R}[\varepsilon])$. Then $\mathsf{im}(f_i)$ is a relative Cartier divisor D_i of C'/T. If $P_i = \{x_i = 0\}$ locally, then

$$D_i = \{x_i - \lambda_i \varepsilon = 0\}$$

locally on C', for some $\lambda_i \in \mathbb{R}$.

One has a naturally split short exact sequence

$$0 \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow \mathsf{Pic}(C') \longrightarrow \mathsf{Pic}(C) \longrightarrow 0.$$

Let $D = \sum e_i D_i$ on C'/T. Since $T\varphi(v) = 0$,

$$\mathsf{cl}(D) \in \mathsf{im}(\mathsf{Pic}(C) \longrightarrow \mathsf{Pic}(C')) = \\ = \ker(\mathsf{Pic}(C') \longrightarrow H^1(C, \mathcal{O}_C)).$$

$$H^1(C, \mathcal{O}_C) = \text{coker}(K \longrightarrow \bigoplus_{Q \in C} K/\mathcal{O}_Q),$$

where $\mathcal{O}_Q$ is the local ring of C at Q and K is the function field of C. Since

$$(x_i - \lambda_i\varepsilon)^{e_i} = x_i^{e_i} - e_i\lambda_i x_i^{e_i - 1}\varepsilon = x_i^{e_i}(1 - e_i\lambda_i \frac{1}{x_i}\varepsilon),$$

the image of $\text{cl}(D)$ in $H^1(C, \mathcal{O}_C)$ is equal to $\rho = (\rho_Q)$, where

$$\rho_Q = \begin{cases} -e_i\lambda_i \frac{1}{x_i} & \text{if } Q = P_i, \\ 0 & \text{otherwise} \end{cases}$$

Let $i \in \{1, \ldots, g\}$. By Riemann-Roch, $\exists \omega_i \in H^0(C, \Omega_C)$, $\omega_i \neq 0$, such that $\omega_i(P_j) = 0$, $j \neq i$. The form ω_i has at least 2 zeros on each of the real branches B_j, $j \neq i$. It follows that $\omega_i \neq 0$ on B_i. In particular, $\omega_i(P_i) \neq 0$.

Let t be the trace map

$$t\colon H^1(C,\Omega_C) \longrightarrow \mathbb{R}.$$

Since $\rho = 0$, one has $t(\rho\omega) = 0$. Hence

$$\mathrm{res}_{P_i}(-e_i\lambda_i\frac{1}{x_i}\omega) = 0.$$

Therefore, $\lambda_i = 0$, and $v = 0$. This proves that φ is unramified. $\square$